Snippets of Physics

20. Random Walk Through Random Walks – II

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We continue our exploration of random walks with some more curious results. We discuss the dimension dependence of some of the features of the random walk, describe an unexpected connection between random walks and electrical networks and finally discuss some remarkable features of random walk with geometrically decreasing step-length.

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In the last instalment, we looked at several elementary features of random walk and, in particular, obtained a general formula for the probability $P_N(\mathbf{x})$, for the particle to be found at position $\mathbf{x}$ after N steps. This result, in the case of random walk in a cubic lattice can be written as the integral

$$P_N(\mathbf{x}) = \int_{-\pi}^{\pi} \frac{\mathrm{d}^D k}{(2\pi)^D} [\cos(\mathbf{k}\cdot\mathbf{x})] \left(\frac{1}{D} \sum_{j=1}^{D} \cos k_j \right)^N . \quad (1)$$

This result – for an arbitrary dimension D – might decieve you into believing that the behaviour of random walk in, say, $D = 1, 2, 3$ is all essentially the same. In fact they are not, as can be illustrated by studying the phenomenon known as *recurrence*.

Recurrence refers to the probability for the random walking partcle to come back to the origin, where it started from, in the course of its perambulation, when we wait forever. Let u_n denote the probability that a particle returns to the origin on the nth step and let $\mathcal{R}$ be the expected number of times it returns to the origin.

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Random walk, decreasing steps, recurrence, Watson integrals, grid of resistors.

Clearly,

$$\mathcal{R} = \sum_{n=0}^{\infty} u_n \quad . \tag{2}$$

We can now distinguish between two different scenarios. If the series in (2) diverges, then the mean number of returns to the origin is infinite and we say that the random walk is recurrent. If the series is convergent, leading to a finite $\mathcal{R}$, then we say that the random walk is transient.

This idea can be reinforced by the following alternative interpretation of $\mathcal{R}$. Suppose u is the probability for the particle to return to the origin. Then the normalized probability for it to return exactly k times is $u^k(1-u)$. The mean number of returns to the origin is, therefore,

$$\mathcal{R} = \sum_{k=1}^{\infty} k\, u^{k-1}(1-u) = (1-u)^{-1} \quad . \tag{3}$$

Obviously, if $\mathcal{R} = \infty$, then $u = 1$, showing that the particle will definitely return to the origin. But if $\mathcal{R} < \infty$, then $u < 1$ and we can't be certain that the particle will ever come back home.

Let us compute u_n and $\mathcal{R}$ for random walks in $D = 1, 2, 3$ dimensions with the lattice spacing set to unity for simplicity. From (1), setting $\mathbf{x} = 0$, we have:

$$u_n(\mathbf{x}) = \int_{-\pi}^{\pi} \frac{\mathrm{d}^D k}{(2\pi)^D} \left(\frac{1}{D} \sum_{j=1}^{D} \cos k_j \right)^n \quad . \tag{4}$$

Doing the sum in (2) we get

$$\mathcal{R} = \sum_{n=0}^{\infty} u_n = \int_{-\pi}^{\pi} \frac{\mathrm{d}^D k}{(2\pi)^D} \left(1 - \frac{1}{D} \sum_{j=1}^{D} \cos k_j \right)^{-1} \quad . \tag{5}$$

We want to know whether this integral is finite or divergent. Clearly, the divergence, if any, can only arise due to its behaviour near the origin in k-space. Using the Taylor series expansion of the cosine function, we see that, near the origin we have the behaviour:

$$\mathcal{R} \approx 2D \int_{k\approx 0} \frac{\mathrm{d}k_1 \mathrm{d}k_2 \ldots \mathrm{d}k_D}{(2\pi)^D} \left(k_1^2 + k_2^2 + \cdots + k_D^2\right)^{-1} \propto \frac{2D}{(2\pi)^D} \int_{k\approx 0} \frac{k^{D-1}\mathrm{d}k}{k^2} \,. \quad (6)$$

The dimension dependence is now obvious. In $D = 1, 2$ the integral is divergent and $\mathcal{R} = \infty$; so we conclude that the random walk in $D = 1, 2$ is recurrent and the particle will definitely return to the origin if it walks forever. But in $D = 3$, the $\mathcal{R}$ is finite and the walk is non-recurrent. There is a finite probabiltity that the particle will come back to the origin but there is also a finite probability that it will not. A drunken man will definitely come home (given enough time) but a drunken bird on flight may or may not!

The mean number of recurrences in $D = 3$ is given by the Watson integral

$$\mathcal{R} = \frac{3}{(2\pi)^3} \int_{-\pi}^{\pi} \mathrm{d}k_1 \int_{-\pi}^{\pi} \mathrm{d}k_2 \int_{-\pi}^{\pi} \mathrm{d}k_3 \left[1 - (\cos k_1 + \cos k_2 + \cos k_3)\right]^{-1}, \quad (7)$$

which is notoriously difficult to evaluate analytically [1]. Since the answer happens to be

$$\mathcal{R} = \frac{\sqrt{6}}{32\pi^3} \Gamma\left(\frac{1}{24}\right) \Gamma\left(\frac{5}{24}\right) \Gamma\left(\frac{7}{24}\right) \Gamma\left(\frac{11}{24}\right) , \quad (8)$$

you anyway need to look it up in a table so one might as well do the integral numerically which is trivial in *Mathematica*. (Of course, if you like the challenge of a definite integral, try it out. I do not know of a simple way of doing it; neither do the experts in random walk I have talked to!) The result is $\mathcal{R} \approx 1.5164$ giving the return probability $u \approx 0.3405$.

In the case of $D = 1, 2$ it is also easy to obtain u_n explicitly by combinatorics. In 1-dimension, the particle can return to the origin only if it has taken an even number of steps, half to the right and half to the left. The probability for this is clearly

$$u_{2n} = {}^{2n}C_n \, \frac{1}{2^{2n}} \,. \quad (9)$$

For sufficiently large n, we can use Sterling's approximation for factorials ($n! \approx \sqrt{2\pi n}\, e^{-n} n^n$) to get $u_{2n} \approx 1/\sqrt{\pi n}$. The series in (2) involves the asymptotic sum which is divergent:

$$m = \sum_n u_{2n} \approx \sum_n \frac{1}{\sqrt{\pi n}} = \infty \ . \tag{10}$$

Obviously, the 1-dimensional random walk is recurrent.

Interestingly enough the result for $D = 2$ is just the square of the result for $D = 1$. The integral in (4) becomes for $D = 2$:

$$u_n(\mathbf{x}) = \frac{1}{(2\pi)^D} \frac{1}{2^n} \int_{-\pi}^{\pi} \mathrm{d}k_1 \int_{-\pi}^{\pi} \mathrm{d}k_2 \, (\cos k_1 + \cos k_2)^n \ . \tag{11}$$

If you now change variables of integration to $(k_1 + k_2)$ and $(k_1 - k_2)$ it is easy to show that this integral becomes the product of two integrals giving

$$u_{2n} = \left[\frac{1}{2^{2n}} {}^{2n}C_n \right]^2 \ , \tag{12}$$

which is the square of the result for $D = 1$. Now the series in (2) will be dominated asymptotically by

$$m \approx \sum_n \frac{1}{\pi n} = \infty \ , \tag{13}$$

making the $D = 2$ random walk recurrent again. You might guess at this stage, that in 3-D, the asymptotic series will involve sum over $n^{-3/2}$ (and hence will converge) making the 3-D random walk non-recurrent. This is partially true and the 3-dimensional series is bounded from above by the sum over $n^{-3/2}$. But the 3-dimensional case is *not* the product of three 1-dimensional cases.

We now turn our attention to another curious result. Summing $P_N(\mathbf{x})$ over all N one can construct the quantity $P(\mathbf{x})$ which is the probability of reaching $\mathbf{x}$. Using (1) and doing the geometric sum, we find in $D = 2$, this quantity to be:

$$P(\mathbf{x}) = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{\mathrm{d}k_1 \mathrm{d}k_2}{(2\pi)^2} [\cos(\mathbf{k} \cdot \mathbf{x})] \left(1 - \frac{1}{2}(\cos k_1 + \cos k_2) \right)^{-1} . \tag{14}$$

Consider now the expression

$$R = \frac{1}{2}(P(\mathbf{x}) - P(\mathbf{0})) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\mathrm{d}k_1 \mathrm{d}k_2}{8\pi^2} \frac{[1-\cos(\mathbf{k}\cdot\mathbf{x})]}{[1-\frac{1}{2}(\cos k_1 + \cos k_2]}. \tag{15}$$

Incredibly enough this provides the solution to a completely different problem! It gives the effective resistance between a lattice point $\mathbf{x}$ and the origin in an infinite grid of 1-ohm resistors connected between the lattice sites. Let us see how this comes about by analysing the grid of resistors.

Let a node $\mathbf{x}$ in the infinite planar square lattice be denoted by two integers (m, n) and let a current $I_{m,n}$ be injected at that node. The flow of current will induce a voltage at each node and, combining Kirchoff's and Ohm's laws for the 1-ohm resistors, we can write the relation:

$$\begin{aligned} I_{m,n} &= (V_{m,n} - V_{m+1,n}) + (V_{m,n} - V_{m-1,n}) + (V_{m,n} - V_{m,n+1}) + (V_{m,n} - V_{m,n-1}) \\ &= 4V_{m,n} - V_{m+1,n} - V_{m-1,n} - V_{m,n+1} - V_{m,n-1}, \end{aligned} \tag{16}$$

where $V_{m,n}$ is the potential at the node (m, n) due to the current. This equation can again be solved by introducing the Fourier transform on the discrete lattice. If we write

$$I_{m,n} = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \mathrm{d}k_1\, \mathrm{d}k_2\, I(k_1, k_2) e^{i(mk_1+nk_2)}, \tag{17}$$

$$V_{m,n} = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \mathrm{d}k_1\, \mathrm{d}k_2\, V(k_1, k_2) e^{i(mk_1+nk_2)}, \tag{18}$$

then one can obtain from (16) the result in the Fourier space:

$$I(k_1, k_2) = 2V(k_1, k_2)\,[2 - \cos(k_1) - \cos(k_2)]\ . \tag{19}$$

Suppose we now inject a current of 1 amp at (0,0) and -1 amp at (N, M). Then $I_{m,n} = \delta_{m,n} - \delta_{m-M,n-N}$, leading to

$$I(k_1, k_2) = 1 - \mathrm{e}^{-i(Mk_1+Nk_2)}\ , \tag{20}$$

so that (19) gives the voltage to be

$$V(k_1, k_2) = \frac{1}{2} \times \frac{1 - \mathrm{e}^{-i(Mk_1+Nk_2)}}{2 - \cos(k_1) - \cos(k_2)}\ . \tag{21}$$

The equivalent resistance between nodes (0,0) and (M,N) with a flow of unit current is just the voltage difference between the nodes:

$$
\begin{aligned}
R_{M,N} &= V_{0,0} - V_{M,N} \\
&= \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \mathrm{d}k_1 \mathrm{d}k_2 \, V(k_1, k_2) \left[1 - e^{i(Mk_1 + Nk_2)}\right] \\
&= \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \mathrm{d}k_1 \, \mathrm{d}k_2 \, \frac{1}{2} \frac{(1 - e^{-i(Mk_1 + Nk_2)})(1 - e^{i(Mk_1 + Nk_2)})}{2 - \cos(k_1) - \cos(k_2)} \\
&= \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \mathrm{d}k_1 \, \mathrm{d}k_2 \, \frac{1 - \cos(Mk_1 + Nk_2)}{2 - \cos(k_1) - \cos(k_2)},
\end{aligned} \tag{22}
$$

which is exactly the same as the integral in (15)!

The infinite grid of square lattice resistors is a classic problem and the effective resistance between two adjacent nodes is a 'trick question' that is a favourite of examiners. The answer (0.5 ohm) can be found by trivial superposition but such tricks are useless to find the effective resistance between arbitrary nodes. In fact, the effective resistance between two diagonal nodes of the basic square – the (0,0 and (1,1), say – is given by the integral

$$
R_{1,1} = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \mathrm{d}k_1 \, \mathrm{d}k_2 \, \frac{1 - \cos(k_1 + k_2)}{2 - \cos(k_1) - \cos(k_2)} \; . \tag{23}
$$

This is doable, but not exactly easy, and the answer is $2/\pi$. (Next time someone lectures you about the power of clever arguments, ask her to use them to get this answer, which has a π in it!)

But why does this work? What has random walk on a lattice to do with resistor networks? There are different levels of sophistication at which one can answer this question and an entire book [2] dealing with this subject exists. The mathematical reason has to do with the fact that both the random walk probability to visit a node and the voltage on a node (which does not have any current injected or removed) are harmonic functions. These are functions whose value at any given node is given by the average of the value of the function on the adjacent lattice sites. This is obvious in the case of random walk because a particle which reaches the node (m, n) must have hopped to that node with equal probability from one of the neighbouring nodes $(m \pm 1, n \pm 1)$. In the case of a resistor network, the same result is obtained from (16) when $I_{mn} = 0$. If you now inject the voltages 1 and 0 at two specific nodes A and B, then the voltage at any other node X can be interpreted as the probability that a random walker starting at X will get to A before B. One can then use this interpretation to make a formal connection between voltage distribution in an electric network and a random walk problem. The interested reader can find more in the book [2] referred to above.

Having done all these in the lattice, we now go back to the random walk in full space for which we had obtained the result in the last installment, which, specialised to one dimension, is given by:

$$P_N(x) = \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{(2\pi)} \mathrm{e}^{ikx} \prod_{n=1}^{N} p_n(k) \ . \tag{24}$$

We want to consider a situation in which the steps are random and uncorrelated but their lengths are decreasing monotonically. (This is what will happen if the drunkard gets tired as he walks!) In particular, we will assume that each step length is a fraction λ of the previous one with $\lambda < 1$ and the first step is of unit length. It is clear that $P_N(x)$ is now given by

$$P_N(x) = \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{(2\pi)} \mathrm{e}^{ikx} \prod_{n=1}^{N} \cos(k\lambda^n) \ . \tag{25}$$

We can now study the limit of $N \to \infty$ and ask how the probability $P(x) \equiv P_\infty(x)$ is distributed. This fuction shows incredibly diverse properties depending on the value of λ and is still not completely analysed. I will confine myself to a simple situation, referring you to the literature if you are interested [3].

Let us first consider the case when $\lambda = 1/2$. In this case the relevant infinite product is given by

$$\prod_{n=1}^{\infty} \cos \frac{k}{2^n} = \frac{\sin k}{k} \ . \tag{26}$$

(This is a cute result which you might want to prove for yourself; All you need to do is to write $\cos(k/2^n) = (1/2)[\sin(k/2^{n-1})/\sin(k/2^n)]$, take a product of N terms cancelling out the sines and then take the limit $N \to \infty$.) Since the fourier transform of $(\sin k/k)$ is just a uniform distribution, we get the tantalising result that $P(x)$ is just a uniform distribution in the interval $(-1, 1)$ and zero elsewhere!

This trick which has been used to get (26) also works for $\lambda = 2^{-1/2}, 2^{-1/4}...$, etc. For example, when $\lambda = 2^{-1/2}$ the infinite product is

$$\prod_{n=1}^{\infty} \cos(k/2^{n/2}) = \frac{\sin k}{k} \frac{\sin \sqrt{2}k}{\sqrt{2}k} \ . \tag{27}$$

The Fourier transform of this involves a convolution of two rectangular distributions and is easily seen to be a triangular probability distribution. (I will let you explore the general case of $\lambda = 2^{-1/k}$.) It turns out that the infinite product of $\cos(k\lambda^n)$ is extraordinarily sensitive to changes in λ. For almost every λ in the interval $(0.5, 1)$ this product is square integrable. But once in a while, it is not so. Further, if $\lambda < (1/2)$, the support for $P(x)$ happens to be the Cantor set. The really bizarre behaviour occurs when λ is the golden ratio $g = (\sqrt{5} - 1)/2$. Clearly, there are enough surprises in store in the study of random walks.

Suggested Reading

[1] This integral was first evaluated by Watson in terms of elliptic integrals and a "simpler" result was obtained by Glasser and Zucker: G N Watson, Three triple integrals, *Quarterly J. Math.*, Vol.10, p.266, 1939;
M L Glasser and I J Zucker, Extended Watson integrals for the cubic latice, *Proc. Natl. Acad. Sci*., USA,Vol.74, p.1800, 1977.

[2] There is a large literature on this subject, most of which can be found from the references in the book: P G Doyle and J Laurie Snell, *Random Walks and Electric Networks*, Mathematical Association of America, Oberlin, OH, 1984. The book is available on the web as a pdf file at: http://math.dartmouth.edu/~doyle docs/walks/walks.pdf.

[3] This interesting topic does not seem to have been explored extensively. A good discussion is available in the paper: P L Krapivsky and S Redner, *Am. J. Phys*., Vol.72(5), p.591, 2004. Also see, K E Morrison, *Random Walks with Decreasing Steps*, available at: http://www.calpoly.edu/~kmorriso/Research/RandomWalks.pdf

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