Snippets of Physics

19. Random Walk Through Random Walks - I

T Padmanabhan

T Padmanabhan works at IUCAA, Pune and is interested in all areas of theoretical physics, especially those which have something to do with gravity.

Few processes in nature are as ubiquitous as the random walk which combines extraordinary simplicity of concept with considerable complexity in the final output. In this and the next installment, we shall examine several features of this remarkable phenomenon.

In 1785, the Dutch physicist Jan Ingenhauez, discoverer of photosynthesis, put alcohol to good use by sprinkling powdered charcoal on it and observing it under a microscope. The random motion of the charcoal particles was probably the first observation of what we now call Brownian motion. The name comes from Robert Brown who published an extensive investigation of similar phenomena in 1828. Eventually, this was heralded as an evidence for the molecular nature of matter and was instrumental in the 1926 Nobel Prize in physics to Jean Perrin for determining the Avogadro number.

It appears that the term 'random walk' was first coined by Carl Pearson in 1905, the same year in which Einstein published his paper on Brownian motion. Pearson was interested in providing a simple model for the spread of mosquito infestation in a forest – which goes to show, right at the outset, the generality of the process! Pearson's letter to *Nature* was answered by Lord Rayleigh who had solved this problem earlier in the case of sound waves in heterogeneous materials. Independently, Louis Bachelor was developing the theory of random walks in his remarkable doctoral thesis *La theorie de la speculation* published in 1900. Here, random walk was suggested as a model for financial time series which has, until recently, helped physicists to get Wall Street jobs

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with the disastrous consequences we all now know only too well! This brief glimpse at history already shows the occurrence of random walk in widely different contexts [1, 2].

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Let us begin by reviewing the simplest of all random walks in which a particle moves from the origin, taking steps of length ℓ, with each step being in a random direction uncorrelated with the previous one. The displacement of the particle after N steps is given by

$$\mathbf{x} = \sum_{n=1}^{N} \mathbf{x}_n \ , \tag{1}$$

where

$$|\mathbf{x}_n| = \ell; \qquad \langle \mathbf{x}_n \rangle = 0; \qquad \langle \mathbf{x}_n \cdot \mathbf{x}_m \rangle = \ell^2 \delta_{nm} \ . \tag{2}$$

The first equation in (2) tells you that each step has a constant magnitude. The second and third equations (the symbol $\langle ... \rangle$ denotes averaging over a probability distribution quantify the uncorrelated nature of the directions of the steps. From these, we can immediately obtain the two key results of such a random walk. First, $\langle \mathbf{x} \rangle = 0$. Further, we have

$$\sigma^2 \equiv \langle \mathbf{x}^2 \rangle = \left\langle \left(\sum_{n=1}^{N} \mathbf{x}_n \right)^2 \right\rangle = \sum_{n,m=1}^{\infty} \langle \mathbf{x}_n \cdot \mathbf{x}_m \rangle = N\ell^2 \ . \tag{3}$$

This shows the key characteristic of the random walk viz., that the root-mean-square displacement σ grows as $\sqrt{N}$.

The key characteristic of the random walk is that the root-mean-square displacement σ grows as $(n)^{1/2}$.

We can think of ℓ as Δx denoting the magnitude of the displacement between any two consecutive steps. If the time interval between the steps is Δt, then $\sigma \propto \sqrt{N}$ suggests that $(\Delta x)^2/\Delta t$ remains a constant in the continuum limit. Clearly, a random walk corresponds

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To see how such a continuum limit emerges in this context, it is better to generalize the concept of random walk slightly by assuming that the probability for the particle to take a step given by the vector $\Delta\mathbf{y}$ is given by some function $p(\Delta\mathbf{y})$ with the properties

$$\begin{aligned}\langle \Delta y^i \rangle &\equiv \int d^D\Delta y\,[\Delta y^i p(\Delta\mathbf{y})] = 0,\\ \langle \Delta y^i \Delta y^j \rangle &\equiv \int d^D\Delta y[\Delta y^i \Delta y^j\, p(\Delta\mathbf{y})] = \langle (\Delta y)^2\rangle \frac{\delta^{ij}}{D}\ ,\end{aligned} \tag{4}$$

where $i, j, ... = 1, 2, ...D$ denote the components of the vector. Let $P_N(\mathbf{x})$ be the probability that the net displacement is $\mathbf{x}$ after N steps. Then, since the steps are uncorrelated, we have the elementary relation:

$$P_N(\mathbf{x}) = \int d^D\Delta y\, P_{N-1}(\mathbf{x}-\Delta\mathbf{y})p(\Delta\mathbf{y})\ . \tag{5}$$

To obtain the continuum limit, we will assume that a Taylor series expansion of $P_{N-1}(\mathbf{x}-\Delta\mathbf{y})$ is possible so that we can write (assuming summation over repeated indices):

$$\begin{aligned}P_N(\mathbf{x}) &\cong \int d^D\Delta y\, p(\Delta\mathbf{y})\Big\{P_{N-1}(\mathbf{x}) - \Delta y^i\partial_i P_{N-1}(\mathbf{x})\\ &\qquad + \frac{1}{2}\Delta y^i\Delta y^j\partial_i\partial_j P_{N-1}(\mathbf{x})\Big\}\\ &= P_{N-1}(\mathbf{x}) + \frac{\langle(\Delta y)^2\rangle}{2D}\nabla^2 P_{N-1}(\mathbf{x})\ ,\end{aligned} \tag{6}$$

where we have used (4). In the continuum limit, we will denote the total time which has elapsed since the beginning of the random walk by $t = N\Delta t$ and define a continuum probability density by $\rho(\mathbf{x},t) = \rho(\mathbf{x},N\Delta t) \equiv P_N(\mathbf{x})$. Since we can take $(\partial\rho/\partial t)$ as the limit $[P_N(\mathbf{x}) - P_{N-1}(\mathbf{x})]/\Delta t$ when $\Delta t \to 0$, we get from (6) the result

$$\frac{\partial \rho}{\partial t} = K\nabla^2\rho \ , \qquad (7)$$

where we have defined a ('diffusion') coefficient $K \equiv \langle(\Delta y)^2\rangle/2D\Delta t$. The continuum limit exists if we can treat K as a constant when $\Delta t \to 0$. Clearly, this is equivalent to $(\Delta y)^2/\Delta t$ being finite in the continuum limit as we indicated earlier. This is quite different from the usual continuum limits we are accustomed to in physics in which the ratio of the differentials *of the same order* are replaced by a derivative. This should warn you that something nontrivial is going on.

The final equation we have obtained, of course, is the diffusion equation which can also be written as $(\partial\rho/\partial t) = -\nabla\cdot\mathbf{J}$, where the current $\mathbf{J} = -K\nabla\rho$ arises due to a gradient in the particle density. (In this form we can even consider a situation with spatially varying diffusion coefficient K.) This indicates that diffusive processes in physics can be modelled at the microscopic level by a random walk of the discrete constituent element. The diffusion equation is also unique in the sense that it is not invariant under time reversal; diffusion gives you a direction of time which is another remarkable fact that arises in the continuum limit.

Being a linear equation, the diffusion equation (7) can be solved by Fourier transforming both sides. Denoting the Fourier transform of $\rho(\mathbf{x},t)$ by $\rho(\mathbf{k},t)$ it is easy to show that $\rho(\mathbf{k},t) = \exp(-Kk^2t)$. Taking a Fourier transform, we get the fundamental solution to the diffusion equation (which is essentially the Green's function) to be

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$$\rho(\mathbf{x},t) = \frac{e^{-x^2/4Kt}}{(4\pi Kt)^{D/2}} \ . \qquad (8)$$

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This shows how particles located close to the origin at $t = 0$ spread in the course of time. The mean square spread is clearly proportional to Kt which is the residue of the discrete result $\sigma^2 \propto N$.

The diffusion of a particle need not always take place in the real 3-dimensional space. An interesting phenomenon which occurs in plasmas as well as gravitating systems – in which long-range, inverse square forces act between particles – involves diffusion in the *velocity space*. A simple version of this can be described as follows. Consider a nearly homogeneous distribution of gravitationally interacting particles (e.g., stars in a globular cluster). When two stars scatter off each other with an impact parameter b, each one undergoes a typical acceleration Gm/b^2 acting for a time b/v. As a result of one such scattering, a typical star will acquire a 'kick' in the velocity space of magnitude $\delta v_\perp \approx Gm/bv$, $\delta v_\perp \ll v$. The effect of a large number of such collisions is to make the star perform a random walk in the velocity space. The net mean-square velocity induced by collisions with impact parameters in the range $(b, b + \mathrm{d}b)$ in a time interval Δt will be the product of the mean number of scatterings in time Δt and $(\delta v_\perp)^2$. The former is given by the number of scatterers in the volume $(2\pi b\, \mathrm{d}b)(v\Delta t)$. Hence

$$\langle(\delta v_\perp)^2\rangle = (2\pi b \mathrm{d}b)\,(v\Delta t)\, n \left(\frac{Gm}{bv}\right)^2 \quad , \tag{9}$$

where n is the number density of scatterers. The total mean-square transverse velocity due to all stars is found by integrating over b within some range (b_1, b_2):

$$\begin{aligned}\langle(\delta v_\perp)^2\rangle_{\text{total}} &\simeq \Delta t \int_{b_1}^{b_2} (2\pi b \mathrm{d}b)\,(vn)\left(\frac{G^2m^2}{b^2v^2}\right) \\ &= \frac{2\pi n G^2 m^2}{v}\Delta t \ln\left(\frac{b_2}{b_1}\right). \end{aligned} \tag{10}$$

We again see the signature of random walk in $\langle \delta v_\perp^2\rangle \propto \Delta t$. The logarithmic factor shows that we cannot take

$b_1 = 0, b_2 = \infty$ and one needs to use some physical criteria to fix b_1 and b_2. It is reasonable to take $b_2 \simeq R$, the size of the system; as regards b_1, notice that the velocity change per collision can become comparable to v itself when $b \simeq b_c \simeq (Gm/v^2)$ and our diffusion approximation breaks down. It is, therefore, reasonable to take $b_1 \simeq b_c \simeq (Gm/v^2)$. Then $(b_2/b_1) \simeq (Rv^2/Gm) = N\,(Rv^2/GM) \simeq N$ for a system in virial equilibrium. From (10) we see that this effect is important over time-scales (Δt) which is long enough to make $\langle(\delta v_1)^2\rangle_{\rm total} \simeq v^2$. Using this condition and solving for (Δt) we get:

$$(\Delta t)_{\rm gc} \simeq \frac{v^3}{2\pi G^2 m^2 n \ln N} \quad . \tag{11}$$

This is the time scale for gravitational relaxation in such systems (or electromagnetic relaxation in plasmas) and the $\ln N$ factor arises due to diffusion in velocity space.

The entire process can be described by a diffusion equation in velocity space – or so it would seem at first sight. A moment of thought, however, shows that if we describe the process by a diffusion equation in velocity space, it will make the root-mean-square velocities of *every* particle in the system to increase as $\sqrt{t}$ as time goes on; this violates some sacred notions in physics [3]. This is one key difference between diffusing in real space compared to velocity space and there must exist a process which prevents this.

This process is called 'dynamical friction'. To understand it, consider a particle ('star') which moves with a velocity V that is significantly larger than the root-mean-square speed of the cloud of stars around it. In the rest frame of the fast star, on the average, other stars will be streaming past it and will be deflected towards it. This will produce a slight density enhancement of stars behind the fast star. This density enhancement produces the necessary force to reduce the speed V of

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the star. This dynamical friction ensures that no runaway disaster occurs in velocity space.

If we take both the processes into account, the evolution in the velocity space is described by an equation which is a variant of what is called the Fokker–Planck equation. A simplified version of this is given by

$$\frac{\partial f(v,t)}{\partial t} = \frac{\partial}{\partial v}\left\{\frac{\sigma^2}{2}\frac{\partial f}{\partial v} + (\alpha v)f\right\} \ . \qquad (12)$$

The first term on the right-hand side has the standard form of a diffusion current proportional to the gradient in the velocity space. As time goes on, this term will cause the mean-square velocities of particles to increase in proportion to t inducing the 'random walk' in the velocity space. Under the effect of this term, *all* the particles in the system will have their $< v^2 >$ increasing without bound. This unphysical situation is avoided by the presence of the second term $(\alpha v f)$ which describes the dynamical friction. The combined effect of the two terms is to drive f to a Maxwellian distribution with an effective temperature $(k_{\rm B}T) = (\sigma^2/\alpha)$ and $(\partial f/\partial t) = 0$. In such a Maxwellian distribution the gain made in (Δv^2) due to diffusion is exactly balanced by the losses due to dynamical friction. When two particles scatter, one gains the energy lost by the other; on the average, we may say that the one which has lost the energy has undergone dynamical friction while the one which gained energy has achieved diffusion to higher v^2. The cumulative effect of such phenomena is described by the two terms in (12).

The above points can be easily illustrated by explicitly solving (12). Suppose we take an initial distribution $f(v,0) = \delta(v - v_0)$ peaked at a velocity v_0. The solution of (12) with this initial condition is easy to find:

$$f(v,t) = \left[\frac{\alpha}{\pi\sigma^2(1-\mathrm{e}^{-2\alpha t})}\right]^{1/2} \exp\left[-\frac{\alpha(v - v_0\mathrm{e}^{-\alpha t})^2}{\sigma^2(1-\mathrm{e}^{-2\alpha t})}\right] \tag{13}$$

which is a Gaussian with the mean $< v >= v_0\mathrm{e}^{-\alpha t}$ and dispersion $< v^2 > - < v >^2 = (\sigma^2/\alpha)(1-\mathrm{e}^{-2\alpha t})$. At late times ($t \to \infty$), the mean velocity $< v >$ goes to zero while the velocity dispersion becomes (σ^2/α). Thus the equilibrium configuration is a Maxwellian distribution of velocities with this particular dispersion, for which $\partial f/\partial t = 0$. To see the effect of the two terms individually on the initial distribution $f(v,0) = \delta(v-v_0)$, we can set α or σ to zero. When $\alpha = 0$, we get pure diffusion:

The equilibrium configuration is a Maxwellian distribution of velocities with this particular dispersion, for which $\partial f/\partial t = 0$.

$$f_{\alpha=0}(v,t) = \left(\frac{1}{2\pi\sigma^2 t}\right)^{1/2} \exp\left\{-\frac{(v-v_0)^2}{2\sigma^2 t}\right\}. \tag{14}$$

Nothing happens to the steady velocity v_0; but the velocity dispersion increases in proportion to t representing a random walk in the velocity space. On the other hand, if we set $\sigma = 0$, then we get

$$f_{\sigma=0}(v,t) = \delta(v - v_0\mathrm{e}^{-\alpha t}). \tag{15}$$

Now there is no spreading in velocity space (no diffusion); instead the friction steadily decreases $< v >$.

Going back to the discrete case, we can make another useful generalization of (5) by assuming that $p(\Delta\mathbf{y})$ itself depends on N so that the fundamental equation becomes

$$P_N(\mathbf{x}) = \int d^D y\, P_{N-1}(\mathbf{x} - \Delta\mathbf{y}) p_N(\Delta\mathbf{y}) \ . \tag{16}$$

This equation, which is a convolution integral, is trivial to solve in Fourier space in which the convolution integral becomes a product. If we denote by $P_N(\mathbf{k})$ and $p_N(\mathbf{k})$ the Fourier transforms of $P_N(\mathbf{x})$ and $p_N(\Delta\mathbf{y})$ then

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this equation becomes $P_N(\mathbf{k}) = P_{N-1}(\mathbf{k})p_N(\mathbf{k})$. Iterating this N times and normalizing the initial probability by assuming the particle was at the origin we immediately get

$$P_N(\mathbf{k}) = \prod_{n=1}^{N} p_n(\mathbf{k}) \quad . \tag{17}$$

Doing an inverse Fourier transform we find the solution to our problem to be

$$P_N(\mathbf{x}) = \int \frac{d^D k}{(2\pi)^D} e^{i\mathbf{k}\cdot\mathbf{x}} \prod_{n=1}^{N} p_n(\mathbf{k}) \quad . \tag{18}$$

Once again, it is possible to make some general comments if the individual probability distributions $p_n(\Delta\mathbf{y})$ satisfy some reasonable conditions. Suppose, for simplicity, that $p_n(\Delta\mathbf{y})$ is peaked at the origin and dies down smoothly and monotonically for large $|\Delta\mathbf{y}|$. Then, its Fourier transform will also be peaked around the origin in k-space and will die down for large values of $|\mathbf{k}|$. Further, because the probability is normalized, we have the condition $p_n(\mathbf{k} = 0) = 1$. When we take a product of N such functions, the resulting function will again have the value unity at the origin. But as we go away from the origin, we are taking the product of N numbers each of which is less than unity. So clearly when $N \to \infty$, the product of $p_n(\mathbf{k})$ will have significant support only close to the origin.

The nontrivial assumption we will now make is that $p_n(\mathbf{k})$ has a smooth curvature at the origin of the Fourier space and is not 'cuspy'. Then, near the origin in Fourier space, we can approximate

$$p_n(\mathbf{k}) \simeq 1 - \frac{1}{2}\alpha_n^2 k^2 \simeq e^{-(1/2)\alpha_n^2 k^2} \tag{19}$$

with some constant α_n. Hence the product becomes

$$\prod_{n=1}^{N} p_n(\mathbf{k}) = \exp -\frac{1}{2}k^2 \sum_{n=1}^{N} \alpha_n^2 \equiv \exp -\frac{N}{2}\sigma^2 k^2 \quad , \qquad (20)$$

where we have defined

$$\sigma^2 = \frac{1}{N}\sum_{n=1}^{N} \alpha_n^2 \quad . \qquad (21)$$

In this limit, the final Fourier transform in (18) is trivial and will give a Gaussian in x with $\langle x^2 \rangle \propto N$.

An observant reader would have noticed that we have essentially proved a variant of the central limit theorem for the sum $(\mathbf{x}_1 + \mathbf{x}_2 + ... + \mathbf{x}_N)$ of N independently distributed random variables each having its own probability distribution $p_n(\mathbf{x}_n)$. In fact, the joint probability for these variables to be in some given interval is given by the product of $p_n(\mathbf{x}_n)\mathrm{d}^D\mathbf{x}_n$ over all $n = 1, 2, ...N$. The probability for their sum to be $\mathbf{x}$ is given by

$$P_N(\mathbf{x}) = \int \prod_{n=1}^{N} p_n(\mathbf{x}_n)\mathrm{d}^D\mathbf{x}_n \delta_D\left(\mathbf{x} - \sum \mathbf{x}_n\right) \quad , \qquad (22)$$

where the Dirac delta function ensures that the sum of the random variables is $\mathbf{x}$. Writing the Dirac delta function in Fourier space, we immediately get

$$\begin{aligned} P_N(\mathbf{x}) &= \int \frac{\mathrm{d}^D k}{(2\pi)^D} e^{i\mathbf{k}\cdot\mathbf{x}} \prod_{n=1}^{N} \int \mathrm{d}^D\mathbf{x}_n p_n(\mathbf{x}_n) e^{-i\mathbf{k}\cdot\mathbf{x}_n} \\ &= \int \frac{\mathrm{d}^D k}{(2\pi)^D} e^{i\mathbf{k}\cdot\mathbf{x}} \prod_{n=1}^{N} p_n(\mathbf{k}), \qquad (23) \end{aligned}$$

which is identical to the result we obtained earlier in (18).

An observant reader would have noticed that we have essentially proved a variant of the central limit theorem for the sum $\mathbf{x}_1 + \mathbf{x}_2 + ... + \mathbf{x}_N$.

A classic example in which our analysis (and central limit theorem) *fails* is given by the case in which each

The key reason for the central limit theorem to fail in this case is that the Lorentzian distribution has a diverging second moment.

of the probability distributions $p_n(\Delta\mathbf{y})$ is given by a Lorentzian

$$p_n(\Delta\mathbf{y}) = \frac{(\beta/\pi)}{(\Delta y)^2 + \beta^2} \ . \tag{24}$$

The Fourier transform now gives $p_n(\mathbf{k}) = \exp(-\beta|\mathbf{k}|)$. Clearly our approximation in (19) fails for this function since it is 'cuspy' due to a linear term in $|\mathbf{k}|$ near the origin. We can, of course, carry out the analysis in (18) to get

$$P_N(\mathbf{x}) = \int \frac{\mathrm{d}^D k}{(2\pi)^D} e^{i\mathbf{k}\cdot\mathbf{x}} e^{-N\beta|\mathbf{k}|} = \frac{(N\beta/\pi)}{|\mathbf{x}|^2 + (N^2\beta^2)} \ . \tag{25}$$

We have the result that the probability distribution for the final displacement is identical to the probability distribution of individual steps when the latter is a Lorentzian – except for the (expected) scaling of the width. The key reason for the central limit theorem to fail in this case is that the Lorentzian distribution has a diverging second moment. You should remember this the next time you think of full width at half maximum of a Lorentzian as 'similar to' the width of a Gaussian! There are physical situations, (e.g., one called anomalous diffusion), which can be modelled along these lines. They are characterized by random walks in which every once in a while the particle takes a large step because of the slow decrease in the probability $p(\Delta\mathbf{y})$.

Very often one considers random walk on a lattice of specific shape, the simplest being the D-dimensional cube. Here the particle hops from one site of the lattice to another nearby site along any one of the axes with the lattice spacing taken to be unity for simplicity. In this case the Fourier integrals in (18) will become Fourier series and we get:

$$P_N(\mathbf{x}) = \int_{-\pi}^{\pi} \frac{\mathrm{d}^D k}{(2\pi)^D} [\cos(\mathbf{k}\cdot\mathbf{x})] \prod_{n=1}^{N} p_n(\mathbf{k}) \ , \tag{26}$$

where all the integrals are in the range $(-\pi, \pi)$ and $\mathbf{x}$ is a vector with integer valued components. If $p_n(\mathbf{k})$ is independent of n and hops in all directions from any site are equally likely, then $p(\mathbf{k}) = (1/D)(\cos k_1 + \cos k_2 + \cdots \cos k_D)$ and we get

$$P_N(\mathbf{x}) = \int_{-\pi}^{\pi} \frac{\mathrm{d}^D k}{(2\pi)^D} [\cos(\mathbf{k}\cdot\mathbf{x})] \left(\frac{1}{D} \sum_{j=1}^{D} \cos k_j \right)^N . \tag{27}$$

As a test, we can reproduce the standard result for one-dimensional lattice using (27). In this case $x = J$, with J being a positive or negative integer. After N steps when the particle has taken n_{L} steps to the left of origin and n_{R} steps to the right, we have $n_{\mathrm{L}} + n_{\mathrm{R}} = N$ and $n_{\mathrm{R}} - n_{\mathrm{L}} = J$. Solving, we get $n_{\mathrm{R}} = (1/2)(N+J)$, $n_{\mathrm{L}} = (1/2)(N-J)$. The probability that out of N steps n_{L} were to the left and n_{R} were to the right is the same as getting, say, n_{L} heads while tossing N coins and is given by

$$P_N(J) = \frac{1}{2^N} {}^N C_{n_{\mathrm{L}}} = \frac{1}{2^N} \frac{N!}{((1/2)(N+J))!((1/2)(N-J))!} . \tag{28}$$

You can amuse yourself by proving that this is also given by the integral in (27) for $D = 1$,

$$P_N(J) = \int_{-\pi}^{\pi} \frac{\mathrm{d}k_1}{(2\pi)} [\cos(k_1 J)](\cos k_1)^N \tag{29}$$

as it should. The result in (27) will be useful in the next installment when we address some interesting dimension-dependent properties of random walks (and an unexpected connection with electrical networks!).

Suggested Reading

[1] An entertaining discussion of history is available in B Hughes, *Random Walks and Random Environments*, Vol.1, Oxford, 1965. Also see E W Montroll and M F Shlesinger, On the wonderful world of random walks, in *Studies in Statistical Mechanics*, edited by J L Lebowitz and E W Montroll, North-Holland, Vol.11, Amsterdam, 1984.

[2] Joseph Rudnick and George Gaspari, *Elements of the Random Walk*, Cambridge University Press, 2004.

[3] There is an interesting history associated with this issue, involving S Chandrasekhar; see T Padmanabhan, Stellar Dynamics and Chandra, *Current Science*, Vol.70, p.784, 1996.

Address for Correspondence
T Padmanabhan
IUCAA, Post Bag 4
Pune University Campus
Ganeshkhind
Pune 411 007, India.
Email: paddy@iucaa.ernet.in,
nabhan@iucaa.ernet.in