Snippets of Physics

14. The Power of Nothing

T Padmanabhan

The vacuum state of the electromagnetic field is far from trivial. Amongst other things it can exert forces that are measurable in the lab, in a curious phenomenon known as *Casimir effect*.

T Padmanabhan works at IUCAA, Pune and is interested in all areas of theoretical physics, especially those which have something to do with gravity.

We all know that classical electromagnetic fields can exert forces on charged particles. The standard expression for the classical force is $q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ which, of course, vanishes when $\mathbf{E} = \mathbf{B} = 0$. That sounds eminently reasonable.

But then, we know that the real world is quantum mechanical in character and not classical, which implies that we need to treat the electromagnetic field as a quantum entity. When we do that, photons emerge as the quanta of electromagnetic field. As in the case of any other quantum system, e.g., the hydrogen atom, one can describe the physics in terms of the quantum states of the electromagnetic field. In this language, one can also define a state with zero photons which could be thought of as the vacuum state of the electromagnetic field. One would have imagined that, if there are no photons, then there will be no measurable physical effects due to the electromagnetic field. While this is more or less true – which is rather reassuring – there are indeed interesting situations in which it is *not* true! We will describe one such context, called Casimir effect, in this installment.

The simplest, though a bit idealized, description of Casimir effect is the following. Suppose you keep two parallel, perfectly conducting plates in the otherwise empty space, separated by a distance L. Then, they will attract

Keywords
Harmonic oscillator, quantum mechanics, electromagnetism.

Remember that there are no net charges put on the plates; we are not talking about a *charged* parallel plate capacitor. The force acts between two plates kept in the vacuum.

each other with a force

$$F = -\frac{\pi^2}{240}\frac{\hbar c}{L^4}A, \tag{1}$$

where A is the cross-sectional area of the plates!! Remember that there are no net charges put on the plates; we are not talking about a *charged* parallel plate capacitor. The force acts between two plates kept in the vacuum. This effect was predicted [1] by the Dutch physicist Hendrick Casimir in 1948 and has actually been measured in the lab [2]. One nice way of understanding this result is as a tangible force exerted by the electromagnetic vacuum. Let us see how.

Before launching into mathematics, let me explain the basic reason for this phenomenon in qualitative terms. Consider the familiar example of a harmonic oscillator, with the Hamiltonian

$$H(p, q) = \frac{1}{2}[p^2 + \omega^2 q^2]. \tag{2}$$

We have set the mass of the particle to unity for simplicity. Classically, the minimum energy for such a system is zero ($E_{\text{class}} = 0$), which is achieved when $q = p = 0$. We know, however, that this is not possible in quantum theory, essentially because of the uncertainty principle. To minimize the potential energy, we need to set $q = 0$; but if we know the position to such infinite precision, the momentum will be infinitely uncertain and we cannot guarantee a low value for $p^2/2$! So to minimize the total energy, we need to allow for some amount of fluctuation in both q and p that is commensurate with the uncertainty principle $\Delta p \Delta q \gtrsim \hbar$. The resulting ground state, as we know, has a non-zero energy $E_{\text{quant}} = (1/2)\hbar\omega$.

To minimize the total energy, we need to allow for some amount of fluctuation in both q and p that is commensurate with the uncertainty principle $\Delta p\, \Delta q \gtrsim \hbar$.

Suppose we consider a different physical system with the Hamiltonian $H_{\text{new}} = H(p, q) - (1/2)\hbar\omega$ where $H(p, q)$ is given by (2). Since the subtraction of a constant from the Hamiltonian does not change the equations of motion, we still again have a harmonic oscillator but with a shift in the energy. Classically, the minimum energy state will still correspond to $q = p = 0$ but with energy $E_{\text{class}} = -(1/2)\hbar\omega$. But quantum mechanically, the ground state will exhibit fluctuations in q and p but will have zero energy; $E_{\text{quant}} = 0$! This is the crucial point. Quantum mechanics allows you to have a state for the harmonic oscillator with the Hamiltonian $H_{\text{new}}(p, q)$ such that $E_{\text{quant}} = 0$ which can host fluctuations in the dynamical variables q and p.

Quantum mechanics allows you to have a state for the harmonic oscillator with the Hamiltonian H_{new} (p,q) such that $E_{\text{quant}} = 0$ which can host fluctuations in the dynamical variables q and p.

Something very analogous happens in the case of an electromagnetic field. As we shall see the electromagnetic field can be thought of as a bunch of harmonic oscillators. The ground state will correspond to a state of zero photons and one can arrange matters such that it has zero energy. But the electric and magnetic fields will play the role analogous to p and q of the oscillator and they will exhibit fluctuations – usually called vacuum fluctuations – in the ground state. Therefore one cannot really say that the electromagnetic fields vanish in the vacuum state even though we can make its energy vanish. Once we recognize this fact, it is not surprising that the electromagnetic vacuum can exert forces on bodies. Actually the situation is a little bit more complicated because the procedure analogous to the subtraction of $(1/2)\hbar\omega$ is more nontrivial in this case but the essential idea is the same.

Let us now try to understand this in a more mathematical language and in a somewhat broader context. As it turns out, the essential idea can be illustrated by ignoring two complications of the real world. First is the vector nature of electromagnetism and the second is the fact that space is three dimensional. We will work out first a simpler picture using just a scalar field with one degree of freedom (rather than with the electromagnetic field) and also ignoring the two transverse directions and treating space as one-dimensional. After we work out the simplified picture, we will describe how it

One cannot really say that the electromagnetic fields vanish in the vacuum state even though we can make its energy vanish.

generalizes to the real life.

Once we ignore the vector nature of the electromagnetic field, we can work with a single scalar field $\phi(t,x)$ which is a function of one space dimension and time. (If you want, you can think of it as analogous to any one component of the electromagnetic field.) In the absence of sources, we know that each component of the electromagnetic field satisfies the wave equation; so we will assume that our scalar field satisfies the equation:

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} = 0. \tag{3}$$

(We have chosen units with $c = 1$. In 3-dimensions, the second term would have been $-\nabla^2\phi$ which becomes $-(\partial^2\phi/\partial x^2)$ when we ignore two spatial coordinates.) This equation can be simplified by introducing the spatial Fourier transform Q_k of $\phi(t,x)$ by

$$\phi(t,x) = \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{(2\pi)} Q_k(t) \exp(ikx). \tag{4}$$

Substituting this in (3) we find that $Q_k(t)$ satisfies the equation $\ddot{Q}_k + k^2 Q_k = 0$. The field $\phi(t,x)$ is completely specified by the function $Q_k(t)$ so that we can think of $Q_k(t)$ as the dynamical variables describing our system. The fact that we are dealing with the field translates into the fact that we now have an infinite number of dynamical variables, one for each value of k. Other than that, we can work directly with $Q_k(t)$ instead of the original field $\phi(t,x)$.

The field ϕ (t,x) is completely specified by the function $Q_k(t)$ so that we can think of $Q_k(t)$ as the dynamical variables describing our system.

One minor problem with $Q_k(t)$ is that it is a complex number (since $\phi(t,x)$ is real) and we would like to work with dynamical variables that are real. This is easily taken care of. As Q_k is complex, we have two degrees of freedom corresponding to the real and imaginary parts of Q_k for each k with the constraint $Q_k^* = Q_{-k}$. If we write $Q_k = (A_k + iB_k)$, then, since ϕ is a real scalar field, we can relate the variables for k to that for $-k$ as $A_k = A_{-k}$ and $B_k = -B_{-k}$. Evidently, only half the modes constitute independent degrees of freedom. Therefore, we can work with a new set of real modes q_k, defined for all values of k with a suitable redefinition, say, by taking $q_k = A_k$ for one half of k and $q_{-k} = B_k$ for the other half. This will, of course, lead to the same equation but for the *real* variable $q_k(t)$.

$$\ddot{q}_k + k^2 q_k = 0. \tag{5}$$

That is, the dynamical variable $q_k(t)$ satisfies the harmonic oscillator equation with frequency $\omega = |k|$, for each value of k. Our field is mathematically the same as an infinite number of harmonic oscillators, one for each k. It follows that everything we know about harmonic oscillators can now be applied to this system. In particular, we can quantize the field by quantizing each of the harmonic oscillators $q_k(t)$. (In fact, that is the essence of quantum field theory of non-interacting fields; rest is just detail.)

In particular, we can quantize the field by quantizing each of the harmonic oscillators $q_k(t)$. (In fact, that is the essence of quantum field theory of non-interacting fields; rest is just detail.)

Classically, we can now construct the ground state by taking $q_k = 0$ for all values of k. This will, of course, make the field vanish along with its energy, as to be expected from a sensible ground state. But, as we discussed earlier, this does not hold for the quantum ground state. The ground state of the harmonic oscillator for a given value of k is described by the ground state energy eigenfunction

$$\psi(q_k) = \left(\frac{\omega_k}{\pi}\right)^{1/4} \exp\left(-\frac{1}{2}\omega_k q_k^2\right). \tag{6}$$

We are using units with $\hbar = 1$ for simplifying the expressions. The ground state wave function for the full system, made of a bunch of independent oscillators, can be described by the product of the ground state wave

This expression can be interpreted along similar lines as we interpret a harmonic oscillator wave function in usual quantum mechanics.

functions of each of the oscillators:

$$\begin{aligned}\Psi\left[\phi(x)\right] &= \prod_k \left(\frac{\omega_k}{\pi}\right)^{1/4} \exp\left(-\frac{1}{2}\omega_k q_k^2\right) \\ &\equiv \bar{N}\exp\left[-\frac{1}{2}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{(2\pi)}\omega_k q_k^2\right]. \qquad (7)\end{aligned}$$

This expression can be interpreted along similar lines as we interpret a harmonic oscillator wave function in usual quantum mechanics. Suppose we have a harmonic oscillator in the ground state and we measure the position q. Then the relative probability that we will get a value $q = a$ compared to a value $q = b$ is given by

$$R = \frac{|\psi(a)|^2}{|\psi(b)|^2} = \exp\left(-\omega[a^2 - b^2]\right). \qquad (8)$$

Now suppose we have a *quantum field* which is in the ground state and we measure the field everywhere at, say, $t = 0$. Then, there is some probability that we will get a field configuration described by the function $\phi(0, x) = f_1(x)$ and some other probability that field configuration is described by the function $\phi(0, x) = f_2(x)$. Just as in the previous case, we want to know the relative probability of getting one configuration compared to another. To find this, we first obtain the spatial Fourier transforms of $f_1(x)$ and $f_2(x)$ and call them a_k and b_k. Then the relative probability is given by

For any choice of $f_1(x)$ and $f_2(x)$ the number R can be computed, allowing us to determine the probabilities of different field configurations in the vacuum state.

$$R = \frac{|\Psi(f_1(x))|^2}{|\Psi(f_2(x))|^2} = \exp\left(-\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{(2\pi)}\omega_k[a_k^2 - b_k^2]\right). \quad (9)$$

For any choice of $f_1(x)$ and $f_2(x)$ the above number can be computed, allowing us to determine the probabilities of different field configurations in the vacuum state.

You would have noticed that we switched to relative probabilities rather than absolute probabilities in this discussion. For a single harmonic oscillator, one could have said that $|\psi(q)|^2\mathrm{d}q$ gives the absolute probability of finding the particle in the interval $(q, q + \mathrm{d}q)$. When we have infinite number of oscillators, the normalization factor $\bar{N}$ in (7) involves an infinite product which is hard to define rigorously. We get around this by talking about relative probabilities in which the normalization factor cancels out.

We get around this by talking about relative probabilities in which the normalization factor cancels out.

Before we proceed further, let me mention the corresponding result in three spatial dimensions. In this case (7) has the obvious generalization to:

$$\begin{aligned} \Psi\left[\phi(\mathbf{x})\right] &= \prod_{\mathbf{k}} \left(\frac{\omega_k}{\pi}\right)^{1/4} \exp\left(-\frac{1}{2}\omega_k|q_{\mathbf{k}}|^2\right) \\ &= \bar{N} \exp\left[-\frac{1}{2}\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\omega_k|q_{\mathbf{k}}|^2\right]. \end{aligned} \qquad (10)$$

In fact, in this case it is nicer to exhibit the result in terms of the field configuration itself by using $\omega_k = |\mathbf{k}|$ and $\omega_k|q_{\mathbf{k}}|^2 = k^2|q_{\mathbf{k}}|^2/|\mathbf{k}|$. Since $i\mathbf{k}q_{\mathbf{k}}$ is essentially the Fourier spatial transform of $\nabla\phi$, we can easily obtain

$$\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi^3)}\omega_k|q_{\mathbf{k}}|^2 = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi^3)}\frac{|\mathbf{k}|^2|q_{\mathbf{k}}|^2}{|\mathbf{k}|}$$

$$= \frac{1}{2\pi^2}\int \mathrm{d}^3\mathbf{x}\int \mathrm{d}^3\mathbf{y}\left\{\frac{\nabla_{\mathbf{x}}\phi\cdot\nabla_{\mathbf{y}}\phi}{|\mathbf{x}-\mathbf{y}|^2}\right\}. \qquad (11)$$

(Prove this!) Substituting this into (10) and taking the modulus, we get the probability distribution in the ground state to be

$$\mathcal{P}[\phi(\mathbf{x})] = |\Psi[\phi(\mathbf{x})]|^2$$

$$= N\exp\left\{-\frac{1}{2\pi^2}\int\int \mathrm{d}^3\mathbf{x}\ \mathrm{d}^3\mathbf{y}\frac{\nabla_{\mathbf{x}}\phi\cdot\nabla_{\mathbf{y}}\phi}{|\mathbf{x}-\mathbf{y}|^2}\right\}, \qquad (12)$$

with $N = |\bar{N}|^2$. Once again, this expression shows clearly that the vacuum state of the field can host, what

Once again, expression (12) shows clearly that the vacuum state of the field can host, what is usually called, zero point fluctuations of the field variable ϕ.

Let us now ask what happens if we introduce two perfectly conducting, parallel plates into the vacuum.

is usually called, zero point fluctuations of the field variable ϕ. The probability that one detects a particular field configuration $\phi(\mathbf{x})$ when the field is in the vacuum state can be obtained by evaluating the value of $\mathcal{P}$ for this particular functional form $\phi(\mathbf{x})$. The result is independent of time because of the stationarity of the vacuum state. Given the ambiguity in the overall normalization factor N this probability should again be interpreted as a relative probability. That is, the ratio $\mathcal{P}_1/\mathcal{P}_2$ will give the relative probability between two field configurations characterized by the functions $\phi_1(\mathbf{x})$ and $\phi_2(\mathbf{x})$.

Let us now ask what happens if we introduce two perfectly conducting, parallel plates into the vacuum. The fact that the plates are perfectly conducting implies that the electromagnetic field – for which our $\phi(t,\mathbf{x})$ is a proxy – must satisfy some non-trivial boundary conditions at $x = 0$ and $x = L$ where the plates are located. For the scalar field, we can take the boundary condition to be that the field vanishes at the plates: $\phi(t,0) = \phi(t,L) = 0$ in one spatial dimension. You cannot describe a field satisfying such a boundary condition using the Fourier integral in (4) with k taking all possible values in $-\infty < k < \infty$. Instead we can restrict it to a discrete, though infinite, set of values given by $k = n(\pi/L)$ with $n = 1, 2, ...$ and write

$$\phi(t,x) = \sum_{n=1}^{\infty} q_n(t) \sin\left[n\frac{\pi x}{L}\right] \tag{13}$$

so that the boundary conditions at $x = 0$ and $x = L$ are satisfied. We still have to deal with an infinite number of oscillators but their frequencies are now given by $\omega_n = k_n = n(\pi/L)$.

If we now work out the corresponding ground state, it will clearly be different from the one described by (7) because the integral over k will be replaced by the sum over n. This is needed because, our boundary condition tells us that the ground state should now have zero probability for field configurations which do not vanish at the plates. The introduction of the plates, through changing the boundary condition, has changed the ground state.

The introduction of the plates, through changing the boundary condition, has changed the ground state.

What about the energy of the ground state with and without the plates? They are also different. In the absence of plates, each harmonic oscillator contributes an energy $(1/2)\hbar\omega = (1/2)\hbar|k|$. So the total ground state energy, per unit length of space, is given by an integral over all k of $(1/2)\hbar|k|$. Therefore, the energy in a region of length L will be:

$$E_0 = \frac{L}{(2\pi)} \int_{-\infty}^{\infty} \mathrm{d}k \frac{1}{2}\hbar|k| = \frac{L}{(2\pi)} \int_0^{\infty} \mathrm{d}k \hbar k. \qquad (14)$$

This is manifestedly infinite, essentially because there are an infinite number of harmonic oscillators. What about the ground state energy in the presence of the plates? This is given by the sum

$$E'_0 = \frac{1}{2}\sum_{n=0}^{\infty} \hbar\omega_n = \frac{1}{2}\sum_{n=0}^{\infty} \hbar(n\pi/L) \qquad (15)$$

which is also infinite, essentially being the sum of all positive integers.

These infinities are bad news but there is a trick to get around them. As we said before, the equation of motion for the k-th oscillator will not change if we substract from the Hamiltonian $(1/2)\hbar\omega_k$ but it will 'regularize the ground state energy to zero. This is equivalent to looking at the difference $(E'_0 - E_0)$ as the physically relevant quantity. To study this, it is convenient to introduce in (14) a continuous variable n via the equation $k = (\pi/L)n$. Then we get from (14) and (15):

$$(E'_0 - E_0) = \frac{\hbar\pi}{2L}\left[\sum_{n=0}^{\infty} n - \int_0^{\infty} \mathrm{d}n\, n\right]. \qquad (16)$$

Both the expressions in (17) as well as their difference are now finite and the idea is to *first* compute the difference as a function of λ and then take the limit of $\lambda \to 0$ hoping for the best.

You may think that this is not much help because this is of the form $(\infty - \infty)$ which does not have a precise meaning. That is true but there are ways of giving meaning to such expressions in a fairly systematic manner. The simplest procedure is to consider, instead of the expression in (16), the expression:

$$E_0'(\lambda) - E_0(\lambda)$$

$$\equiv \frac{\hbar\pi}{2L}\left[\sum_{n=0}^{\infty} n\ \exp(-n\lambda) - \int_0^{\infty} dn\, n \exp(-n\lambda)\right]. \quad (17)$$

Here we have multiplied both the expressions by a 'regulator function' $\exp(-n\lambda)$ where λ is just a parameter. Both the expressions as well as their difference are now finite and the idea is to *first* compute the difference as a function of λ and then take the limit of $\lambda \to 0$ hoping for the best. That is, we interpret the expression in (16) as the limit of the expression in (17) when $\lambda \to 0$. I will let you work out the expressions. You should first get:

$$\sum_{n=0}^{\infty} n \exp(-n\lambda) = \frac{e^{-\lambda}}{(1-e^{-\lambda})^2} = \frac{1}{\lambda^2} - \frac{1}{12} + \frac{\lambda^2}{240} + \mathcal{O}(\lambda^4) \quad (18)$$

which diverges when $\lambda \to 0$ as to be expected. Similarly

$$\int_0^{\infty} dn n \exp(-n\lambda) = \frac{1}{\lambda^2} \quad (19)$$

which also diverges when $\lambda \to 0$. But, astonishingly enough, the difference between (18) and (19) remains finite as $\lambda \to 0$:

$$\sum_{n=0}^{\infty} n \exp(-n\lambda) - \int_0^{\infty} dn n \exp(-n\lambda)$$

$$= -\frac{1}{12} + \mathcal{O}(\lambda^2) = -\frac{1}{12} \quad (20)$$

when $\lambda \to 0$. This allows us to obtain the following remarkable result:

$$E(L) \equiv (E_0' - E_0) \equiv \lim_{\lambda \to 0}(E_0'(\lambda) - E_0(\lambda)) = -\frac{\pi \hbar}{24L}. \tag{21}$$

So we see that the ground state energy of the system with the plates – when regularized by subtracting away the energy in the absence of the plates – is a negative number[1] and is inversely proportional to the separation between the plates! Clearly, this will lead to an attractive force $F = -(\mathrm{d}E/\mathrm{d}L) \propto L^{-2}$ between the plates, since reducing the separation between the plates leads to the lowering of the energy. A more physical way of thinking about this result is as follows. If we change the separation between the plates by an amount ΔL, the energy of the configuration will change by $(\mathrm{d}E/\mathrm{d}L)\Delta L$ which must be accounted by the work done by the agency separating the plates acting against the attractive force F. Equating it to $-F\Delta L$, we find that $F = -\mathrm{d}E/\mathrm{d}L$.

[1] With this 'regularization', quantum field theorists often conclude that the sum of all positive integers is not only finite but is a negative fraction $(-1/12)$! If you are familiar with Riemann-zeta function, $\zeta(x) = \sum_{n=1}^{\infty} n^{-x}$, you will recognize that the sum of all positive integers is formally the same as $\zeta(-1)$. One can define this quantity by analytic continuation in the complex plane and one does recover the result $\zeta(-1) = -1/12$. Of course, this does not make one any wiser as to what is going on.

In the mythical world of one spatial dimension, the plates are zero-dimensional points which is not of much use. The corresponding calculation for electromagnetic field in 3-dimensions is more complicated algebraically but all the concepts remain the same. The final result in this case is an expression for energy per unit transverse area of the plates, given by:

$$\frac{(E_0' - E_0)}{A} = -\frac{\pi^2}{720}\frac{\hbar c}{L^3}, \tag{22}$$

where we have re-introduced the c factor. The force per unit area acting between the plates is given by

$$\frac{F}{A} = -\frac{\mathrm{d}}{\mathrm{d}L}\frac{(E_0' - E_0)}{A} = -\frac{\pi^2}{240}\frac{\hbar c}{L^4}. \tag{23}$$

This tiny force has actually been measured in the lab! Note that, though the result is electromagnetic by nature, it is independent of the electronic charge e. The

The essential lesson is that the pattern of quantum fluctuations is sensitive to the boundary conditions we impose, both mathematically and practically.

electromagnetism only enters through the boundary condition on the perfect conductors, which is one reason we could mimic it with a scalar field.

The whole phenomenon is quite bewildering and if you are shaking your head in disbelief, I will not blame you! But the reality of this effect is beyond dispute and it has been derived from several different perspectives over years. The essential lesson is that the pattern of quantum fluctuations is sensitive to the boundary conditions we impose, both mathematically and practically. The ground state of the electromagnetic field in the presence of two parallel, conducting, plates is quite different from the ground state in the absence of the plates. This much alone is easy to understand because the ground state in the presence of the plates must ensure that, the field configurations which do not satisfy the boundary conditions at the plates, have zero probability for their existence. But what is rather curious is that this ground state has an energy which differs from that in the absence of the plates by a *finite* amount. There is no simple explanation for this fact, which makes Casimir effect all the more fascinating.

Address for Correspondence
T Padmanabhan
IUCAA, Post Bag 4
Pune University Campus
Ganeshkhind
Pune 411 007, India.
Email:
paddy@iucaa.ernet.in
nabhan@iucaa.ernet.in

Suggested Reading

[1] H B G Casimir, *Proc. Kon. Ned. Akad. Wetensch.*, Vol.B51, p.793, 1948.

[2] S K Lamoreaux, *Phys. Rev. Lett.*, Vol.78, p.58 1997.
G Bressi, G Carugno, R Onofrio, G Ruoso, *Phys. Rev. Lett.*, Vol.88, p.041804, 2002.