Snippets of Physics

16. Lagrange has (more than) a Point!

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A solution to the 3-body problem in gravity, due to Lagrange, has several remarkable features. In particular, it describes a situation in which a particle, located at the maxima of a potential, can remain stable against small perturbations.

Motion of bodies under their mutual gravitational attraction is of historical, theoretical and even practical (thanks to the space-age and satellites) importance. The simplest case of two bodies – corresponding to the so-called Kepler problem – already possesses several interesting features, like the existence of an extra integral and the fact that the trajectory of the particle in the velocity space is a circle (see, for example, [1]). The situation becomes more interesting, but also terribly complicated, when we add a third particle to the fray. The 3-body problem, as it is called, has attracted the attention of several dynamicists and astronomers but, unfortunately, it does not possess a closed solution.

When an exact problem cannot be solved, physicists look around for a simpler version of the problem which will at least capture some features of the original one. One such case corresponds to what is known as the *restricted three-body problem* which could be described as follows. Consider two particles of masses m_1 and m_2 which orbit around their common centre of mass, exactly as in the case of the standard Kepler problem. We now consider a third particle of mass m_3 with $m_3 \ll m_1$ and $m_3 \ll m_2$ which is moving in the gravitational field of the two particles m_1 and m_2. Since it is far less massive than the other two particles, we will assume that it behaves like a test particle and does not affect the

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original motion of m_1 and m_2. You can see that this is equivalent to studying the motion of m_3 in a time-dependent external gravitational potential produced by m_1 and m_2. Given the fact that we lose both the time translation invariance and axial symmetry, any hope for simple analytic solutions is misplaced. But there is a special case – described in this installment – for which a beautiful solution can be obtained.

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This corresponds to a situation in which all the three particles maintain their relative positions with respect to one another but rotate rigidly in space with an angular velocity ω! In fact, the three particles are located at the vertices of an equilateral triangle irrespective of the ratio of the masses m_1/m_2. If you think about it, you will find that this solution, first found by Lagrange, is quite elegant and somewhat counter-intuitive. How do you balance the forces, which depend on mass ratios, without adjusting the distance ratios but always maintaining the equilateral configuration? What is more, the location of m_3 happens to be at the local *maximum* of the effective potential in the frame co-rotating with the system. Traditionally, the maxima of a potential have bad press due to their tendency to induce instability. It turns out that, in this solution, stability can be maintained (for a reasonable range of parameters) because of the existence of Coriolis force – which is one of the things many students do not have an intuitive grasp of. I will now obtain this solution and describe its properties leaving (as usual!) the detailed algebra for you to work out.

If the separation between m_1 and m_2 is a, the standard Kepler solution tells us that they can rotate in circular orbits around the centre of mass with the angular velocity given by

$$\omega^2 = \frac{G(m_1 + m_2)}{a^3}. \qquad (1)$$

The Coriolis force has the form identical to the force exerted by a magnetic field $(2m/q)\omega$ on a particle of charge q.

Since Lagrange has shown that a rigidly rotating solution exists with the third body, we will save work by studying the problem in the coordinate system co-rotating with the masses, in which the three bodies will be at rest. We will first work out the equations of motion in a rotating frame before proceeding further.

This is most easily done by starting from the Lagrangian for a particle $L(\mathbf{x}, \dot{\mathbf{x}}) = (1/2)m\dot{\mathbf{x}}^2 - V(\mathbf{x})$ and transforming to a rotating frame by using the transformation law $\mathbf{v}_{\text{inertial}} = \mathbf{v}_{\text{rot}} + \omega \times \mathbf{x}$. This leads to the Lagrangian of the form

$$L = \frac{1}{2}m\mathbf{v}^2 + m\mathbf{v} \cdot (\omega \times \mathbf{x}) + \frac{1}{2}m(\omega \times \mathbf{x})^2 - V(\mathbf{x}) \quad (2)$$

and equations of motion

$$m\frac{d\mathbf{v}}{dt} = -\frac{\partial V}{\partial \mathbf{x}} + 2m\mathbf{v} \times \omega + m\omega \times (\mathbf{x} \times \omega) \quad (3)$$

We see that the transformation to a rotating frame introduces two additional force terms in the right-hand side of (3) of which the $2m(\mathbf{v} \times \omega)$ is called the Coriolis force and $m\omega \times (\mathbf{x} \times \omega)$ is the more familiar centrifugal force. The Coriolis force has the form identical to the force exerted by a magnetic field $(2m/q)\omega$ on a particle of charge q. It follows that this force cannot do any work on the particle since it is always orthogonal to the velocity. The centrifugal force, on the other hand, can be obtained as the gradient of an effective potential which is the third term in the right-hand side of (2).

We are now ready to find the rigidly rotating solution in which all the three particles are at rest in the rotating frame in which (3) holds. We will choose a coordinate system in which *the test particle is at the origin* and denote by $\mathbf{r}_1, \mathbf{r}_2$ the position vectors of masses m_1 and m_2. The position of the centre of mass of m_1 and m_2 will be denoted by $\mathbf{r}$ so that

$$(m_1 + m_2)\mathbf{r} = m_1\mathbf{r}_1 + m_2\mathbf{r}_2. \quad (4)$$

In the solution we are looking for, all these three vectors are independent of time in the rotating frame and the Coriolis force vanishes because $\mathbf{v} = 0$. Since m_1 and m_2 are already taken care of (and are assumed to be oblivious to m_3), we only need to satisfy the equation of motion for m_3 which demands:

$$\frac{Gm_1}{r_1^3}\mathbf{r}_1 + \frac{Gm_2}{r_2^3}\mathbf{r}_2 = \omega^2\mathbf{r}. \qquad (5)$$

To work out the exact position of equilibrium, one has to solve a fifth-order equation which will lead to three real roots.

You should now be able to see the equilateral triangle emerging. If we assume $r_1 = r_2$, and take note of (4), the left-hand side of (5) can be reduced to $(G/r_1^3)(m_1 + m_2)\mathbf{r}$ which is in the direction of $\mathbf{r}$. If we next set $r_1 = a$, this equation is identically satisfied, thanks to (1). (The cognoscenti would have realized that making the location of the test particle the origin is an algebraically clever thing to do.) This analysis clearly shows how the mass ratios go away through the proportionality of both sides to the radius vector between the centre of mass and the test particle.

To make sure we catch *all* the equilibrium solutions, we can do this a bit more formally. We define the vector $\mathbf{q}$ by the relation $m_1\mathbf{r}_1 - m_2\mathbf{r}_2 = (m_1 + m_2)\mathbf{q}$. A little bit of algebraic manipulation allows us to write (5) as:

$$\frac{G(m_1+m_2)}{2r_1^3r_2^3}\left[(r_1^3 + r_2^3)\mathbf{r} + (r_2^3 - r_1^3)\mathbf{q}\right] = \frac{G(m_1+m_2)}{a^3}\mathbf{r}. \qquad (6)$$

For this equation to hold, all the vectors appearing in it must be collinear. One possibility is to have $\mathbf{r}$ and $\mathbf{q}$ to be in the same direction. It then follows that $\mathbf{r}_1, \mathbf{r}_2$ and $\mathbf{r}$ are all collinear and the three particles are in a straight line. The equilibrium condition can be maintained at three locations usually called $\mathrm{L}_1, \mathrm{L}_2$ and L_3. To work out the exact position of equilibrium, one has to solve a fifth-order equation which will lead to three real roots. We are, however, not interested in these (at least, not in this installment!) though L_2 of the Sun–Earth system has lots of practical applications.

If we do *not* want $\mathbf{r}$ and $\mathbf{q}$ to be parallel to each other, then the only way to satisfy (6) is to make the coefficient of $\mathbf{q}$ vanish which requires $r_1 = r_2$. Substituting back, we find that each should be equal to a. So we get the rigidly rotating equilateral configuration of three masses with:

$$r_1 = r_2 = a. \tag{7}$$

Obviously, there are two such configurations corresponding to the two equilateral triangles we can draw with the line joining m_1 and m_2 as one side. The locations of the m_3 corresponding to these two solutions are called L_4 and L_5, giving Lagrange a total of five points.

Incidentally, there are several examples in the solar system in which nature uses Lagrange's insight. The most famous among them is the collection of more than a thousand asteroids called Trojans which are located at the vertex of an equilateral triangle, the base of which is formed by Sun and Jupiter – the two largest gravitating bodies in the solar system. Similar, but less dramatic, features are found in the L_5 point of Sun–Mars system and in the satellites of Saturn. The entire configuration goes around in rigid rotation since the orbit of Jupiter is approximately circular.

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The existence of such real life solutions tells us that the equilateral solution must be stable in the sense that if we displace m_3 from the equilibrium position L_5 slightly, it will come back to it. (It turns out that the other three points L_1, L_2, L_3 are not.) Our next job is to study this stability; for this a different coordinate system is better. It will also help to rescale variables to simplify life. We will now take the origin of the rotating coordinate system to be *at the location of the centre of mass* of m_1 and m_2 with the x-axis passing through the two masses and the motion confined to the x–y plane.

Measuring all masses in terms of the total mass m_1+m_2, we can denote the smaller mass by μ and the larger by $(1-\mu)$. Similarly, we will measure all distances in terms of the separation a between the two primary masses and choose the unit of time such that $\omega = 1$. (If these appear strange for you, just write down the equations in normal units and re-scale them; such tricks are worth learning.) The position of m_3 is (x, y) and r_1 and r_2 will denote the (scalar) distances to m_3 from the masses $(1-\mu)$ and μ respectively. (Note that these are *not* the distances to m_3 from the origin.) It is now easy to see that the equations of motion in (3) reduce to the set:

$$\ddot{x} - 2\dot{y} = -\frac{\partial \Phi}{\partial x}, \qquad \ddot{y} + 2\dot{x} = -\frac{\partial \Phi}{\partial y}, \tag{8}$$

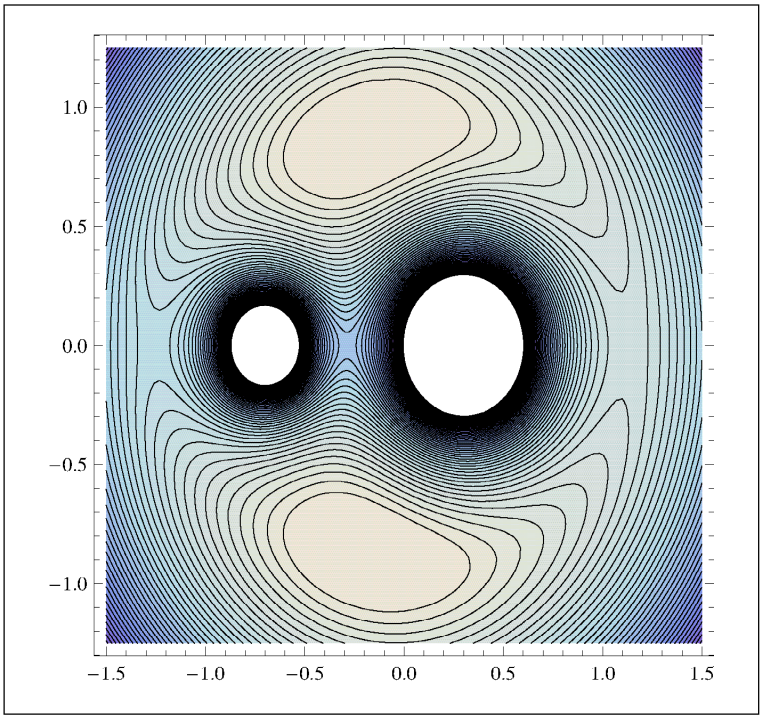
where

$$\Phi = -\frac{1}{2}(x^2 + y^2) - \frac{(1-\mu)}{r_1} - \frac{\mu}{r_2} \tag{9}$$

is the effective potential in the rotating frame which includes a term from the centrifugal force. The only known integral of motion is the rather obvious one corresponding to the energy function $(1/2)v^2 + \Phi = \text{constant}$. A little thought shows that $\nabla\Phi = 0$ at L_4 and L_5, confirming the existence of a stationary solution. To study the stability, we normally would have checked whether these correspond to maxima or minima of the potential. As we shall see (*Figure* 1), it turns out that L_4 and L_5 actually correspond to maxima, so if that is the whole story L_4 and L_5 should be unstable.

But, of course, that is not the whole story since we need to take into account the Coriolis force term corresponding to $(2\dot{y}, -2\dot{x})$ in (8). To see the effect of this term clearly, we will take the Coriolis force term to be $(C\dot{y}, -C\dot{x})$ so that the real problem corresponds to $C = 2$. But this trick allows us to study the stability

Figure 1. A contour plot of the potential Φ (x,y) when μ = 0.3. The L_4 and L_5 are at the potential maxima. One can also see the saddle points L_1, L_2, L_3 along the line joining the two primary masses.



for any value of C, in particular for $C = 0$, to see what happens if there is no Coriolis force. We now have to do a Taylor series expansion of the terms in (8) in the form $x(t) = x_0 + \Delta x(t)$, $y(t) = y_0 + \Delta y(t)$ where the point (x_0, y_0) corresponds to the L_5 point with $y_0 > 0$. We also need to expand Φ up to quadratic order in Δx and Δy to get the equations governing the small perturbations around the equilibrium position. This is straightforward but a bit tedious. If you work it through, you will get the equations

$$\frac{d^2}{dt^2}\Delta x = \frac{3}{4}\Delta x + \left(\frac{3\sqrt{3}}{4}\right)(1-2\mu)\,\Delta y + C\frac{d}{dt}\Delta y, \tag{10}$$

$$\frac{d^2}{dt^2}\Delta y = \frac{9}{4}\Delta y + \left(\frac{3\sqrt{3}}{4}\right)(1-2\mu)\,\Delta x - C\frac{d}{dt}\Delta x. \tag{11}$$

To check for stability, we try solutions of the form $\Delta x = A\exp(\lambda t)$, $\Delta y = B\exp(\lambda t)$ and solve for λ. An elementary calculation gives

$$\lambda^2 = \frac{3 - C^2 \pm [(3 - C^2)^2 - 27\mu(1-\mu)]^{1/2}}{3}. \qquad (12)$$

Stability requires that we should not have a positive real part to λ; that is, λ^2 must be real and negative. For λ^2 to be real, the term in (12) containing the square root should have a positive argument which requires

$$(C^2 - 3)^2 > 27\mu(1-\mu). \qquad (13)$$

Further, if both roots of λ^2 are negative, then the product of the roots must be positive and the sum should be negative. It is easily seen that this requires the condition $C > \sqrt{3}$. Thus we conclude that the motion is unstable if $C < \sqrt{3}$; in particular, in the absence of the Coriolis force ($C = 0$), the motion is unstable because the potential at L_5 is actually a maximum. But when $C > \sqrt{3}$ and in particular for the real case we are interested in with $C = 2$, the motion is stable when condition in (13) is satisfied. Using $C = 2$ we can reduce this condition to $\mu(1-\mu) < (1/27)$. This leads to

$$\mu < \left(\frac{1}{2} - \sqrt{\frac{23}{108}}\right) \approx 0.0385. \qquad (14)$$

This criterion is met by the Sun–Jupiter system with $\mu \approx 0.001$ and by the Earth–Moon system with $\mu \approx 0.012$. Stability of Trojans is assured. In fact, the L_5s and L_4s are the favourites of science fiction writers and some NASA scientists for setting up space colonies. (There is even a US-based society called the 'L_5 society', which was keen on space colonization based on L_5!)

So how does Coriolis force actually stabilize the motion? When the particle wanders of the maxima, it acquires a non-zero velocity and the Coriolis force induces an

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Box 1. Geometrical Proof of Lagrange's Equilateral Solution

If you like matters geometrical, you might find the following proof interesting. In *Figure* A, the triangle ABL_5 is Lagrange's equilateral triangle of unit side with mass μ located at A, mass $(1-\mu)$ located at B, and the test particle located at L_5. The centre of mass of the primary bodies is C and all the three masses rotate rigidly around C. We need to prove that the resultant of the gravitational attraction along L_5A and L_5B will be precisely along L_5C and will have a magnitude equal to the (outward) centrifugal force acting on L_5. With our choice of units, $\omega^2 = 1$ and the centrifugal force is numerically the same as the length L_5C. To prove this, draw DE perpendicular to CL_5 and drop perpendiculars AD and BE as shown. Also draw a perpendicular from A to BE (shown by dashed line). If AD is equal to x and EB is equal to $x+y$, it follows from elementary geometry that $L_5F = x$ and $FC = y(1-\mu)$. We can also write DL_5 and L_5E as $(1-\mu)l$ and μl respectively for some l.

To prove that forces match at L_5, we need to show that the component of the gravitational force along L_5D due to the mass μ is balanced by the component of the gravitational force along L_5E due to the mass $(1-\mu)$. This leads to the condition $\mu(1-\mu)\,l = (1-\mu)\,\mu\,l$ which is true. (While taking cosines and sines of angles, recall that the equilateral triangle has unit side.) Next consider the component of the force along L_5C. The sum of the two gravitational forces along L_5C is given by $\mu x + (1-\mu)(x+y)$ which should balance the outward centrifugal force equal to the length of L_5C, viz., $x + y(1-\mu)$. Since $\mu x + (1-\mu)(x+y) = x + y(1-\mu)$, we are again through with the proof. This proves that one can achieve force balance in the equilateral configuration for any value of μ. The fact that C divides AB in the inverse ration of the masses is, of course, crucial.

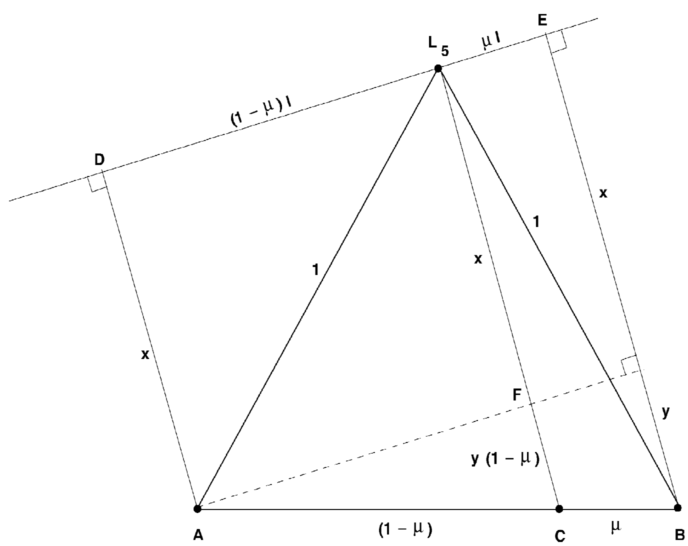

Figure A

acceleration in the direction perpendicular to the velocity. As we noted before, this is just like the motion in a magnetic field and the particle just goes around L_5. The idea that a force which does not do work can still help in maintaining the stability may appear a bit strange but is completely plausible. In fact, the analogy between Coriolis and magnetic forces tells you that one may be able to achieve similar results with magnetic fields too. This is true and one example is the so-called 'Penning trap', which you might like to read about with the current insight.

To be absolutely correct and for the sake of experts who may be reading this, I should add a comment regarding another peculiarity which this system possesses. A more precise statement of our result on stability is that, when (14) is satisfied, the solutions are *not linearly unstable*. The characterization "not unstable" is qualified by saying that this is a result in linear perturbation theory. A fairly complex phenomenon (which is too sophisticated to be discussed here, but see [2] if you are interested) makes the system unstable for two precise values of μ which do satisfy (14). These values happen to be $(1/30)[15-\sqrt{213}]$ and $(1/90)[45-\sqrt{1833}]$. (Yes, but I said the phenomenon is complex!) While of great theoretical value, this is not of much practical relevance since one cannot fine-tune masses to any precise values.

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Suggested Reading

[1] T Padmanabhan, Planets move in circles, *Resonance*, Vol.1, No.9, pp.34–40, 1996.

[2] D Boccaletti and G Pucacco, *Theory of Orbits*, Springer, Vol. 1, p. 271, 1996.