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Mach's principle and the notion of time

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Abstract. The role of time coordinate in the realization of Mach's principle is highlighted. It is shown that Mach's principle is linked to the definition of a 'particle'. These results suggest a deep connection between quantum gravity and Mach's principle.

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In classical Newtonian mechanics, the "frame of fixed stars" (S) is of crucial importantce. This is the frame of reference in which distant matter in the universe (stars, galaxies, or some such predecided set of objects; here they will be referred to as "stars") is at rest. Newton's laws are usually stated in such a frame. A particle shielded from all external influences will follow an unaccelerated trajectory in this frame. This result can be stated as follows: Let $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)\cdots\mathbf{x}_N(t)$ be the position vectors of N stars and let $\mathbf{x}(t)$ be the position of a test particle shielded from external influences. In a reference frame in which $\mathbf{x}_i(t)=\mathbf{x}_i(0)$ ['distant stars are fixed'], $\mathbf{x}(t)$ satisfies the equation

$$\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2}=0. \tag{1}$$

Thus, by connecting up the local behaviour of test particles with the state of motion of distant matter, we have brought in (a particular version of) Mach's principle (Mach 1912). In contrast, consider another frame S' in which the distant stars are *not* at rest but moves according to the law

$$\mathbf{x}_i'(t)=\frac{1}{2}\mathbf{g}t^2. \tag{2}$$

In this frame—in which distant stars are not fixed—we cannot use (1). However, it is easy to make a coordinate transformation which will bring these stars to rest; we can, then, use (1) in such a frame. Transforming back we can find the equation of motion for the free test particle in our original frame S'. By this procedure we will find that $\mathbf{x}'$ satisfies the equation:

$$\frac{\mathrm{d}^2\mathbf{x}'}{\mathrm{d}t^2}=-\mathbf{g}. \tag{3}$$

It is usual to call S as 'inertial frame' and S' as 'non-inertial frame'; and the acceleration g [in (3)], experienced by a test particle in S', as due to a 'pseudo-force'. In its true form, Mach's principle does not distinguish between coherent motion of all the distant matter in the universe and a local transformation to a non-inertial frame.

All this is well-known. But the Mach's principle has another facet which is not often emphasized. To bring this up, consider again the frame in which $\mathbf{x}_i(t) = \mathbf{x}_i(0)$—i.e. distant stars are fixed. Suppose *all* test particles, shielded from external influences, follow in this frame, a trajectory $\mathbf{x}(t)$, such that:

$$\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2} = -\alpha(t)\dot{\mathbf{x}}. \tag{4}$$

What do we make of this? Since the distant stars are now fixed, we will think of this frame as inertial. We are thus forced to conclude that the particles are experiencing a velocity dependent drag force and is following an *accelerated* trajectory. This conclusion, however, would have been wrong. A transformation of the time coordinate from t to T such that

$$T = \int^t \mathrm{d}t \left[\exp - \int \alpha(t)\,\mathrm{d}t\right] \tag{5}$$

will bring the trajectory into that of familiar free particle!

$$\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}T^2} = 0. \tag{6}$$

We have here two coordinate systems for spacetime: $(\mathbf{x}, t)$ and $(\mathbf{x}, T)$ with t and T related by (5). Fixed stars stay fixed in both these frames but Newton's law picks up a pseudo-force in one of these frames.

In the usual discussions of Mach's principle, one never bothers about time transformations like the one in (5). This is because in Newtonian physics, there is a 'God given', 'absolute time' which 'flows uniformly'. Motion is described using this particular time coordinate, which we are not allowed to tamper with. Then—and, only then—the frame of fixed stars define for us a useful inertial frame. *If we have no information about the time coordinate used then we cannot exclude pseudo-forces even in a frame of fixed stars.*

This realization raises several interesting questions. It is almost a miracle that we are gifted with such a time coordinate in which "motion appears to be simple". Where does such a time coordinate stem from? From cosmological observations, we know that matter on the very large scale does not have coherent motion. In other words, a frame with fixed stars does exist. *But why is it that in this particular frame of fixed stars test particles obey equations (1) rather than an equation like (4)?* Everyday experience, of course, shows that (1) is correct rather than (4). What is not often realized is that this requires an explanation.

If the nature was governed by strictly Newtonian laws then there would have been no problem. As we said before, Newtonian physics permits arbitrary transformations of the space coordinates $\mathbf{x}$ but forbids transformations of time (except for scaling and translation, $t' = at + b$, under which (1) is invariant). By prescribing an absolute time in which (1) is valid and forbidding transformations of t, Newtonian physics has effectively by passed this question.

But the world is not Newtonian. It is quantum mechanical and it is general relativistic. In a universe obeying the laws of quantum mechanics and general relativity there is no place for a sacred time coordinate. *Neither is there the concept of a particle obeying a particular trajectory.* From such an exact world obeying the laws of relativistic quantum theory, we construct our approximate 'everyday world' by taking two limits. First we take the limit of weak gravitational field ($G \to 0$) and then we take the limit of classical physics ($\hbar \to 0$). In this limit, particles with definite trajectories exist; so does a reference frame in which the space coordinates of distant stars fixed. But what is incredible in the emergence of a time coordinate such that free particles obey (1) rather than (4). This breaks the invariance of the physical laws under the reparametrization of time coordinate, which is present at the higher levels of description of the theory. If the exact theory is fully invariant under arbitrary time transformations, then a special time coordinate can emerge only if it was put in by hand. In what follows we will describe how this 'miracle' happens and what it means.

Let us begin by reversing the process and proceed from (1) to more exact descriptions. A classical free particle obeying (1) has its quantum mechanical equivalent, described by the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla_x^2\psi. \tag{7}$$

Arbitrary transformations of the time coordinate—of the form $t \to t' = F(t)$—is still forbidden. The Schrödinger equation retains the above form only when the 'sacred time coordinate' is used. The transition from (7) to (1) is via the expectation value

$$\langle x\rangle = \int \psi^* \hat{x}\psi \, \mathrm{d}^3 x \tag{8}$$

and Ehrenfest's theorem. In fact, the evolution equation for the expectation value

$$\left(\frac{\partial\langle x\rangle}{\partial t}\right) = \left\langle \frac{i}{\hbar}[H, x]\right\rangle \tag{9}$$

remains valid (for a given Hamiltonian) only if the time coordinate satisfying (7) is used.

In the next stage—that of quantum field theory—the situation becomes trickier. Let us suppose we are working in flat spacetime and that our particle is described by a scalar field *operator* obeying the Klein-Gordon equation:

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2c^2}{\hbar^2}\right]\hat{\phi}(t, \boldsymbol{x}) = 0. \tag{10}$$

How do we make the transition from (10) to (7)? Note that (10) is an equation in Heisenberg picture for the *operator* $\hat{\phi}$ while (7) is an equation in Schrödinger picture for a c-number state connection ψ. To make the proper identification we have to first define the Fock basis corresponding to $\hat{\phi}$ in (10). Let $|0\rangle$ and $|1_k\rangle$ be the vacuum and one-particle states of the quantum field theory described by $\hat{\phi}(t, \boldsymbol{x})$. One way of making the necessary identification between (10) and (7) will be to use the transition element $\langle 0|\hat{\phi}|1_k\rangle$:

$$\langle 0|\hat{\phi}(x,t)|1_k\rangle = \frac{1}{\sqrt{2\omega_k}}\exp(i\omega_k t + i\boldsymbol{k}\cdot\boldsymbol{x}); \quad \hbar\omega_k = (k^2c^2 + m^2c^4)^{1/2}. \tag{11}$$

In the non-relativistic limit of $c \to \infty$, we should identify the expression $\langle 0|\hat{\phi}|1_k\rangle \exp(imc^2 t/\hbar)$ with the Schrödinger wave function $\psi(x,t)$ (see e.g. Roman 1968). This is easily seen from noting that $\langle 0|\hat{\phi}|1_k\rangle \exp(imc^2 t/\hbar)$ has the limiting form

$$\langle 0|\hat{\phi}|1_k\rangle \exp(imc^2 t/\hbar) \simeq \exp\left(i\boldsymbol{k}\cdot\boldsymbol{x} - \frac{ik^2}{2m\hbar}t\right) \tag{12}$$

which is just the free particle wave function with momentum $\hbar\boldsymbol{k}$. In other words, correct limiting form for free particle is obtained *only after we have defined the one-particle Fock state* $|1_k\rangle$. [It is also possible to reach the same conclusion by working in the Schrödinger picture of the field theory; but the analysis is simpler to understand in the Heisenberg picture]. As long as we work within the framework of special relativity and Lorentz transformations, the Fock basis is unique. Though time can be mixed with space in Lorentz transformations, inertial frames retain their identity.

This uniqueness is lost once we go beyond the realm of special relativity, and allow arbitary coordinate transformations. Consider, for example, the Rindler frame in which the line element has the form

$$ds^2 = \left(1 + \frac{gx}{c^2}\right)^2 c^2 d\tau^2 - dx^2 - dy^2 - dz^2. \tag{13}$$

As is well known the Fock basis defined using τ is not the same as the one defined using the inertial time t (Fulling 1973). The transition element constructed from the Rindler states has the following form:

$$_R\langle 0|\hat{\phi}(x,\tau)|1_k\rangle_R = \frac{(\sinh \omega\pi)^{1/2}}{2\pi^2}\exp(-i\omega\tau + ik_y y + ik_x z)K_{i\nu}(k(1+gx)) \tag{14}$$

where $K_{i\nu}(z)$ is the modified Bessel function. In the non-relativistic limit, $(c \to \infty)$ the function $\psi = {}_R\langle 0|\phi(x,\tau)|1_k\rangle_R \exp(+imc^2 t/\hbar)$ satisfies the Schrödinger equation for a uniformly accelerated particle:

$$i\hbar\frac{\partial\psi}{\partial\tau} = -\frac{\hbar^2}{2m}\nabla^2\psi + mgx\psi. \tag{15}$$

Therefore, in the corresponding Newtonian limit, these particles obey eq. (3)—with a pseudo-force! Suppose we had described our flat spacetime using x and τ. Then, in the appropriate non-relativistic limit all our test 'particles' [defined using the mode functions in (14)] will experience an acceleration.

In flat spacetime, of course, this difficulty is easy to cure: we could make a rule that it is the Minkowski frame, rather than (13) which should be used to define the particles. But this example illustrates how a particular time coordinate achieves preference in the non-relativistic limit. It is the definition of 'particle' which breaks the invariance under time transformations. *We choose the quantum vacuum state in such a way that "particles" in the classical limit experience no pseudo-force.*

Real universe, of course, is not flat. To define particles in the real universe, we have to quantise the fields in a given background. [In fact, if we stick to the usual principles of flat spacetime field theory and the inertial co-ordinates, then it is possible to define particle states in an axiomatic manner, without taking a non-relativistic limit. This, of course, is of no use to us because in order to study questions like Mach's principle, one necessarily needs to deal with curved spacetime and curvilinear co-ordinates. It is extremely difficult to extend the constructions based on axiomatic field theory to an arbitrary curved spacetime.] Since quantum state is a global concept—defined on a spacelike hypersurface cutting through the universe—we now have a chance to introduce 'distant matter' in the discussion. We must demand that: "one particle state should be defined in our universe in such a way that, in the non-relativistic limit, these particles must be unaccelerated with respect to fixed stars".

This version of the Mach's principle, viz. that all the test particles in the universe should respect an equation like (1) rather than (3), imposes some restrictions on the form of metric describing the large scale universe.

Suppose our universe in the large scale is described by a metric g_{ik}. Quantum field theory now needs to be done in this background metric. When we take the weak-field approximation of gravity the generic form of the line element can be taken as:

$$ds^2 = \left(1 + \frac{2u(\boldsymbol{x}, t)}{c^2}\right)c^2 dt^2 - dx^2 - dy^2 - dz^2. \tag{16}$$

Repeating an analysis similar to the one in (13), (14), (15) one can work out the Schrödinger equation which will result for particles defined using t in (16) as the time coordinate. This equation will be

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + mu(\boldsymbol{x}, t)\psi. \tag{17}$$

If Mach's principle—as stated before—has to be realized, then we would, minimally require the following condition: we would expect a coordinate system to exist in which $g_{00}(\boldsymbol{x}, t) = g_{00}(t)$ so that no pseudo-force ∇u arises in the classical limit.

More generally, we expect two conditions to be satisfied: (1) There should exist a system of coordinates in which $g_{00}(\boldsymbol{x}, t) = g_{00}(t)$ *and* (2). In this frame, in which $g_{00}(\boldsymbol{x}, t) = g_{00}(t)$, the distant stars should be unaccelerated, i.e. $\boldsymbol{x} = \text{constant}$ should be geodesics. When and only when—these two conditions are simultaneously satisfied—can the time coordinate t be used to define particles which will not experience any pseudo-force.

Our universe admits precisely such a coordinate system in the form of the FRW line element:

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]. \tag{18}$$

Particles defined using Robertson–Walker time coordinate will remain as "good particles". For comparison, consider FRW metric transformed into the "locally inertial" form (see e.g. Narlikar 1983):

$$ds^2 = \exp(\nu)\,dT^2 - \exp(\lambda)\,dR^2 - R^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{19}$$

where

$$\exp(-\lambda) = 1 - kr^2 - r^2\dot{a}^2;\ \exp(\nu) = \exp(\lambda)a^2\dot{a}^2(1 - kr^2);\ R = ra(t); \tag{20}$$

$$T = \int^r \frac{x\,dx}{1 - kx^2} + \int^t \frac{dy}{a(y)\dot{a}(y)} \tag{21}$$

The non-relativistic limit of a quantum field theory in these two coordinate systems are very different. Using (18) we get the Schrödinger equation

$$i\hbar\frac{\partial f}{\partial t'} = -\frac{\hbar^2}{2m}\nabla_x^2 f \tag{22}$$

where $t' = \int dt a^2(t)$ and $\psi = a^{-3/2}f$, whereas (21) leads to

$$i\hbar\frac{\partial \psi}{\partial T} = -\frac{\hbar^2}{2m}\nabla_R^2\psi - \frac{m}{2}\frac{\ddot{a}}{a}R^2\psi \tag{23}$$

which has a pseudo-force term. Thus. from the point of view expressed above, Machian particles should be defined using FRW time coordinate t rather than using T. (In practice, of course, any difference will be insignificantly small; but we are discussing a question of principle).

Lastly, quantum cosmology raises some more questions as regards Mach's principle. Any prescription to assign a "wavefunction for the universe" also contains information about the quantum state of the matter fields (for a review of quantum cosmology, see e.g. Hartle 1986; Padmanabhan 1989; Halliwell 1990). To incorporate Mach's principle, we have to choose the quantum state of the universe carefully. Firstly it should be peaked at those geometries in which the coordinate system with $g_{00} = 1$, $g_{0\alpha} = 0$ is allowed. Secondly, the vacuum state of the matter fields should be consistent with the one defined naturally by this coordinate system. It would be interesting to see how much of freedom Mach's principle still allows for the wave function of the universe.

In popular quantum cosmological models the quantum state of the matter fields is determined during the initial deSitter phase. The quantum state is taken to be the Bunch–Davies vacuum which is defined with respect to the FRW time coordinate t. The reason for such a choice is very different from the considerations described above. It is gratifying to see that this is precisely the choice which leads to Mach's principle in the correspondence principle limit. It may not be far fetched to imagine that the combination of gravity (relevant for describing pseudo-force) and quantum theory (with its inherently global nature) provides us with a useful version of Mach's principle.

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