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A definition for time in quantum cosmology

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Abstract. A definition for the intrinsic time co-ordinate is proposed, using the phase of the wave function of the universe. This definition generalizes the notion of time co-ordinate which arises in the semiclassical cosmology. It also leads to acceptable results for the evaluation of expectation values of physical variables.

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Among the ten field equations which describe the gravitational field in Einstein's theory, four are constraint equations related to the freedom in the choice of the co-ordinates on the spacetime manifold. In the attempts to quantise gravity, it is usual to impose these constraints on the physically allowed states. The hamiltonian constraint, $H = 0$, leads to the Wheeler-DeWitt equation of quantum gravity:

$$\left[-\frac{1}{2M}\nabla^2 + MV(Q^a) + H_{\text{matter}}(p_\phi, \phi, Q^a)\right]\Psi(Q^a, \phi) = 0 \tag{1}$$

where Q^a denotes the coordinates in the superspace representing the 3-metrics; ∇^2 is the superspace laplacian

$$\nabla^2 = \frac{1}{\sqrt{G}}\partial_a(\sqrt{G}G^{ab}\partial_b) \tag{2}$$

constructed out of the DeWitt metric $G_{ab}(Q^c)$ in the superspace; V is the 'potential' term in the ADM hamiltonian related to the 3-curvature; M stands for $(16\pi G)^{-1}$ and H_{matter} is the matter hamiltonian with some generic set of matter fields $\phi = \{\phi^A\}$ (DeWitt 1967; Wheeler 1968). To arrive at this form of equation from the classical ADM hamiltonian, it is necessary to make a particular choice of factor ordering. I have chosen this factor ordering in such a way that the equation is covariant under the co-ordinate transformations in the superspace. The 'scalar' nature of the laplacian ensures this property. The notation, of course, is highly condensed; Latin – and later on, Greek – indices will be used to denote co-ordinates in the infinite dimensional superspace. I will obey the traditions of this subject by proceeding with formal manipulations of the above quantities without worrying about mathematical rigor.

The above equation has been studies in detail by many people in the past. These studies reveal a major difficulty in the interpretation of Ψ because eq. (1) does not have a $(i\partial/\partial t)$ term; (there is an extensive amount of literature dealing with this subject; detailed references can be found e.g. in Hartle 1988a, b; Halliwell 1990; Padmanabhan 1989). If quantum cosmology is based on this equation then it is not obvious as to how a notion of time emerges from the solution $\Psi(Q^a)$. (To be precise, Ψ also depends on the matter field ϕ as well; we will not display this dependence explicitly unless it is crucial to the discussion.)

In this letter, I will suggest a prescription for solving this problem. The idea is essentially the following: It is well-known that a formal definition for the time co-ordinate can be given in the context of *semiclassical* cosmology, i.e. when Ψ is a WKB solution (Lapchinsky and Rubakov 1979; Banks 1985; Hartle 1986; Singh and Padmanabhan 1989). I generalize this definition for *arbitrary* Ψ and use it to introduce a new set of co-ordinates $\{t, x^\alpha, \phi\}$ in the superspace, in terms of the old set of variables $\{Q^a, \phi\}$. The new co-ordinate t, behaves as a good 'intrinsic' time variable. Such co-ordinate transformations do not change (1) because of my choice of factor ordering.

The semiclassical approximation to the Wheeler-DeWitt equation is obtained by a WKB expansion in powers of M. To the leading (non-trivial) order this gives the wave-function

$$\Psi(Q, \phi) = Nf(Q, \phi) \exp iMS(Q) \tag{3}$$

where N is the WKB prefactor related to det $(\partial^2 S/\partial Q^a \partial Q^a)$, S is the solution to the Hamilton-Jacobi equation

$$\frac{1}{2}M(\nabla S)^2 + MV(Q) = 0 \tag{4}$$

and $f(Q, \phi)$ satisfies the following equation:

$$i(\nabla S)\cdot(\nabla f) = H_{\text{matter}}(p_\phi, \phi, Q) f(Q, \phi). \tag{5}$$

[We use the single symbol Q for the set $\{Q^a\}$ when no confusion is likely to arise.] It is clear that (5) is the functional Schrödinger equation for the matter field in a given background geometry if we identify the left hand side of (5) with a time derivative. This suggests that we define the time co-ordinate, t, to be such that the following condition is satisfied:

$$\left(\frac{\partial}{\partial t}\right) = (\nabla S)\cdot(\nabla) = G^{ab}\left(\frac{\partial S}{\partial Q^a}\right)\left(\frac{\partial}{\partial Q^b}\right). \tag{6}$$

The function S defines the vector field ∇S in the superspace; equation (6) implies that t parametrises the integral curves of this vector field.

Once a time co-ordinate has been identified, it is natural to redefine the co-ordinates so that t is one of them. [This will anyway be the simplest way to attack eq. (5)]. Let $\{t, x^\alpha\}$ be a new set of co-ordinates in the superspace, determined by the co-ordinate transformations $Q^a = Q^a(t, x^\alpha)$. [To avoid any possible misunderstanding, let me emphasize that the quantities (t, x^α) are co-ordinates in the *superspace* and, of course, has nothing to do with any co-ordinate charts in the spacetime; in other words, time is being defined here in terms of some combination of geometric degrees of freedom Q^a realising the doctrine: 'three-geometry is the carrier of information about time'.]

We already know, from (6), that

$$\left(\frac{\partial Q^b}{\partial t}\right) = G^{ab}\left(\frac{\partial S}{\partial Q^a}\right). \tag{7}$$

Further, from the invariance of the superspace line element, we know that

$$\mathrm{d}L^2 = G_{ab}\mathrm{d}Q^a\mathrm{d}Q^b = \mathscr{G}_{tt}\mathrm{d}t^2 + 2\mathscr{G}_{t\alpha}\mathrm{d}t\,\mathrm{d}x^\alpha + \mathscr{G}_{\alpha\beta}\mathrm{d}x^\alpha\mathrm{d}x^\beta. \tag{8}$$

Assuming that the (t, x^α) co-ordinates form an orthogonal set we can choose

$$\mathscr{G}_{t\alpha} = 0 = G_{ab}\left(\frac{\partial Q^a}{\partial t}\right)\left(\frac{\partial Q^b}{\partial x^\alpha}\right). \tag{9}$$

Combining (7) and (9) we get

$$0 = G_{ab}G^{am}\left(\frac{\partial S}{\partial Q^m}\right)\left(\frac{\partial Q^b}{\partial x^\alpha}\right) = \left(\frac{\partial S}{\partial x^\alpha}\right) \tag{10}$$

showing that S depends only on t. Inverting this relation, we realize that t depends on Q's only through $S{:}t(Q) = t[S(Q)]$. (This fact has a simple geometric meaning. Notice that the vector field ∇S is orthogonal to the surfaces of constant S and t parmetrises the integral curves of this vector field. Clearly the orthogonality implies natural parametrisation in terms of any function of S along.). Using this fact in (6), we get

$$\frac{\mathrm{d}S}{\mathrm{d}t} = G^{ab}\left(\frac{\partial S}{\partial Q^a}\right)\left(\frac{\partial S}{\partial Q^b}\right) \tag{11}$$

which, in turn, implies that

$$\mathscr{G}_{tt} = G^{ab}\left(\frac{\partial S}{\partial Q^a}\right)\left(\frac{\partial S}{\partial Q^b}\right) = \frac{\mathrm{d}S}{\mathrm{d}t}; \quad \mathscr{G}^{tt}\frac{\mathrm{d}S}{\mathrm{d}t} = 1. \tag{12}$$

Notice that the above analysis did not use the fact that S was a solution to the Hamilton-Jacobi equation. Therefore, it should be possible to introduce a similar prescription even in the general case by using the phase of the exact solution Ψ to the Wheeler-DeWitt equation.

Before discussing this general case, it is preferable to simplify the notation a bit. Notice that, the matter hamiltonian will also have a ∇^2_ϕ term and a potential term $U(Q, \phi)$. We may, therefore extend our superspace by using a set of co-ordinates $q^a = \{Q^a, \phi^A\}$ which treats the gravitational and matter degrees of freedom in a single footing. The metric in this extended superspace will have the block diagonal form $g_{ab} = \{G_{ab}, F_{AB}\}$ where F_{AB} is the metric in the matter sector. [Normally it will just be proportional to δ_{AB} but we do not need its explicit form]. The Wheeler-DeWitt equation can be now written compactly as

$$\left[-\frac{1}{2}\nabla^2 + MV(Q) + U(q)\right]\Psi(q^a) = 0 \tag{13}$$

where the extended laplacian

$$\nabla^2 = \frac{1}{\sqrt{g}}\partial_a(\sqrt{g}g^{ab}\partial_b) \tag{14}$$

hides all the horror by taking care of the kinetic term for gravity and the matter fields. We have also rescaled the co-ordinates to take care of the M factor. The general solution can now be written in the form

$$\Psi = R(q)\exp i\theta(q). \tag{15}$$

I shall now construct the new coordinate system (t, x^α) from the previous set $\{q^a\}$ using the phase θ in accordance with the prescription we followed earlier. The defining conditions will be

$$\left(\frac{\partial q^a}{\partial t}\right) = g^{ab}\left(\frac{\partial \theta}{\partial q^a}\right); \quad \left(\frac{\partial \theta}{\partial x^\alpha}\right) = 0 \tag{16}$$

from which one can show that

$$\mathscr{G}^{tt}\left(\frac{\partial \theta}{\partial t}\right) = 1. \tag{17}$$

These conditions allow further freedom in the choice of transforming x^α among themselves; this freedom can be used, for example, to retain the identity of the matter fields ϕ's. [Note that, in the *exact* description of gravity interacting with the matter fields, we should treat the Q^α's and ϕ^A's at an equal footing; in that case, the time co-ordinate, defined through the phase of the total wave function will depend on all the degrees of freedom. In the semiclassical limit of course, the phase will have a dominant term which is independent of matter field (as in (3)) and the dependence of the time co-ordinate on the matter variables will be suppressed by factors of M.].

It is now necessary to demonstrate that the above definition makes sense physically and that the time co-ordinate defined above does behave in a reasonable manner. In the semiclassical limit it was obvious because we defined the time co-ordinate in such a way that a Schrödinger equation for the matter field will emerge. Since we cannot do this here, I will proceed in a slightly different manner.

If the co-ordinate t is 'really' the time, then the expectation value of some observable $A(x^\alpha)$, constructed out of the x^α's can be calculated as

$$< A > = \int [dx]\sqrt{\mathscr{G}}\Psi^*(t, x)A\Psi(t, x) \tag{18}$$

[It can be easily verified that the hamiltonian is hermitian, with the measure $\sqrt{\mathscr{G}}$ in the expectation value, provided $\mathscr{G}_{t\alpha}$ is zero.] The time derivative of this quantity is

$$\left(\frac{\partial < A >}{\partial t}\right) = \int [dx]\sqrt{\mathscr{G}}\left[\left(\frac{\partial \ln\sqrt{\mathscr{G}}}{\partial t}\right)\Psi^* A\Psi + \left(\frac{\partial \Psi^*}{\partial t}\right)A\Psi + \Psi^* A\left(\frac{\partial \Psi}{\partial t}\right)\right]. \tag{19}$$

On the other hand, we would have expected the time derivative of $< A >$ to be given by

$$< i[H, A] > = \int [\mathrm{d}x] \sqrt{\mathscr{G}} \Psi^*(t, x) i \left[-\frac{1}{2} \frac{1}{\sqrt{\mathscr{G}}} \partial_\alpha (\sqrt{\mathscr{G}} \mathscr{G}^{\alpha\beta} \partial_\beta), A \right] \Psi(t, x). \quad (20)$$

Both these expressions can be evaluated independently (since time derivatives can be converted to q-derivatives using (16)). *It turns out that, the prescription for time co-ordinate suggested above ensures the equality of these two expressions.* This shows that t is a physically reasonable time co-ordinate.

To prove the above assertion, we only need to verify the equality of (19) and (20) for $A = x^\alpha$; since the system respects co-ordinate transformations among x^α's, A is as good a co-ordinate. Using our experience with ordinary quantum mechanics, we can easily write down the condition for the equality of (19) and (20) for the case of $A = x^\alpha$. We must have

$$i \left(\frac{\partial \Psi}{\partial t} \right) + \frac{i}{2} \frac{1}{\sqrt{\mathscr{G}}} \left(\frac{\partial \sqrt{\mathscr{G}}}{\partial t} \right) \Psi = \left[-\frac{1}{2} \frac{1}{\sqrt{\mathscr{G}}} \partial_\alpha (\sqrt{\mathscr{G}} \mathscr{G}^{\alpha\beta} \partial_\beta) + W \right] \Psi \quad (21)$$

where W is a real quantity. We now multiply this equation by Ψ^* and take the real and imaginary parts. The real part only defines W; but the imaginary part of this equation gives the constraint on our co-ordinate transformation:

$$\frac{1}{\sqrt{\mathscr{G}}} \left(\frac{\partial \sqrt{\mathscr{G}}}{\partial t} \right) = -\frac{1}{R^2} \left[\mathrm{Im} \left(\Psi^* \frac{1}{\sqrt{\mathscr{G}}} \partial_\alpha (\sqrt{\mathscr{G}} \mathscr{G}^{\alpha\beta} \partial_\beta) \Psi \right) + 2R \left(\frac{\partial R}{\partial t} \right) \right]. \quad (22)$$

The right hand side of (22) can be reduced by straightforward manipulation to the form

$$\frac{1}{R^2} \left[\left(\frac{\partial \ln \sqrt{\mathscr{G}}}{\partial t} \right) R^2 \mathscr{G}^{tt} \left(\frac{\partial \theta}{\partial t} \right) + R \left(\frac{\partial}{\partial t} \right) \left[\mathscr{G}^{tt} \left(\frac{\partial \theta}{\partial t} \right) R \right] + \mathscr{G}^{tt} R \left(\frac{\partial \theta}{\partial t} \right) \left(\frac{\partial R}{\partial t} \right) - 2R \left(\frac{\partial R}{\partial t} \right) \right]$$

which is identically equal to the left hand side of (22) because of the condition (17). Thus our choice of the time-co-ordinate allows a sensible interpretation of the expectation values. Needless to say, the prescription agrees with the notion of time in the semiclassical limit.

The operational definition of time always originates from the correlation with the changes observed in physical observables. I believe this fact adds strength to the prescription discussed here. It will be worthwhile to construct the time co-ordinate suggested above in some suitable (multidimensional) minisuperspace models. This and related issues are under investigation. Similar suggestion has been put forward independently by Greensite (1990) whose analysis indicates that the *only* prescription which leads to proper evolution of expectation values is the one suggested here.

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