Snippets of Physics

11. Isochronous Potentials

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Oscillatory motion of a particle in a one dimensional potential belongs to a class of exactly solvable problems in classical mechanics. In this installment, we examine some lesser known aspects of the oscillations in some potentials.

The motion of a particle of mass m in one dimension under the action of a potential $V(x)$ is the simplest problem which one studies in classical mechanics. In fact, a formalist will consider this as a solved problem, in the sense that the differential equation governing the motion can be reduced to a quadrature; ie., the trajectory of the particle can be expressed as an indefinite integral. In spite of this apparent triviality of the problem there are some interesting surprises one encounters in their study.

Using the constancy of the total energy, $E = (1/2)m\dot{x}^2 + V(x)$, one can write down the equation determing the trajectory of the particle $x(t)$ in the form of the integral

$$t(x) = \sqrt{\frac{m}{2}} \int^x \frac{\mathrm{d}x}{\sqrt{E - V(x)}}. \qquad (1)$$

This determines the inverse function $t(x)$ for a given $V(x)$ and the problem is completely solved. In this installment, we are interested in the case of bounded oscillations of a particle in a potential well $V(x)$ which has the general shape shown in *Figure* 1. The potential has a single minimum and increases without bound as $|x| \to \infty$. For a given value of energy E, the particle will oscillate between the two turning points $x_1(E)$ and $x_2(E)$ which are the roots of the equation $V(x) = E$. The period of oscillation can be immediately written

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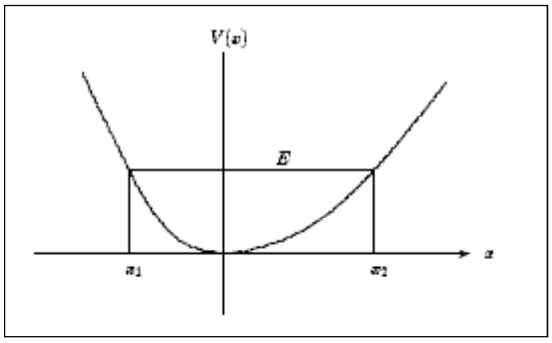

Figure 1. A one-dimensional potential with a single minimum which supports oscillations.

down using equation (1). We get

$$T(E) = \sqrt{\frac{m}{2}} \int_{x_1(E)}^{x_2(E)} \frac{\mathrm{d}x}{\sqrt{E - V(x)}}. \qquad (2)$$

For a general potential $V(x)$, the result of integration on the right hand side will depend on the value of the energy E. In other words, the period of oscillation will depend on the energy of the particle; equivalently, if one imagines releasing the particle from the location $x = x_1$, say, one might say that the period depends on the amplitude of oscillation.

For a simple class of potentials, it is quite easy to determine the scaling of the period T with the energy E. Consider, for example, a class of potentials of the form $V(x) = kx^{2n}$ where n is an integer. These potentials are symmetric in the x-axis and have a minimum at $x = 0$ with the minimum value being $V_{\min} = 0$. In this case, by introducing a variable q such that $q = (k/E)^{1/2n}x$, the energy dependence of the integral in (2) can be easily identified to give

$$T(E) \propto \frac{1}{\sqrt{E}} E^{1/2n} \int_0^1 \frac{\mathrm{d}q}{\sqrt{1 - q^{2n}}} \;\propto\; E^{\frac{1}{2}\left(\frac{1-n}{n}\right)}. \qquad (3)$$

We find that, for all values of n other than $n = 1$, the period T has a non-trivial dependence on the energy. When $n = 1$, which corresponds to the harmonic oscillator potential, $V(x) = kx^2$, we find that the period

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is independent of the energy. This, of course, is the well-known result that the period of a harmonic oscillator does not depend on the amplitude of the oscillator. The above analysis also shows that amongst all the *symmetric* potentials of the form $V(x) \propto x^{2n}$, only the harmonic oscillator has this property.

Let us now consider the inverse problem. Suppose we are given the function $T(E)$. Is it then possible for us to determine the potential $V(x)$? For example, if we are told that the period is independent of the amplitude, what can one say about the form of the potential $V(x)$? Should it necessarily be a harmonic oscillator potential or can it be more general?

Before launching into a mathematical analysis, let me describe a simple example which deserves to be better known than it is. Consider a potential of the form

$$V(x) = ax^2 + \frac{b}{x^2} \tag{4}$$

in the region $x > 0$. In this region, this potential has a distinct minimum at $x_{\min} = (b/a)^{1/4}$ with the minimum value of the potential being $2\sqrt{ab}$. (Being symmetric in x, the potential has two minima in the full range $-\infty < x < \infty$, but we shall confine our attention to the range $x > 0$. By shifting the origin suitably we can make the potential in this range to look like the one in *Figure* 1). For any finite energy, a particle will execute periodic oscillations in this potential. It turns out that the period of oscillation in this potential is independent of the amplitude just as in the case of a harmonic oscillator potential! So clearly harmonic oscillator is not unique in having this property.

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There are several ways to prove this result, the hardest route being to evaluate the integral in (2) with $V(x)$ given by (4); the cutest procedure is probably the following. Consider a particle moving not in one dimension

but in two (say in the xy plane) under the action of the two-dimensional harmonic oscillator potential

$$V(x,y) = \frac{1}{2}m\omega^2(x^2+y^2). \tag{5}$$

Clearly, such a particle will oscillate with a period which is independent of its energy. Now consider the same problem in polar coordinates instead of Cartesian coordinates. The conservation of energy now becomes

$$\begin{aligned} E &= \frac{1}{2}m(\dot{x}^2+\dot{y}^2) + \frac{1}{2}m\omega^2(x^2+y^2) \\ &= \frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2) + \frac{1}{2}m\omega^2 r^2. \end{aligned} \tag{6}$$

Using the fact that for such a motion – under the central force $V(r) \propto r^2$ – the angular momentum $J = mr^2\dot{\theta}$ is conserved, the energy can be expressed in the form

$$E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m\omega^2 r^2 + \frac{1}{2}\frac{J^2}{mr^2} = \frac{1}{2}m\dot{r}^2 + Ar^2 + \frac{B}{r^2} \tag{7}$$

with $A = (1/2)m\omega^2, B = J^2/2m$. We now see that, mathematically, this is identical to the problem of a particle moving in one dimension under the action of a potential of the form in (4). But we know by construction that the period of oscillation does not depend on the conserved energy E in the case of (7). It follows that the potential in (4) must have this property. The actual frequency of oscillation is $\omega_0 = (8a/m)^{1/2}$ which is most easily found by using the fact that the frequency must be the same as that for very small oscillations near the minimum. One may think that since ω_0 is independent of b, it must be $(2a/m)^{1/2}$ for $b = 0$. This is, however, not true because however small b may be, the potential does rise to infinity at $x = 0$ thereby doubling the frequency.

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the amplitude. It is not difficult to see that there are actually an infinite number of such potentials. In fact, for every function $T(E)$, one can construct an infinite number of potentials $V(x)$ such that equation (4) holds. There is an elementary way by which one can construct them which we will now describe [1].

Note that the period $T(E)$ is determined by the integral in (2) which is essentially the area under the curve $(E - V(x))^{-1/2}$. Suppose we are given a potential $V_1(x)$ for which the energy dependence of the period is given by a function $T(E)$. Let us now construct another potential $V_2(x)$ by 'shearing' the original potential $V_1(x)$ parallel to the x-axis. This is done by shifting the potential curve horizontally by an amount $\Delta(V)$ at every value of V using some arbitrary function $\Delta(V)$. The only restriction on the function $\Delta(V)$ is that the resulting potential should be single valued everywhere. A moment of thought shows that such a shift leaves the area under the curve invariant and hence $T(E)$ does not change. In other words, given any potential $V(x)$, there are an infinite number of other potentials for which we will get the same period–energy dependence $T(E)$; each of these potentials is determined by the form of the function $\Delta(V)$.

In the case of the harmonic oscillator potential, the distance $h(V)$ between the two turning points ('width') varies as $\sqrt{V}$ when the potential is measured from its minima. Since (4) has the isochronous property, we would suspect that it is obtained from the harmonic oscillator potential by a shearing motion keeping the width $h(V)$ varying as $(V - V_{\min})^{1/2}$. This is indeed true and can be demonstrated as follows. From (4), we can determine the inverse, double-valued function $x(V)$ through the equation

$$ax^4 + b - Vx^2 = 0. \tag{8}$$

If the roots of this equation are x_1^2 and x_2^2, we immedi-

ately have $x_1^2 + x_2^2 = V/a$ and $x_1^2 x_2^2 = b/a$. Elementary algebra now gives

$$h(V)^2 = (x_1 - x_2)^2 = \frac{V}{a} - 2\sqrt{\frac{b}{a}}. \qquad (9)$$

Or, equivalently,

$$h(V) = \frac{1}{\sqrt{a}} (V - V_{\min})^{1/2}. \qquad (10)$$

This shows that the potential in (4) is indeed obtained by a shearing of the harmonic oscillator potential.

For those of you who do not like such a geometric argument, here is a more algebraic derivation of the same result [2]. Let us suppose that we are given the function $T(E)$ and are asked to determine the potential $V(x)$ which is assumed to have a single minimum and a shape roughly like the one in *Figure* 1. We can always arrange the coordinates such that the minimum of the potential lies at the origin of the coordinate system. The shape of the curve in the regions $x > 0$ and $x < 0$ will, of course, be different. In order to maintain single valuedness of the inverse function $x(V)$, we will denote the function as $x_1(V)$ in the region $x < 0$ and $x_2(V)$ in the region $x > 0$. Once this is done, we can replace $\mathrm{d}x$ in the integral in (2) by $(\mathrm{d}x/\mathrm{d}V)\mathrm{d}V$. This allows us to write

$$T(E) = \sqrt{2m} \int_0^E \left[\frac{\mathrm{d}x_2}{\mathrm{d}V} - \frac{\mathrm{d}x_1}{\mathrm{d}V}\right] \frac{\mathrm{d}V}{\sqrt{E - V}}. \qquad (11)$$

This is an integral equation which, fortunately, can be inverted by a standard trick. We divide both sides of the equation by $(z - E)^{1/2}$, where z is a parameter and integrate with respect to E from 0 to z. This gives

$$\int_0^z \frac{T(E)\,\mathrm{d}E}{\sqrt{z - E}} =$$

$$\sqrt{2m} \int_0^z \int_0^E \left[\frac{\mathrm{d}x_2}{dV} - \frac{\mathrm{d}x_1}{dV}\right] \frac{\mathrm{d}V\, dE}{\sqrt{[(z - E)(E - V)]}}. \qquad (12)$$

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The family of curves which has the same width will give rise to the same $T(E)$ and vice versa.

In the right hand side we can change the order of integration and use the fact that the integral over E just gives π. The rest is trivial and we obtain

$$\int_0^z \frac{T(E)\,\mathrm{d}E}{\sqrt{z-E}} = \pi\sqrt{2m}\,[x_2(z) - x_1(z)]\,. \qquad (13)$$

This is an implicit equation valid for any z. Calling the variable z as V, gives the functional form of $x_2(V) - x_1(V)$. We get the final result

$$x_2(V) - x_1(V) = \frac{1}{\pi\sqrt{2m}} \int_0^V \frac{T(E)\,dE}{\sqrt{V-E}}. \qquad (14)$$

This result shows explicitly that the function $T(E)$ can only determine for us the 'width' of the curve $x_2(V) - x_1(V)$. The family of curves which has the same width will give rise to the same $T(E)$ and vice versa. The shearing motion by which we transform one potential to another preserves this width and hence the functional form of $T(E)$.

One can also obtain some interesting relations in quantum mechanics for the corresponding systems. In quantum theory, the potentials like the one in *Figure* 1 will have a set of discrete energy levels E_n. Formally inverting the function $E(n)$ – which is originally defined only for integral values of n – one can obtain the inverse function $n(E)$ for this system. This function essentially plays the role analogous to $T(E)$ in the case of quantum theory. We can now ask whether one can determine the potential $V(x)$ given the energy levels E_n or, equivalently, the function $n(E)$. It turns out that one can do this fairly easily in the semi-classical limit corresponding to large n. To see this, recall that the energy E_n of the n-th level of a quantum mechanical system is given by the Bohr quantization condition

$$n(E) \simeq \frac{1}{\hbar}\int_{x_1}^{x_2} p\mathrm{d}x = \sqrt{\frac{2m}{\hbar^2}} \int_{x_1}^{x_2} \sqrt{E-V}\;\mathrm{d}x. \qquad (15)$$

Treating x as a function of V, we can transform this relation to give

$$\begin{aligned} n(E) &\simeq \sqrt{\frac{2m}{\hbar^2}} \int_{x_1}^{x_2} \sqrt{E-V}\, \mathrm{d}x \\ &= \sqrt{\frac{2m}{\hbar^2}} \int^{E} \sqrt{E-V}\, \frac{\mathrm{d}x}{\mathrm{d}V}\, \mathrm{d}V \\ &= \sqrt{\frac{m}{2\hbar^2}} \int^{E} (E-V)^{-1/2}\, x(V)\, \mathrm{d}V. \quad (16) \end{aligned}$$

Here we have done an integration by parts and have treated the integral as a function of the upper limit. This integral equation can again be solved by exactly the same trick which we used in the case of (11). This will lead to the result

$$x(V) = \sqrt{\frac{2\hbar^2}{m}} \int^{n(V)} \frac{\mathrm{d}n}{\sqrt{V-E(n)}}. \quad (17)$$

which determines the form of the potential $V(x)$ – in terms of the inverse function $x(V)$ – such that in the semi-classical limit it will have the energy levels given by the function $E(n)$.

Even though we worked it out for a one-dimensional motion with a Cartesian x-axis, it is obvious that the same formula should be applicable for energy levels in a spherically symmetric potential $V(r)$ provided we only consider the zero angular momentum quantum states. As a curiosity, consider the potential which will reproduce the energy levels (which we know is the one arising in the case of the Coulomb problem) given by

$$E_n = -\frac{me^4Z^2}{2\hbar^2 n^2}. \quad (18)$$

This gives $n(V) = (-2\hbar^2 V/me^4Z^2)^{-1/2}$ so that an elementary integration using (17) with a suitable choice for

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the constant of integration gives

$$r = \sqrt{\frac{2\hbar^2}{m}} \int^{n(V)} \left[V + \left(\frac{me^4 Z^2}{2\hbar^2 n^2} \right) \right]^{-1/2} \mathrm{d}n = -\frac{Ze^2}{V} \tag{19}$$

thereby leading to $V(r) = -Ze^2/r$ which, of course, we know is exact. (This is one of the many curiosities in the Coulomb problem which we will turn to in a future installment.)

We will now describe another interesting feature in quantum theory related to isochronous potentials. It is well known that when we move from classical to quantum mechanics, the harmonic oscillator potential leads to equidistant energy levels. Curiously enough, all the isochronous potentials have this property *in the semi-classical limit*. This is most easily seen by differentiating (15) with respect to E and using (2) so as to obtain

$$\frac{\mathrm{d}n}{\mathrm{d}E} = \sqrt{\frac{m}{2\hbar^2}} \int_{x_1}^{x_2} \frac{\mathrm{d}x}{\sqrt{E - V}} = \frac{T(E)}{\hbar}. \tag{20}$$

In other words, the quantum numbers are given by the equivalent formula

$$n(E) \simeq \frac{1}{\hbar} \int T(E) \mathrm{d}E \tag{21}$$

In the case of the potential in equation (4), something more surprising happens: The exact solution to the Schrödinger equation itself has equally spaced energy levels!

which nicely complements the first equation in (15). If the potential is isochronous, then $T(E) = T_0$ is a constant independent of E and the integral immediately gives the linear relation between E and n of the form $E = \alpha n + \beta$, where $\alpha = (\hbar/T_0)$. Clearly, these energy levels are equally spaced just as in the case of harmonic oscillators.

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indicate briefly how this analysis proceeds leaving the details for you to work out. (You cannot reach this conclusion by the two-dimensional trick used earlier in classical physics.) To begin with, we can redefine the potential to the form

$$V(x) = \left[Ax - \frac{B}{x}\right]^2; \qquad A^2 \equiv a,\ B^2 \equiv b \tag{22}$$

by adding a constant so that the minimum value of the potential is zero at $x = (B/A)^{1/2}$. The frequency of oscillations in this potential is $\omega_0 = (8a/m)^{1/2}$. To study the Schrödinger equation for the potential in (22), it is convenient to introduce the usual dimensionless variables $\xi = (m\omega_0/\hbar)^{1/2}\, x$, $\epsilon = 2E/(\hbar\omega_0)$ and $\beta = B(2m)^{1/2}/\hbar$, in terms of which the Schrödinger equation takes the form:

$$\psi'' + \left[\epsilon - \left(\frac{1}{2}\xi - \frac{\beta}{\xi}\right)^2\right]\psi = 0. \tag{23}$$

As $\xi \to \infty$, the β/ξ term becomes negligible and – as in the case of the standard harmonic oscillator – the wavefunctions will die as $\exp[-(1/4)\xi^2]$. Near the origin, the Schrödinger equation can be approximated as $\xi^2\psi'' \approx \beta^2\psi$ which has solutions of the form $\psi \propto \xi^s$ with s being the positive root of $s(s-1) = \beta^2$. We now follow the standard procedure and write the wavefunction in the form $\psi = \phi(\xi)[\xi^s \exp(-(1/4)\xi^2)]$ and look for a power-law expansion for ϕ of the form

$$\phi(\xi) = \sum_{n=0}^{\infty} c_n \xi^n. \tag{24}$$

Substituting this form into the Schrödinger equation will lead, after some algebra, to the recurrence relation

$$\frac{c_{n+2}}{c_n} = \frac{n + s - \epsilon - \beta + (1/2)}{(n+2)(n+2s+1)}. \tag{25}$$

Do all isochronous potentials lead to evenly spaced energy levels as exact solutions to Schrödinger equation rather than only in the asymptotic limit?

Asymptotically, this will lead to the behaviour $c_{n+2}/c_n \simeq (1/n)$ so that $\phi(\xi) \simeq \exp[(1/2)\xi^2]$ making ψ diverge unless the series terminates. So, ϵ must be so chosen that the numerator of (25) vanishes for some value of n. Clearly, only even powers of ξ appear in $\phi(\xi)$ allowing us to write $n = 2k$, where k is an integer. Putting everything back, the energy of the k-th level can be written in the form

$$E_k = (k + C)\hbar\omega_0; \quad C = \frac{1}{2}\left[1 - \beta + \left(\beta^2 + \frac{1}{4}\right)^{1/2}\right] \tag{26}$$

showing that the energy levels are equally spaced with the width $\hbar\omega_0$ but with C replacing $(1/2)$ in the case of the harmonic oscillator. You can convince yourself that all the limiting behaviour is correctly reproduced.

Do all isochronous potentials lead to evenly spaced energy levels as exact solutions to Schrödinger equation rather than only in the asymptotic limit? The answer is "no". The simple counter-example is provided by two parabolic wells connected together smoothly at the minima with $V(x) = (1/2)m\omega_R^2 x^2$ for $x \geq 0$ and $V(x) = (1/2)m\omega_L^2 x^2$ for $x \leq 0$. It is obvious that this potential is isochronous classically. Solving the Schrödinger equation requires a bit of effort because you need to ensure continuity of ψ and ψ' at the origin. This leads to a set of energy levels which need to be solved for numerically. One finds that the energy levels are not equally spaced but the departure from even spacing is surprisingly small. To the extent I know, there is no simple characterization of potentials which lead to evenly spaced energy levels in quantum theory.

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Suggested Reading

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