Stellar dynamics and Chandra

T. Padmanabhan

Inter-University Centre for Astronomy and Astrophysics, Post Bag 4, Ganeshkhind, Pune 411 007, India

Some contributions of Chandrasekhar to stellar dynamics are reviewed. The article concentrates on two particular effects which arise due to stochastic gravitational fluctuations in stellar systems, viz. (i) diffusion in velocity space, and (ii) dynamical friction. The key ideas are discussed and a brief historical account of the derivation of the formula for dynamical friction is given.

1. Introduction

ASTRONOMERS often have to deal with systems containing a large number of objects which interact through Newtonian gravity. Such systems include star clusters, galaxies and clusters of galaxies. The brute-force approach for studying such a system is by integrating the equations of motion for each of the constituents numerically. Such a method, of course, will not be the preferred course of action except in special circumstances. Ideally, one would like to develop a formalism, analogous to the statistical mechanics of the normal gaseous systems, to understand the gross features of a gravitating system. The development of such a theory, however, is far from straightforward since the gravitational force has a long range. If a gaseous system of total energy E is divided into two parts, each with energies E_1 and E_2, then – to a high degree of accuracy – $E = E_1 + E_2$. Such 'extensivity of energy' does not hold for systems interacting via gravitational forces thereby creating several difficulties in using standard notions from statistical mechanics. What is more, the relaxation timescale for gravitating systems is very large, implying that one often has to deal with quasi-equilibrium situations.

Given the above facts, it is not surprising that progress towards a complete statistical theory of gravitating systems has been rather slow. The nature of the problem demands physical insight coupled with mathematical sophistication. Chandra, in his characteristic style, has attacked several issues arising in the study of such systems. A summary of some of his earlier investigations in this subject can be found in his treatise *Principles of Stellar Dynamics*[1] published in 1942. It is often said that Chandra had the habit of making a detailed study of a subject, writing a definitive treatise on it and moving on to a different area. Stellar dynamics, possibly, is one field where he did *not* do this; in fact some of the major contributions to this subject from Chandra himself came *after* the publication of the monograph mentioned above. From this point of view, stellar dynamics is rather unique compared to other areas in which Chandra has worked during his impressive career.

In this article I shall highlight some specific problems in stellar dynamics which Chandra had addressed and will try to present them within a wider historical context. Section 2 discusses the collisional time scale involved in gravitating systems and section 3 deals with the issue of dynamical friction. The last section summarizes the conclusions.

2. Timescale for gravitational collisions

The first conceptual problem which needs to be addressed in developing the Statistical Mechanics of Gravitating Systems (SMGS, for short) revolves around the definition of 'gravitational collision'. In a gaseous system interacting via molecular forces, it is easy to define the concept of a collision. If the gas is sufficiently dilute, each molecule moves with constant velocity for most of the time; only when two molecules come within the range of their interaction, are they deflected significantly from their original trajectories by a 'collision'. Thereafter, both the molecules continue with uniform velocity until the occurrence of the next collision etc. The mean time between collisions in such a system is large compared to the time during which the collision takes place. Such a division of the trajectory of the molecule – into 'straight line motion between collisions' and 'hard collisions with significant deflection' – is possible because the molecular force is left only when two molecules are sufficiently close together. In the case of a gravitating system like, say, a globular cluster, each star feels the gravitational force of all other stars all the time. The concept of a collision in such a case needs to be defined differently in order to be of any use.

The usual procedure is as follows. The gravitational potential at any given point in a stellar system can be approximated, to the lowest order of accuracy, as that due to a smooth density distribution of stars. This potential will lead to a class of systematic orbits for the stars. The actual potential, at any point, will differ from the smooth potential due to the granularity of the system. Some amount of randomness in the motion of a star is introduced due to the difference between the actual gravitational force acting on any one star and the mean gravitational force calculated from a smoothed-out distribution of mass in the system. This difference can significantly affect the distribution of stars over some timescale, say, t_{gc}. Since there is an inherent stochasticity in this process, it is reasonable to consider the above effect as being analogous to molecular collisions. We may, therefore, think of t_{gc} as the timescale for gravitational collisions. The first issue in SMGS is to estimate this timescale t_{gc}.

Consider a gravitational encounter between two stars, each of mass m and relative velocity v with the impact parameter b. The typical transverse velocity induced by such an encounter is $\delta v_\perp \simeq (Gm/b^2)(b/v) = (Gm/bv)$. For most of the collisions $(\delta v_\perp/v)$ will be small compared to unity. But the cumulative effect of large number of such encounters is to make the star perform a random walk in the velocity space. The net mean-square-velocity induced by collisions with impact parameters in the range $(b, b+db)$ in a time interval Δt is

$$\langle(\delta v)_\perp^2\rangle = (2\pi b db)(v\Delta t)b\,(Gm/bv)^2. \tag{1}$$

The total mean-square transverse velocity due to all stars is found by integrating this expression over b within some range (b_1, b_2):

$$\langle(\delta v)_\perp^2\rangle_{\rm total} \simeq \Delta t \int_{b_1}^{b_2} (2\pi b db)(vn)(G^2m^2/b^2v^2)$$

$$= \frac{2\pi n G^2 m^2}{v}\,\Delta t\,\ln(b_2/b_1). \tag{2}$$

It is reasonable to take $b_2 \simeq R$, the size of the system; As regards b_1, notice that the concept of slow diffusion in the velocity space fails when $(\delta v_\perp/v) \cong 1$; that is when the impact parameter is less than (Gm/v^2). So we may take $b_1 \simeq b_c \simeq (Gm/v^2)$. Then $(b_2/b_1) \simeq (Rv^2/Gm) = N(Rv^2/GM) \simeq N$ for a system obeying the virial theorem. (Here $M = Nm$ is the total mass of the system and N is the total number of stars.) This effect is important over timescales at which $\langle(\delta v)_\perp^2\rangle_{\rm total} \simeq v^2$, giving the timescale for gravitational 'collision' to be:

$$t_{gc} \simeq \frac{v^3}{2\pi G^2 m^2 n\,\ln N} \simeq \left(\frac{N}{\ln N}\right)\left(\frac{R}{v}\right). \tag{3}$$

This is the timescale over which the cumulative effect of distant stellar encounters will significantly affect the distribution of stars. Since (R/v) is the typical dynamical timescale of the system, we see that collisional timescale is larger by a factor $(N/\ln N)$. For galaxies and clusters of galaxies, t_{gc} is larger than the age of the universe making this effect somewhat irrelevant. But in the cores of globular clusters t_{gc} can be less than the age of the universe and this effect will have significant dynamical consequences.

The result obtained above reflects the stochastic nature of the actual gravitational force acting on any one given star in a stellar system. Over a timescale $t \cong t_{gc}$, the fluctuations in the gravitational force (around the mean value contributed by a smooth density distribution of stars) will make the stellar orbits very different from what one would have computed from the mean force. Quite understandably, Chandra stresses this result in his book and applies it to different astrophysical scenarios.

Considering the importance of this result, it is worthwhile to put it in historical perspective. At least two other authors have discussed this phenomenon before Chandrasekhar. The formula given above occurs in section 287 of the treatise *Astronomy and Cosmogony* by James Jeans, published in 1929 [see equation (287.5) of ref. 2]. The derivation given by Jeans is substantially the same as the one given here, except for the fact that Jeans takes the lower cut-off to correspond to the mean distance between stars. The Russian astrophysicist, Ambartsumian, also arrived at similar conclusions independently and used the expression for time of relaxation to study the dynamics of open clusters[3] as early as 1938. In particular, he expresses the logarithmic term in the form $\ln N$ which neither Jeans nor Chandra does explicitly.

Chandra cites the earlier work by Jeans and says that 'though the physical ideas were correctly formulated by Jeans and Schwarzschild*, a completely rigorous evaluation of the time of relaxation was not available until recently'. It is interesting to note that Chandra's 'rigorous' evaluation spans more than 20 pages in his book (compared to 4 pages in the derivation by Jeans) and uses a 'three-dimensional figure' (page 52 of ref. 1) which many readers have found intriguing, to say the least. Chandra's derivation of t_{gc} is characteristic of his approach to astrophysical problems – steeped in mathematical rigour and complete to the extent possible even when the physical situation may not require such a mathematical *tour de force.*

3. The case of the dynamical friction

While the estimate of timescale given above is correct from a practical point of view, there is a serious conceptual issue which has not been addressed in the above derivation by Chandra, when he first completed it. If the above analysis is really true, then the velocity dispersion of the stellar system will increase without bound in the course of time! This conclusion arises essentially from the fact that any random walk in velocity space will lead to a mean square velocity increasing linearly with time. The absurdity of this result shows that we are missing some essential aspect of the physics in the above analysis.

There exists another effect, now called 'dynamical

*The investigation by Schwarzschild was published posthumously in 1924; it appears the work was done prior to 1917.

friction' which saves the day. When the velocity of any single star is significantly larger than that of the surrounding stars, the medium of stars exerts a retarding force to decrease the velocity of the fast-moving star. The cumulative effect of the two processes – diffusion in velocity space and dynamical friction – leads to a Maxwellian distribution for stellar velocities. For a Maxwellian distribution, the two effects cancel each other precisely.

The fact that one requires the existence of dynamical friction, to avoid physical absurdities, must have been realized by Chandra shortly after the publication of *Principles of Stellar Dynamics* in 1942. The first edition of this book does *not* mention dynamical friction. However, the *Reviews of Modern Physics* article by Chandra[4] as well as his article 'New methods in stellar dynamics'[5] which was awarded the AC Morrison prize in 1942, discuss this phenomenon. In the later article, Chandra specifically mentions the fact that unbounded diffusion in velocity space can only be corrected by introducing a dynamical friction term and provides a derivation of the coefficient of dynamical friction by studying the two-body encounters. In contrast, the *Reviews of Modern Physics* article and the paper by Chandrasekhar and Von Neuman[6] provide a statistical argument to justify the phenomenon of dynamical friction. There are, of course, several later papers by Chandra and collaborators, elaborating on this phenomenon.

To the extent I know, this is probably the only example in Chandra's career in which he published a treatise which is incomplete and misses an essential piece of the physics. It, however, goes to his credit that he realized the need for dynamical friction and went on to develop the theory of stochastic processes as applied to astrophysical phenomena during 1942 to 1945.

The story of dynamical friction, however, has another interesting twist to it.

A careful reading of the derivation of dynamical friction by Chandra and co-workers makes one feel that there is something basically unsatisfactory about this approach to dynamical friction. Since the diffusion in velocity space and dynamical friction are just different manifestations of the same stochastic process, there must exist a direct and elegant derivation of both together. Such a derivation should also show that Maxwellian distribution of velocities will lead to a precise cancellation of the two effects, on the average. It is, indeed, possible to provide such a derivation along the following lines.

Let us consider the time evolution of the distribution function $f(\mathbf{x}, \mathbf{v}, t)$ describing the stars in a stellar system. The gravitational force acting on any star in a self-gravitating system can be divided into two parts, $\mathbf{f}_{\mathrm{smooth}} + \mathbf{f}_{\mathrm{fluc}}$: The $\mathbf{f}_{\mathrm{smooth}}$ is due to the gravitational potential arising from the *smooth* distribution of matter. Since the matter is made of individual particles, there will be a deviation from the smooth force $\mathbf{f}_{\mathrm{smooth}}$ and this deviation is denoted by the fluctuating part of the force $\mathbf{f}_{\mathrm{fluc}}$. It is the latter part that produces a slow diffusion of particles in the velocity space. We need to derive the equation satisfied by the distribution function $f(\mathbf{x}, \mathbf{p}, t)$ describing the system, taking into account this slow diffusion in the momentum space. In general, such a diffusion process can be described by an equation of the kind

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} - \nabla\phi \cdot \frac{\partial f}{\partial \mathbf{v}} = - \frac{\partial J^{\alpha}}{\partial p^{\alpha}} . \quad (4)$$

The right side is the divergence of a particle current J^{α} in the momentum space which is characteristic of a diffusive process arising from $\mathbf{f}_{\mathrm{fluc}}$; the smooth gravitational force $\mathbf{f}_{\mathrm{smooth}}$ appears in the left hand side of the equation as the $\nabla\phi$ term. By considering the transfer of momentum in individual collisions, it is possible to show that[8] the current J_{α} can be expressed in the form

$$J_{\alpha} = \int \mathrm{d}l' B_{\alpha\beta}(\mathbf{l},\mathbf{l}') \left(f \frac{\partial f'}{\partial l'_{\beta}} - f' \frac{\partial f}{\partial l_{\beta}} \right) \quad (5)$$

where

$$B_{\alpha\beta} = \frac{1}{2} \frac{B_0}{|\mathbf{k}|} \left(\frac{\delta_{\alpha\beta} - k_{\alpha}k_{\beta}}{k^2} \right); \qquad \mathbf{k} = \mathbf{l} - \mathbf{l}' \quad (6)$$

and $B_0 = 4\pi G^2 m^5 \ln(b_{\max}/b_{\min})$. The logarithmic term arises due to the long-range nature of the interaction and has been estimated earlier to be:

$$\ln\left(\frac{b_{\max}}{b_{\min}}\right) \simeq \ln N . \quad (7)$$

This expression for J_{α} provides the complete description of the evolution of the system driven by soft collisions. It follows that this expression *must* describe both the diffusion and the dynamical friction; and indeed it does. By straightforward algebraic manipulation, one can transform J_{α} into the form[8]

$$J_{\alpha}(\mathbf{l}) = a_{\alpha}(\mathbf{l}) f(\mathbf{l}) - \frac{1}{2} \frac{\partial}{\partial l_{\beta}} (\sigma^2_{\alpha\beta} f) , \quad (8)$$

where $a_{\alpha}(\mathbf{l}) = B_0(\partial\eta/\partial l_{\alpha})$, $\sigma^2_{\alpha\beta} = B_0(\partial^2\psi/\partial l_{\alpha}\partial l_{\beta})$ and the potentials η and ψ satisfy the equations

$$\nabla^2_l \eta(\mathbf{l}) = -8\pi f(\mathbf{l}); \qquad \nabla^2 \psi = \eta . \quad (9)$$

A comparison of equations (4) and (8) shows that the term linear in $f(\mathbf{l})$ describes dynamical friction and the derivative term describes diffusion in the velocity space. To see this more clearly, consider a simpler equation of the form

$$\frac{\partial f(v,t)}{\partial t} = \frac{\partial}{\partial v}\left\{(\alpha v)f + \frac{\sigma^2}{2}\frac{\partial f}{\partial v}\right\} \equiv -\frac{\partial J}{\partial v}, \quad (10)$$

where α and σ^2 are constants. (J in this equation is similar in structure to J_α in equation (8) if we confine our attention to one dimension and set $a = \alpha v$.) Let us consider the effect of the two terms. The second term $(\sigma^2/2)(\partial f/\partial v)$ has the standard form of a 'diffusion current' proportional to the gradient in the velocity space. As time goes on this term will cause the mean square velocities of particles to increase in proportion to t inducing the 'random walk' in the velocity space. Under the effect of this term, the system will have its $\langle v^2 \rangle$ increasing without bound. *This unphysical situation is avoided by the presence of the first term ($\alpha v f$) in* J. This term acts as a 'dynamical friction' term. The combined effect of the two terms is to drive f to a Maxwellian distribution with velocity distribution (σ^2/α). *In such a Maxwellian distribution the gain made in (Δv^2) due to diffusion is exactly balanced by the losses due to dynamical friction.* When two particles scatter, one gains the energy lost by the other; on the average, we may say that the one which has lost the energy has undergone 'dynamical friction' while the one which gained energy has achieved 'diffusion' to higher v^2. The cumulative effect of such phenomena is described by the two terms in $J(v)$.

The above points can be easily illustrated by solving equation (10). Suppose we take an initial distribution $f(v,0) = \delta(v - v_0)$ peaked at a velocity v_0. The solution of equation (10) with this initial condition is

$$f(v,t) = \left[\frac{\alpha}{\pi\sigma^2(1-e^{-2\alpha t})}\right]^{1/2} \exp\left[-\frac{\alpha(v - v_0 e^{-\alpha t})^2}{\sigma^2(1-e^{-2\alpha t})}\right], \quad (11)$$

which is a Gaussian with the mean

$$\langle v \rangle = v_0 e^{-\alpha t} \quad (12)$$

and dispersion

$$\langle v^2 \rangle - \langle v \rangle^2 = (\sigma^2/\alpha)(1 - e^{-2\alpha t}). \quad (13)$$

At late times ($t \to \infty$), the mean velocity $\langle v \rangle$ goes to zero while the velocity dispersion becomes (σ^2/α). Thus the equilibrium configuration is a Maxwellian distribution of velocities with this particular dispersion, for which $J = 0$. To see the effect of the two terms individually on initial distribution $f(v,0) = \delta(v - v_0)$, we can set α or σ to zero. When $\alpha = 0$, we get pure diffusion:

$$f_{\alpha=0}(v,t) = \left(\frac{1}{2\pi\sigma^2 t}\right)^{1/2} \exp\left\{-\frac{(v-v_0)^2}{2\sigma^2 t}\right\}. \quad (14)$$

Nothing happens to the steady velocity v_0; but the velocity dispersion increases in proportion to t representing a random walk in the velocity space. On the other hand, if we set $\sigma = 0$, then we get

$$f_{\sigma=0}(v,t) = \delta(v - v_0 e^{-\alpha t}). \quad (15)$$

Now there is no spreading in velocity space (no diffusion); instead the friction steadily decreases $\langle v \rangle$. Note that the timescale for the operation of the dynamical friction is $\alpha^{-1} = (a/v)^{-1}$.

The behaviour of the system driven by J_α in equation (8) will be similar except for the mathematical complexity arising from the fact that a_α and $\sigma^2_{\alpha\beta}$ are now functionals of f. We can, however, use equation (8) to compute the coefficient of dynamical friction if f is known. For a Maxwellian distribution of velocities

$$f(v) = A e^{-v^2/q^2} \quad (16)$$

we get,

$$a(v) \cong -\frac{32\sqrt{\pi}}{3}\, n \cdot (Gm)^2\, L\, \frac{v}{q^3}\left(1 - \frac{3}{5}\frac{v^2}{q^2} + \ldots\right) \quad (17)$$

for small (v/q). Since $a \propto v$ for small velocities, we can estimate the timescale for dynamical friction based on the previous analysis. We find

$$t_{\rm df} \equiv [\alpha(v)]^{-1} \equiv -\frac{v}{a(v)} \cong \frac{3\sqrt{\pi}}{16} \cdot \left(\frac{N}{\ln N}\right) \cdot \left(\frac{R}{q}\right). \quad (18)$$

As to be expected, this timescale is of the same order as $t_{\rm gc}$.

The elegant structure of the equation (5) also allows us to see immediately that J_α vanishes for the Maxwellian distribution function. For any Maxwellian distribution, with $f(\mathbf{l}) \propto \exp(-\mu l^2)$, we have

$$f\frac{\partial f'}{\partial l'_\beta} - f'\frac{\partial f}{\partial l_\beta} = f(\mathbf{l})(-2\mu l'_\beta) f(\mathbf{l}') - f(\mathbf{l}')(-2\mu l_\beta) f(\mathbf{l})$$

$$= 2\mu f(\mathbf{l}) f(\mathbf{l}')(\mathbf{l} - \mathbf{l}')_\beta = 2\mu f(\mathbf{l}) f(\mathbf{l}')\, k_\beta. \quad (19)$$

Hence J_α vanishes due to the relation $k_\beta B^{\alpha\beta} = 0$.

Incredibly enough, the expression in (5) (containing implicitly, as it does, both diffusion and dynamical friction) was due to L. D. Landau[9] and was published in 1936 – nearly 6 years before the work of Chandra. Landau was dealing with collisions in plasmas but it is trivial to translate the result to the case of gravitational encounter. The original derivation of Landau is now available in pages 168–172 of *Physical Kinetics*[7]. To the extent I can ascertain, not only Chandra but also many later workers have missed the characteristic elegance in Landau's derivation. For example, the much cited paper by Rosenbuth *et al.*[10], which attempts to derive the Fokker–Planck equation for an inverse square force starting from two-body encounters, does not bother to cite Landau's work. They do cite Chandrasekhar's earlier work and also state that Cohen *et al.*[11] has a more 'complete list of references'. The paper by Cohen *et al.* cites Chandrasekhar *and* Landau but goes on to comment that 'a similar but incomplete approach was made somewhat earlier by Landau .... In this reference the important terms representing dynamical friction which should appear in the diffusion equation are set equal to zero as a result of certain approximations'. Assuming that the English version of Landau's derivation, given in *Physical Kinetics* (which specifically cites Landau 1936), is a faithful reproduction of Landau's original derivation, one must conclude that Cohen *et al.* and later on Rosenbuth *et al.* completely misunderstood Landau's work. The result quoted in equation (5) remains to date the most elegant and complete description of the stochastic process we are discussing and was published six years before Chandra's work appeared.

The concept of dynamical friction has been used extensively in later years in the study of many astrophysical problems. In particular, the evolution of cores of globular clusters and the slow inward spiralling of galaxies in clusters are dictated, to a large extent, by the phenomenon of dynamical friction. Such studies, in turn, raise interesting conceptual issues as regards the final stage of a gravitating system evolving collisionally. By and large, one would try to answer such a question by locating configurations which maximize a suitably defined entropy. It is, however, easy to see that there is no global maxima for the entropy of a self-gravitating system made of point particles. What is more intriguing is that such systems exhibit a critical behaviour even as regards the existence of local maxima for entropy. There have been several studies in the literature (see ref. 8 for a review) dealing with this question and it goes to the credit of Chandra that he popularized the language of statistical mechanics and stochastic processes among the astronomical community.

Another important offshoot of the work by Chandra was the realization by astronomers that the timescale for gravitational collisions is enormously large for galactic systems. As a consequence, the Maxwellian distribution of velocities – seen in a class of galactic systems – could not have possibly arisen due to collisional relaxation. This realization motivated Lynden-Bell in 1967 to suggest a collisionless relaxation process as the primary mechanism in gravitating system[12]. This process, now known as 'violent relaxation', arises due to the mixing of distribution function in phase space and is quite different from the collisional processes considered by Chandra and others. With the rising popularity of collisionless dark mark halos around galaxies, violent relaxation has assumed increasing importance in the study of structure formation.

4. Conclusions

The statistical mechanics of gravitating systems remains as an incomplete formalism even today and lacks a systematic approach at the same level as, say, the study of radiative transport. Chandra's contribution should be judged keeping in mind the above fact. There is no doubt that Chandra made this subject popular among the astrophysicists and drew the attention of the community to the use of stochastic methods in stellar dynamics and astrophysics. The two major physical processes in stellar dynamics – the diffusion in velocity space and dynamical friction – were both attacked by Chandra in his characteristic style and he obtained essentially the correct results. What was probably uncharacteristic of Chandra's contribution in this subject are two facts: firstly, he seems to have overlooked the need for dynamical friction when he produced his monograph on stellar dynamics. Secondly, even when he accounted for dynamical friction in his later work, he did not do it in the most elegant and comprehensive style, which is what we usually expect from Chandra.

1. Chandrasekhar, S., *Principles of Stellar Dynamics*, University of Chicago Press, USA, 1942.
2. James H. Jeans, *Astronomy and Cosmogony*, Cambridge University Press, UK, 1929.
3. Ambartsumian, V. A., Uch. Zap. L.G.V. No. 22, p. 19; English translation in *Dynamics of Star Clusters* (eds Goodman, J. and Hut, P.), D. Reidel Publishing Company, Holland, IAU Symposium No. 113, 1985, pp. 521.
4. Chandrasekhar, S., *Rev. Mod. Phys.*, 1943, **21**, 383.
5. Chandrasekhar, S., *Ann. N. Y. Acad. Sci.*, 1943, **45**, 131–162.
6. Chandrasekhar, S. and Von Neumann, J., *Astrophys. J.*, 1942, **95**, 489.
7. Lifshitz, E. M. and Pitaeveskii, L. P., *Physical Kinetics*, Pergamon, UK, 1981.
8. Padmanabhan, T., *Phys. Rep.*, 1990, **188**, 285.
9. Landau, L., *Physik. Zeits. Sowjetunion*, 1936, **10**, 154.
10. Rosenbluth, M. N., MacDonald, W. M. and Judd, D. L., *Phys. Rev.*, 1957, **107**, 1.
11. Cohen, R. S. *et al.*, *Phys. Rev.*, 1950, **80**, 230.
12. Lynden-Bell, D., *Monthly Not. R. Astron. Soc.*, 1967, **136**, 101.

ACKNOWLEDGEMENT. I thank R. Nityananda for sharing with me his knowledge of the history of this subject which has helped me tremendously while preparing this article.