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Gaussian States in de Sitter Spacetime and the Evolution of Semiclassical density perturbations. 2. Inhomogeneous Modes

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Abstract. In the computation of density perturbation in inflation it is conventional to assume the inflation field to be in the vacuum state. There are, however, some advantages in relaxing this assumption. In an earlier paper we have estimated the density perturbations in a Gaussian coherent state using a toy-model. Here we extend this work by doing an exact analysis of this problem. The advantages of this method is discussed and the results are compared with earlier results.

Key words: cosmology—early universe—density perturbations

1. Introduction

After the inflationary universe was suggested (Guth 1981; see also Sato 1981; and Kazanas 1980) and subsequently modified (Linde 1982) various people have computed density perturbations arising out of this scenario (Guth & Pi 1982; Starobinsky 1982; Hawking 1982; Bardeen Steinhardt & Turner 1983). All the conventional models (generically) produce too large a magnitude for the density contrast unless some sort of fine-tuning or Planck length cut-off (Padmanabhan, Seshadri & Singh 1989) is resorted to. The conventional model, however, produces the correct spectral dependence; suggested by Harrison & Zeldovich (Harrison 1970; Zeldovich 1972).

In this paper we have analyzed the various subtleties and drawbacks of these approaches. We have studied the density perturbations using Gaussian states. In earlier works (Padmanabhan & Seshadri 1986; Seshadri & Padmanabhan 1989), we had worked with toy-models for inflation which used only the homogeneous mode. In this paper we have considered the full scalar field including the inhomogeneous modes.

2. Gaussian state in inflationary universe

The conventional approach and its drawbacks have been discussed in an earlier paper (Seshadri & Padmanabhan 1989). We will briefly summarize these difficulties. The conventional approach assumes that the scalar field is in a vacuum state. This leads to problems at two levels. (i) In the ‘pre-inflationary days’ it was believed that the expectation value of the energy-momentum tensor is the source of semiclassical gravity. Such a prescription however leads to difficulties in the inflationary scenario. Since the vacuum state is translationally invariant, the above method cannot give rise to density perturbations, (ii) The conventional picture makes use of an intermediate classical field, $\phi'_{\rm cl}$, to construct density perturbations. This classical field is constructed using the two-point correlation function of the field operator in the vacuum state. This method is a bit *ad hoc.* (It is more proper to use the energy-momentum tensor operator directly instead of using the intermediate field.) Clearly these difficulties arise because the quantum state in which the field is assumed to be is translationally invariant. It is worthwhile to try out Gaussian states as an alternative to vacuum state. With this in view, we will now study the evolution of Gaussian states in an exponentially expanding background.

We will work in the Schrödinger picture field theory. Using the action, we can define the momentum π conjugate to the field ϕ. In the Schrödinger picture, the momentum operator π is to be represented by the functional derivative operator

$$\pi = -i\hbar \frac{\delta}{\delta\phi(x,t)} \tag{1}$$

so that the field operator, ϕ and its momentum, π, are related by the commutation relation

$$[\phi, \pi] = ih \tag{2}$$

(see *e.g.* Narlikar & Padmanabhan 1986; Chapter 4, Section 4.4). The quantum dynamics of the system is described by the wavefunctional $\psi[\phi(x), t]$; $|\psi|^2$ represents the probability for the field to have the field configuration $\phi(x)$ at time t. We will use the Schrödinger picture field theory since it is convenient to describe coherent excitations and semiclassical limit in this approach.

Instead of the approach based on wavefunctional, we will follow a simpler procedure. We will first expand the scalar field in terms of its Fourier modes. The action for the scalar field can now be expressed in terms of its Fourier components. If we consider a potential which is at most quadratic in ϕ, the total action is just the sum of the action for each mode.All these modes behaveas harmonic oscillators.Denoting by ψ_k the wavefunction for the kth mode we can write a Schrödinger equation which governs the evolution of ψ_k. The full wavefunction for the field is just the (infinite) product of the wavefunctions of all the modes.

Consider a scalar field in a quadratic potential of the form

$$V(\phi) = -\tfrac{1}{2}\omega^2\phi^2 + V_0. \tag{3}$$

The action for the field is given by

$$\mathscr{A} = \int \left(\tfrac{1}{2}(\partial_i\phi)(\partial^i\phi) + \tfrac{1}{2}\omega^2\phi^2 - V_0\right) S^3\, \mathrm{d}^3x\,\mathrm{d}t. \tag{4}$$

The scalar field can be expressed in terms of Fourier modes.

$$\hat{\phi}(\mathbf{x}, t) = \int \hat{\phi}_{\mathbf{k}}(t)\, e^{i\mathbf{k}\cdot\mathbf{x}} \frac{\mathrm{d}^3 k}{(2\pi)^3}. \tag{5}$$

Dropping the constant term in the potential (because the evolution of the wave-function is independent of V_0), we may write the action in terms of $\phi_k(t)$ as

$$\mathscr{A} = \int \left(\frac{1}{2}|\dot{\hat{\phi}}_{\mathbf{k}}|^2 - \frac{1}{2}\left(\frac{k^2}{S^2} - \omega^2\right)|\hat{\phi}_{\mathbf{k}}|^2\right) \mathrm{d}t \frac{\mathrm{d}^3 k}{(2\pi)^3} S^3. \tag{6}$$

The Fourier modes, $\hat{\phi}_k$, are complex quantities. Instead of working with these it is simpler to work with the real abd imaginary parts of $\hat{\phi}_k$. These are real quantities. We will denote then generically by $\hat{q}_k$. In terms of $\hat{q}_k$ the action can be written as

$$\mathscr{A} = \int \left(\frac{1}{2}\dot{\hat{q}}_k^2 - \frac{1}{2}\left(\frac{k^2}{S^2} - \omega^2\right)\hat{q}_k^2 \right) S^3 \frac{\mathrm{d}^3 k}{(2\pi)^3}\,\mathrm{d}t. \tag{7}$$

This is basically the sum of the actions for all the modes. Using a new time coordinate defined as

$$T = \int \frac{\mathrm{d}t}{S^3} \tag{8}$$

we get

$$\mathscr{A} = \int \left[\frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}T}\hat{q}_k\right)^2 - \frac{1}{2}\left(\frac{k^2}{S^2} - \omega^2\right) S^6 \hat{q}_k^2 \right] \frac{\mathrm{d}^3 k}{(2\pi)^3}\,\mathrm{d}T. \tag{9}$$

In the Schrödinger picture, the dynamics is described by a wavefunction ψ_k for the mode $\hat{q}_k$. (Tomonaga 1946; Wheeler 1962; Narlikar & Padmanabhan 1986, Chapter 4, Section 4.3; Feynman & Hibbs 1965, Section 8, Chapter 9). Using the action in Equation (9) we write the Schrödinger equation for the evolution of ψ_k as

$$i\frac{\partial \psi_k}{\partial T} = -\frac{1}{2}\frac{\partial^2}{\partial q_k^2}\psi_k + \frac{1}{2} S^6 \left(\frac{k^2}{S^2} - \omega^2\right) q_k^2 \psi_k. \tag{10}$$

At any time t, the complete wavefunctional ψ [φ(x,) t] of the field is the direct product of the wavefunction of all the modes

$$\psi[\phi(x)] = \prod_k \psi_k \tag{11}$$

In this paper we will be interested only in the case $\omega \ll k/S$. So from now on, we will neglect the term ω with respect to k/S. Transforming back to the time coordinate t we can write the Schrödinger equation in this limit as

$$i\frac{\partial}{\partial t}\psi_k = -\frac{1}{2S^3}\frac{\partial^2}{\partial q_k^2}\psi_k + \frac{1}{2} S^3 \left(\frac{k^2}{S^2}\right) q_k^2 \psi_k. \tag{12}$$

(It can be easily seen that we get the same Schrödinger equation (effectively) for the case of a linear potential as well. If

$$V(\phi) = (\eta\phi_f)^4 \left[1 - \lambda\frac{\phi}{\phi_f}\right], \tag{13}$$

substituting this in the action and expressing it in terms of the Fourier modes of ϕ, we will see that the linear term gives only a δ-function in k. Since we are only interested in nonzero k, we can ignore the delta-function. The gradient term will still give the k^2/S^2 term. So the Schrödinger equation in this case is the same as in Equation 12).

The Schrödinger equation can be solved by the ansatz

$$\psi_k = A_k(t)\exp\left[-B_k(q_k - f_k)^2\right]. \tag{14}$$

Following the procedure we have been using, this ansatz can be plugged into the

Schrödinger equation. This yields the equations for A_k, B_k, and f_k to be

$$i\dot{B} = \frac{2B^2}{S^3} - \frac{1}{2}\frac{k^2}{S^2}S^3, \tag{15}$$

$$i(\dot{B}f + B\dot{f}) = \frac{2B^2}{S^3}\dot{f}, \tag{16}$$

$$i\frac{\hat{A}}{A} = i\dot{B}f^2 + 2iBf\dot{f} + \frac{B}{S^3} - \frac{2B^2f^2}{S^3}. \tag{17}$$

(We have suppressed the index k.) These equations can be solved by the substitution,

$$B = -\frac{i}{2}S^3\frac{\dot{Q}}{Q}. \tag{18}$$

This gives

$$f = (\text{const})(S^3\dot{Q})^{-1} \tag{19}$$

$$A = (\text{const})\,Q^{-\frac{1}{2}}\exp\left[-\frac{i}{2}\int\frac{k^2}{S^2}S^3f^2\,\mathrm{d}t\right] \tag{20}$$

where Q satisfies the equation

$$\ddot{Q} + 3H\dot{Q} + \frac{k^2}{S^2}Q = 0. \tag{21}$$

From Equation (14), the probability density $|\psi|^2$ can be computed to be

$$|\psi_k|^2 = N\exp\left[-\frac{(q_k - \bar{q}_k)^2}{2\sigma_k^2}\right] \tag{22}$$

where

$$\sigma_k^2 = \tfrac{1}{2}(B_k + B_k^*)^{-1}, \tag{23}$$

$$\bar{q}_k = \frac{B_k f_k + B_k^* f_k^*}{B_k + B_k^*}. \tag{24}$$

$\bar{q}_k$ is the mean value of the Gaussian. It obeys the classical equation of motion

$$\ddot{\bar{q}}_k + 3H\dot{\bar{q}}_k + \frac{k^2}{S^2}\bar{q}_k = 0. \tag{25}$$

This equation has the simple solution,

$$\bar{q}_k = a_k\left(1 - i\frac{k}{HS}\right)e^{i\frac{k}{HS}} + b_k\left(1 + i\frac{k}{HS}\right)e^{-i\frac{k}{HS}} \tag{26}$$

where a_k and b_k are constants.

To obtain the solution for the spread σ_k one has to first solve the Equation (21) for Q_k. Solving this equation we get the solution for Q_k to be

$$Q_k = c_k\left(1 - i\frac{k}{HS}\right)e^{i\frac{k}{HS}} + d_k\left(1 + i\frac{k}{HS}\right)e^{-i\frac{k}{HS}}. \tag{27}$$

We see that $\bar{q}_k$ and Q_k obey the same differential equation and have similar solutions. Each of these have two constants. While we have a handle on Q_k by comparison with the flat space vacuum, the constants in the expression for $\bar{q}_k$ are in our hand.

We have two constants to be fixed in the expression for Q_k. We will now fix these by comparing the wavefunction with the flat-space wavefunction. Let us first evaluate our expression for Q in the limit $H \to 0$, $S \to 1$, i.e., the flat-space limit. In this limit Q becomes,

$$\lim_{H\to 0} Q_k = \alpha_k e^{-ikt} + \beta_k e^{ikt} \tag{28}$$

where

$$\alpha_k = -ic_k \frac{k}{H} e^{i\frac{k}{H}}, \tag{29}$$

and

$$\beta_k = id_k \frac{k}{H} e^{-i\frac{k}{H}}. \tag{30}$$

From equations (14) and (18) we see that ψ goes as

$$\psi \simeq Q^{-\frac{1}{2}} \exp\left\{\frac{i}{2}\frac{\dot{Q}}{Q}q^2\right\} \tag{31}$$

in the limit $H \to 0$. In flat spacetime, the vacuum state wavefunction must have the form (Landau & Lifshitz 1985, Section 23).

$$\psi \sim \exp(-\tfrac{1}{2}kq^2). \tag{32}$$

So for our wavefunction to have the correct flat-space limit we require,

$$\frac{\dot{Q}}{Q} = ik \tag{33}$$

at all times. From Equation (28) it follows that we need $\alpha_k = 0$.

So in the inflationary phase, our solution is

$$Q_k = d_k\left(1 + i\frac{k}{HS}\right)\exp\left(-i\frac{k}{HS}\right). \tag{34}$$

B can now be computed from Equations (18) and (34).

$$B_k = \frac{1}{2}\frac{k^3}{H^2}\frac{1}{(1 + k^2/H^2S^2)}\left[1 + i\frac{HS}{k}\right]. \tag{35}$$

Substituting this in Equation (23) we get

$$\sigma_k^2 = \frac{H^2}{2k^3}\left(1 + \frac{k^2}{H^2S^2}\right). \tag{36}$$

The σ_k is the same as the power spectrum defined via two-point function. This is to be expected as can be seen from the following argument: σ_k which we have derived here is the spread of the vacuum-state wavefunction. The power spectrum, P_k, is related to the two-point correlation function of the scalar field via the relation:

$$P_k^2 = \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}} \langle 0|\phi(\mathbf{x}+\mathbf{y}, t)\,\phi(\mathbf{y}, t)|0\rangle. \tag{37}$$

Substituting for ϕ in terms of its Fourier modes q_k we get

$$P_k^2 = \langle 0 || \phi_k |^2 | 0 \rangle$$
$$= \langle 0 | \phi_k | 0 \rangle^2 + \sigma_k^2. \tag{38}$$

Since the expectation value of ϕ_k in the vacuum state is zero, we get $P_k = \sigma_k$.

We will use these results for the computation of density perturbations in inflationary universe.

3. Density inhomogeneities in Gaussian states

The goal is to construct $\delta\rho/\rho$ (which is a *c*-number) starting from a quantum field $\hat{\phi}(x, t)$. This passage from a quantum to classical quantity is not unambiguous: There are two ways in which this may be done. One way is to define a classical field, ϕ_{cl}, and use that to compute the density inhomogeneity. The other method is to compute $\delta\rho/\rho$ directly from the energy-momentum tensor, T_k^i of the scalar field. The latter method will be studied in Section 7. In this section we will confine ourselves to the computation of $\delta\rho/\rho$ via an intermediate classical field $\phi_{\rm cl}$.

In the conventional approach also, the density inhomogeneity is computed using an intermediate classical field. However, the way $\phi_{\rm cl}$ is conventionally defined from the field operator $\hat{\phi}(\mathbf{x}, t)$ is somewhat *ad hoc* and arbitrary. In this section we will consider a more natural way of constructing $\phi_{\rm cl}$ from $\hat{\phi}(x, t)$.

Our ultimate aim is to construct the cnumber $\delta\rho/\rho$ starting from the field operator $\hat{\phi}(x, t)$. In order to understand the physics involved in a clear way; it is preferable to study the semiclassical evolution of the scalar field. In flat spacetime this could have been done by assuming the field to be in a coherent state. Since we are interested in the case of expanding universe, coherent states cannot be defined in a strict sense. So we assume the field to be in a Gaussian state. We shall study the origin of density perturbation using these Gaussian states.

The evolution of the field is described by the wavefunctional $\psi[\hat{\phi}(x, t)]$ of the field configuration. It is easier to study the evolution by expanding $\hat{\phi}(x, t)$ in terms of its Fourier modes as

$$\hat{\phi}(\mathbf{x}, t) = \int \frac{d^3k}{(2\pi)^3} \hat{\phi}_k e^{i\mathbf{k}\cdot\mathbf{x}} \tag{39}$$

and study the evolution of each mode. This approach has been studied in Section 2. We will be using the following two kinds of Gaussian states:

(a) *A state in which* $f_k = 0$ *for* $k \neq 0$ *and* $f_k \neq 0$ *for* $k = 0$: For these states the mean value for the inhomogeneous modes is zero. The homogeneous mode, on the other hand, has a nonzero mean. So the homogeneous mode is in a general Gaussian state. Hence for such states we have

$$\langle \hat{q}_k \rangle = 0 \qquad \text{for } k \neq 0, \tag{40}$$

$$\langle \hat{q}_k \rangle \neq 0 \qquad \text{for } k = 0. \tag{41}$$

We will denote these states by $|\psi_1\rangle$.

(b) *A state in which the mean value is non-zero for all modes*: We will denote these states by $|\psi_2\rangle$ .For these states

$$\langle q_k \rangle \neq 0 \qquad \text{for all } \mathbf{k}.$$

In this case, all the modes are in a general Gaussian state.

We will now compute density perturbations in these two states.

4. Density perturbations in $|\psi_1\rangle$

As pointed out earlier, we will construct an intermediate classical field ϕ_{cl} (x, t) and define the density perturbations via this classical field. ϕ_{cl} is a sum of a homogeneous part ϕ_0 (t) and an inhomogeneous part $\delta\phi$ $(\mathbf{x}, t)$.

$$\phi_{cl}(\mathbf{x}, t) = \phi_0(t) + \delta\phi(\mathbf{x}, t). \tag{42}$$

The spatial dependence of ϕ_{cl} comes completely from $\delta\phi(x, t)$. Having done this we give a prescription to compute ϕ_0 and $\delta\phi(x, t)$ (and hence ϕ_{cl}).

The inhomogeneous part of ϕ_{cl} (which is $\delta\phi$ (x, t)) is defined as the Fourier transform of the spread σ_k.

$$\delta\phi(\mathbf{x}, t) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\mathbf{k}\cdot\mathbf{x}}\, \sigma_k(t). \tag{43}$$

As has been shown in Section 2, σ_k^2 is the power spectrum of ϕ. Hence this definition of $\delta\phi$ (x, t) is the same as the conventional definition.

We define $\phi_0(t)$ as the expectation value of the field in the state $|\psi_1\rangle$ Since $\langle \hat{q}_k \rangle$ 0 for $k \neq 0$ in this state, ϕ_0 defined in this way is space-independent. Since the expectation value of an operator obeys the classical equation of motion, ϕ_0 obeys the equation

$$\ddot{\phi}_0 + 3H\,\dot{\phi}_0 + \frac{\partial V(\phi_0)}{\partial \phi_0} = 0. \tag{44}$$

The ϕ_0 can be connected up with the expectation value of the homogeneous mode of the scalar field (Padmanabhan & Seshadri 1986; Seshadri & Padmanabhan 1988). We note that the expectation value of the inhomogeneous modes in the state $|\psi_1\rangle$ are zero. Hence the expectation value of $\hat{\phi}$ (x, t) in the state $|\psi_1\rangle$ is just the expectation value of value of the homogeneous mode of $\hat{\phi}$ (x, t) in a general Gaussian state.The equation satisfied by ϕ_0 is the same as the one satisfied by the expectation value of the homogeneous mode. Having defined ϕ_0 (t) and $\delta\phi(x, t)$ we can construct ϕ_{cl} via Equation (42).

Using ϕ_{cl}, the energy density of the scalar field is given by

$$\rho(\mathbf{x}, t) = \frac{1}{2}\dot{\phi}_{cl}^2 + \frac{1}{2}\frac{1}{S^2}(\bar{\nabla}\phi_{cl})^2 + V(\phi_{cl}). \tag{45}$$

This is a spatially dependent quantity. Substituting for ϕ_{cl} in terms of ϕ_0 and $\delta\phi$ we have

$$\rho(\mathbf{x}, t) = \frac{1}{2}\dot{\phi}_0^2 + \frac{1}{2}\left(\frac{d}{dt}(\delta\phi)\right)^2 + \dot{\phi}_0\left(\frac{d}{dt}(\delta\phi)\right) + \frac{1}{2}\frac{1}{S^2}(\bar{\nabla}\delta\phi)^2 + V(\phi_0 + \delta\phi). \tag{46}$$

We assume that $\delta\phi < \phi_0$. Hence we will retain only those terms which are up to linear order in $\delta\phi$. We will split $p(x, t)$ as a sum of a homogeneous part $\rho_o(t)$ and an inhomogeneous part $\delta\rho(x, t)$. We can now write $p(x, t)$ as

$$\delta\rho(\mathbf{x}, t) = \dot{\phi}_0 \frac{\mathrm{d}(\delta\phi)}{\mathrm{d}t}. \tag{47}$$

Taking the Fourier transform (with respect to the spatial coordinates) of the above equation we have

$$\rho_k(t) = \dot{\phi}_0 \, \dot{\sigma}_k. \tag{48}$$

(Recall that we defined $\delta\phi$ as the Inverse Fourier transform of σ_k.)

In arriving at the above equation for ρ_k we have dropped the term involving the potential. Though this is always done in standard literature (See *e.g.* Brandenberger 1985), it is not strictly correct. However, for reasons given in Appendix 1, ρ_k remains to be of the same order of magnitude even if we include the extra terms. Hence that term will not change the results significantly.

From equation (47) we can compute the density inhomogeneity at the time t_k, when the perturbation leaves the Hubble radius during inflation. Using Bardeen's formalism (Bardeen, Steinhardt & Turner 1983) we can relate the density perturbation at the epoch, t_k, of re-entry into the horizon with the density perturbation at the epoch, t_k,

$$\left.\frac{\Delta\rho}{\rho}\right|_{t_r} \equiv k^{3/2}|\delta_k|\Big|_{t_r} \equiv k^{3/2}\left.\frac{\rho_k}{\rho_0}\right|_{t_r} = \frac{4}{3}\left.\frac{k^{3/2}|\delta_k|}{1+w}\right|_{t=t_k} \tag{49}$$

where p_0 is the homogeneous background density.

At the epoch $t = t_k$ the physical size of the perturbation is equal to the Hubble radius. Therefore at t_k

$$\frac{k}{HS} = 2\pi. \tag{50}$$

We will now compute the right-hand side of Equation (49). At t_k the universe is in an inflationary phase. At this epoch, the background density is dominated by the constant term V_0 . Thus

$$\delta_k(t_k) = \frac{\rho_k(t_k)}{V_0}. \tag{51}$$

In Equation (49) w is the ratio of the pressure to the density. During inflation

$$1 + w = \frac{\dot{\phi}_0^2}{V_0}. \tag{52}$$

Using equations (48), (49), (51) and (52) we get the density contrast at the epoch when the perturbation re-enters the horizon to be

$$\left.\frac{\Delta\rho}{\rho}\right|_{t_r} = k^{3/2}|\delta_k|\Big|_{t_r} = \frac{4}{3}\left.\frac{k^{3/2}|\dot{\sigma}_k|}{\dot{\phi}_0}\right|_{t=t_k}. \tag{53}$$

In order to compare the results derived above with the one which we used in the toy-models (Padmanabhan & Seshadri 1986; Seshadri & Padmanabhan 1989), it is instructive to write Equation (53) in a slightly different form. As we have seen in Section (2) the spread, σ_k, of the wave-packet for the kth mode is given by Equation (36).

$$\sigma_k = \frac{1}{\sqrt{2}} \frac{H}{k^{3/2}} \left(1 + \frac{k^2}{H^2 S^2}\right)^{1/2} \tag{54}$$

where $k = |k|$. Taking its time derivative, we get

$$\dot{\sigma}_k = -\frac{1}{\sqrt{2}} \frac{H^2}{k^{3/2}} \frac{(k/HS)^2}{[1+(k/HS)^2]^{1/2}}. \tag{55}$$

$\dot{\sigma}_k$ can be expressed in terms of σ_k as

$$\dot{\sigma}_k = -H \frac{\theta_k^2}{1+\theta_k^2} \sigma_k \tag{56}$$

where $\theta_k = k/(HS)$. At the epoch of Hubble radius leaving during inflation (*i.e.* at $t = t_k$), we have

$$\theta_k = \frac{k}{HS} = 2\pi. \tag{57}$$

Using this in Equation (56) we get

$$\dot{\sigma}_k(t_k) = -H \frac{4\pi^2}{1+4\pi^2} \sigma_k \bigg|_{t_k}. \tag{58}$$

Using Equation (53) we can express density contrast in terms of σ_k and $\dot{\phi}_0$ as

$$\frac{\Delta\rho}{\rho}\bigg|_{t_f} = k^{3/2}|\delta_k|\bigg|_{t_f} = \frac{4}{3} \frac{4\pi^2}{1+4\pi^2} H \frac{|k^{3/2}\sigma_k|}{\dot{\phi}_0}\bigg|_{t_f} \simeq \frac{4}{3} H \frac{k^{3/2}|\sigma_k|}{\dot{\phi}_0}\bigg|_{t_k}. \tag{59}$$

At this stage we may recall the expression for density inhomogeneity in our toy-models (Padmanabhan & Seshadri 1986; Seshadri & Padmanabhan 1989). The toy-model consisted of a homogeneous scalar field ϕ which depended only on time. The quantum state for ϕ was assumed to be a Gaussian. The density perturbation was estimated using the spread of this quantum state and the mean value of the field $\bar{\phi}$ in that state using the relation (we use the symbol $\delta\rho/\rho$),

$$\frac{\delta\rho}{\rho}\bigg|_{t_f} = \mathcal{O}(1) H \frac{\sigma(t)}{\dot{\phi}_0(t)}\bigg|_{t_k}. \tag{60}$$

Let us compare the Equations (59) and (60). We may note that the two expressions match up to an order of magnitude if we identify the quantum spread $\sigma(t)$ in the Gaussian for the homogeneous mode in the toy-model, with $k^{3/2}$ σ_k of our rigorous analysis. We will now see the condition under which these two can be identified.

If σ_0 is of the order of the de Sitter temperature at the epoch $t = t_k$, then $\sigma(t_k) \simeq H/2\pi$. Using the value of σ in Equation (60) we get

$$\frac{\delta\rho}{\rho}\bigg|_{t_f} = \frac{\mathcal{O}(1)}{2\pi} H \frac{H}{\dot{\phi}_0}\bigg|_{t_k}. \tag{61}$$

We next estimate the value of $\delta\rho/\rho$ in our analysis with spatial dependence. Equation (54) gives

$$k^{3/2}|\sigma_k|\bigg|_{t_k} = \sqrt{2}\,\pi H. \tag{62}$$

Using this in Equation (59), we get

$$\left.\frac{\Delta\rho}{\rho}\right|_{t_f} = \frac{4\pi\sqrt{2}}{3} H \left.\frac{H}{\dot{\phi}}\right|_{t_k}. \tag{63}$$

Comparing Equations (61) and (63) we come to the following conclusion. If we equate the quantum spread of the homogeneous mode in the toy-models to the de Sitter temperature, then the density contrast obtained from the toy-model is of the same order as that obtained from a complete analysis.

We next compute the density perturbations in two kinds of potentials: (i) Steady-slope model (Padmanabhan & Seshadri 1986) and (ii) Inverted oscillator potential model (Seshadri & Padmanabhan 1988). We may recall that for the steady-slope model, the potential is of the form

$$\begin{aligned} V(\phi) &= (\eta\phi_f)^4\left[1-\lambda\frac{\phi}{\phi_f}\right] && 0 \leqslant \phi \leqslant \phi_f, \\ &= 0 && \phi_f \leqslant \phi < \phi_b, \\ &= \infty && \phi > \phi_b, \end{aligned} \tag{64}$$

where n, λ are dimensionless and ϕ_f is a dimensional constant and the inverted oscillator potential is of the form.

$$V(\phi) = V_0 - \tfrac{1}{2}\omega^2\phi^2. \tag{65}$$

We will compare the results with the one obtained in the toy-model.

5. Specific examples

In this section we will consider the specific cases of steady slope and the inverted oscillator potentials. We have derived the equation governing the evolution of the homogeneous mode ϕ_0 and the spread σ_k of the wave-function of the kth mode. (Equation 44 & 56). Using these in (59) we can arrive at the expression for $\Delta\rho/\rho$ at the epoch of re-entry. For the steady-slope case, this comes out to be

$$\left.\frac{\Delta\rho}{\rho}\right|_{t_f} \simeq 4\pi\sqrt{2}\left(\frac{H}{\phi_f}\right)^3 \frac{1}{\eta^4\lambda}. \tag{66}$$

This clearly has no k-dependence. For the inverted oscillator case this comes to be

$$\left.\frac{\Delta\rho}{\rho}\right|_{t_f} = \frac{32\pi\sqrt{2}}{3}\,\frac{p}{2p+3}\,\frac{H}{\phi_i}\,\frac{1}{2p-3}\left(\frac{2\pi H}{k}\right)^{\frac{1}{2}(2p-3)} \tag{67}$$

where

$$p^2 = (\omega/H)^2 + 9/4. \tag{68}$$

For sufficient inflation we require that $(2p–3)$ should be small. This automatically ensures a weak k-dependence of $\Delta\rho/\rho$. For example, $(2p–3) \simeq 0.1$ implies $\Delta p/p|_{tf} \simeq k^{0.05}$. A smaller value for k is more conducive for sufficient inflation. It also weakens the k-dependence of $(\Delta\rho/\rho)$.

6. Density perturbations in the state $|\psi_2\rangle$

Till now in this paper we have followed a hybrid approach. The inhomogeneous modes were assumed to be in a vacuum state, *i.e.* in a Gaussian state with zero mean value. The homogeneous mode was assumed to be in a Gaussian state with nonzero mean. In this section we will consider Gaussian states for which the mean value for all the modes is nonzero.

We know from quantum mechanics that given an operator $\hat{\theta}$ the corresponding classical quantity is the expectation value of θ in the given quantum state. This suggests that we should define $\phi_{\rm cl}$ as

$$\phi_{\rm cl} = \langle \psi_2 | \hat{\phi} | \psi_2 \rangle. \tag{69}$$

This straightforward procedure to compute ϕ_{cl} would not have been of any use when we worked with the state $|\psi_1\rangle$. In that state only the homogeneous mode is in a general Gaussian state. The inhomogeneous modes are all in a Gaussian state with zero mean value. Hence the expectation value of $\hat{\phi}$ in the state $|\psi_1\rangle$ will not have any spatial dependence and cannot be used for computing density perturbations. So we had resorted to a different approach to construct $\phi_{\rm cl}$.

When the field is in the state $|\psi_2\rangle$ the above-mentioned problem will not arise because all the modes are in a general Gaussian state. Hence $\phi_{\rm cl}$ defined in Equation (69) does carry information about spatial dependence and hence can be used to compute density inhomogeneity.

The scalar field operator $\hat{\phi}$ can be expressed in terms of its Fourier modes $\hat{\phi}_k$ (t). The classical field $\phi_{\rm cl}$ corresponding to the field operator $\hat{\phi}$ can be expressed as

$$\phi_{\rm cl}(\mathbf{x}, t) = \int \frac{d^3k}{(2\pi)^3} \, \bar{\phi}_{\mathbf{k}}(t) \, e^{i\mathbf{k}\cdot\mathbf{x}} \tag{70}$$

where $\bar{\phi}_k$ is, the expectation value of $\hat{\phi}_k$ in the quantum state $|\psi_2|\rangle$. $\bar{\phi}_k$ is however, complex. As mentioned in section 2; it is more convenient to work with the real quantities $\bar{q}_k$ rather than $\bar{\phi}_k$

The fourier transform, p_k, of the density excess $\delta\rho$, can be related to $\bar{\phi}_0$ and $\dot{q}_k$ as

$$\rho_{\mathbf{k}} = \dot{\phi}_0(t) \, \dot{\bar{q}}_{\mathbf{k}}(t). \tag{71}$$

ϕ_0 is simply the expectation value of the homogeneous mode which was used in studying the toy-model. The expression for q_k has already been derived in Section 2.

We can consider two kinds of potentials: the steady slope and the inverted oscillator potentials. For both the cases we get the solutions of $\bar{q}_k$ be that given by Equation (26). (For inverted oscillator potential we have used $\omega \ll k/S$. This is so because for sufficient inflation we require $\omega \ll 2\pi H$. Further, we are interested in a wavelength till it grows to the size of the Hubble-radius: $k/S \geqslant 2\pi H$. So we require $\omega \ll k/S$.) The density contrast at the epoch of re-entry, comes out to be,

$$\left.\frac{\Delta\rho}{\rho}\right|_{tr} = k^{3/2}\delta_k\Big|_{tr} = \frac{32\pi^2}{3} H \frac{k^{3/2}|C_k|}{\dot{\phi}_0(t_k)}. \tag{72}$$

where ϕ_0 is just the expectation value of the homogeneous mode of the scalar field.

In the case of the steady slope model, for late times, we have

$$\dot{\phi}_0 = \frac{1}{3}(\eta^4\lambda)\left(\frac{\phi_f}{H}\right)\phi_f^2. \tag{73}$$

Using Equation (72) the density contrast turns out to be

$$\left.\frac{\Delta\rho}{\rho}\right|_{t_f} = 32\pi^2\left(\frac{H}{\phi_f}\right)^3 \frac{1}{\eta^4\lambda}\frac{k^{3/2}|C_k|}{H}. \tag{74}$$

From the above equation, it is clear that the density perturbation depends on the quantum state that has been chosen. This dependence comes about through the parameter C_k. The k-dependence of C_k governs the spectrum of density perturbations. In order to get the correct k-dependence for $(\Delta\rho/\rho)$ we must have

$$|C_k| = \alpha H k^{-3/2} \tag{75}$$

where α is a dimensionless constant.

For the inverted oscillator case the density contrast at t_f turns out to be

$$\left.\frac{\Delta\rho}{\rho}\right|_{t_f} \simeq \frac{64\pi^2}{3}\left(\frac{k^{3/2}|C_k|}{H}\right)\left(\frac{H}{\phi_i}\right)\frac{1}{2p-3}\left(\frac{2\pi H}{k}\right)^{\frac{1}{2}(2p-3)}. \tag{76}$$

For sufficient inflation we require $\omega \ll H$, *i.e.*, $(2p - 3) \ll 1$. Thus the term $(2\pi H/k)^{1/2(2p-3)}$ has a weak k-dependence. Further, if C_k goes as $Hk^{-3/2}$ the density contrast has a nearly flat spectrum. This is what we require.

We have seen that there are various alternatives for the quantum state for the scalar field which can produce the required k-independent spectrum for density perturbation. In particular we used two kinds of states denoted by $|\psi_1\rangle$ and $|\psi_2\rangle$. In the case of $|\psi_2\rangle$, however, we observe that the density perturbation spectrum depends on C_k and hence on the initial state of the field.

The alternatives which we have discussed have a major advantage over the vacuum state which is used conventionally. The way ϕ_0 is constructed from the operator is somewhat *ad hoc* and unnatural in the conventional scenario. However, when we choose the quantum state to be $|\psi_1\rangle$ or $|\psi_2\rangle$, ϕ_0 *can* be defined as the expectation value of the homogeneous mode of $\hat{\phi}$. This is certainly more straightforward and natural.

7. Density perturbation using energy-momentum tensor

Till now we have computed density perturbations using an intermediate classical field. It is, however more natural to compute density contrast directly from the energy-momentum tensor. We will, further assume that the field is in a Gaussian state $|\psi_2\rangle$ which we discussed in the previous section. In this state all the Fourier modes of ϕ are in a Gaussian state with nonzero mean. Classical energy density is identified with the expectation value of T^0_0 in the state $|\psi_2\rangle$

$$\rho(\mathbf{x}, t) = \langle\psi_2|T^0_0|\psi_2\rangle. \tag{77}$$

This straightforward method of transition from quantum to classical physics would not have been possible in the conventional approach, since the field is conventionally assumed to be in a vacuum state. Since the vacuum state is translationally invariant, $\langle 0|T^0_0|0\rangle$ is homogeneous and cannot give rise to density perturbations. This problem, however, does not arise in our method since we have assumed the quantum state to be a general Gaussian state.

We first express the scalar field $\hat{\phi}(x, t)$ in terms of the Fourier modes. We then describe each mode by a wave function ψ_k. As we have pointed out earlier, we are interested only in the case of the free field. So ψ_k satisfies the Schrödinger Equation (12). The complete wavefunctional $\psi(\phi(x))$ is obtained by computing the product of the modes (as in Equation 11). The mean value $\langle q_k \rangle$ and the spread is given by Equations (23) and (24). The energy-momentum tensor for the scalar field is given by

$$T^i_k = (\partial^i\phi)(\partial_k\phi) - \tfrac{1}{2}\delta^i_k(\partial^\alpha\phi)(\partial_\alpha\phi). \tag{78}$$

Since we are only interested in the density perturbations we need to consider only the component T^0_0. From equation (78) we have

$$T^0_0 = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\frac{1}{S^2}(\nabla\phi)^2. \tag{79}$$

The expectation value of T^0_0 in the state $|\psi_2\rangle$ is interpreted as the energy density $\rho(\mathbf{x}, t)$· As we have shown in Appendix 2, spatially dependent part of $\rho(\mathbf{x}, t)$ turns out to be just

$$\rho(\mathbf{x}, t) = \frac{1}{2}\dot{\bar{\phi}}^2 + \frac{1}{2S^2}(\nabla\bar{\phi})^2. \tag{80}$$

(The spread in the quantum state also contributes to the density $\rho(\mathbf{x}, t)$. It, however, contributes terms which do not have any spatial dependence. Hence these terms will not give rise to density inhomogeneities. So we have dropped the terms arising out of the spread in the quantum state. For further details we may refer to Appendix 2.) Here ϕ is the expectation value of $\hat{\phi}_0$ in the Gaussian state $|\psi_2\rangle$. So $\bar{\phi}$ is the same as ϕ_{cl} which we defined earlier (*cf.* Equation 69). As we did there for ϕ_{cl} we can express ϕ as a sum of the homogeneous part ϕ_0 and the inhomogeneous part $\delta\phi$. The Fourier transform of ρ_k is given by

$$\rho_k(t) = \dot{\phi}_0(t)\,\dot{\bar{q}}_k(t) \tag{81}$$

where $\bar{q}_k$ is the Fourier transform of $\delta\phi$. The density contrast in this case turns out to be exactly the same as in Section 6. The spectrum of density perturbations depends on the initial conditions.

In this paper we have shown that it is possible to find Gaussian states which do not give rise to the Zel'dovich spectrum. The spectrum of density perturbations depends on the initial Gaussian state. Conventionally it is believed that a scale-free spectrum for density perturbations is generic to inflation. We have shown that the scale-free spectrum is obtained only for certain special initial conditions on the quantum states of the scalar field, even in the inflationary scenario.

Appendix 1

In Section 1 of this paper we have defined $\rho(\mathbf{x}, t)$ in terms of ϕ_{cl} (x, t). We have constructed ϕ_{cl} as the sum of ϕ_0 (t) and $\delta\phi$ $(\mathbf{x}, t)$ where $\delta\phi$ $(\mathbf{x}, t)$ is the Fourier transform of σ_k (Recall that σ_k is the spread of the Gaussian state for the kth mode.) In arriving at expression for p_k in equation (48) we had dropped the potential-dependent term. Strictly speaking this term should have been included. We show in this appendix that even after dropping that term the result has the correct order of magnitude and that the results are not significantly different from the correct value.

The energy density of the scalar field is given by

$$\rho(\mathbf{x}, t)=\frac{1}{2}\dot{\phi}_0^2+\frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}t}(\delta\phi)\right)^2+\dot{\phi}_0\frac{\mathrm{d}}{\mathrm{d}t}(\delta\phi)+\frac{1}{2}\frac{1}{S^2}(\nabla\delta\phi)^2+V(\phi_0+\delta\phi). \quad \text{(A1.1)}$$

In the approximation $\delta\phi \ll \phi_0$ we will retain terms only upto linear order in $(\delta\phi)$. Dropping the higher powers of $\delta\phi$, we get,

$$\rho(\mathbf{x}, t)=\left[\frac{1}{2}\dot{\phi}_0^2+V(\phi_0)\right]+\left[\left(\dot{\phi}_0\frac{\mathrm{d}\delta\phi}{\mathrm{d}t}\right)+\frac{\partial V(\phi_0)}{\partial\phi_0}\delta\phi\right] \quad \text{(A1.2)}$$

where we have Taylorexpanded $V(\phi_0+\delta\phi)$ around ϕ_0 and retained only upto linear terms in $\delta\phi$. The first term in Equation (A1.2) is space-independent. Hence the complete expression for $\delta\phi$ is given by

$$\delta\rho(\mathbf{x}, t)=\dot{\phi}_0\frac{\mathrm{d}}{\mathrm{d}t}(\delta\phi)+\frac{\partial V(\phi_0)}{\partial\phi_0}\delta\phi. \quad \text{(A1.3)}$$

Fourier transform of both sides of the above expression gives us the following expression for p_k:

$$\rho_k(t)=\dot{\phi}_0\dot{\sigma}_k+\sigma_k V'(\phi_0) \quad \text{(A1.4)}$$

where V' denotes the derivative of $V(\phi)$ with respect to ϕ.

Let us first consider the simple case of an inverted oscillator potential:

$$V(\phi)=V_0-\tfrac{1}{2}\omega^2\phi^2. \quad \text{(A1.5)}$$

We will work in the limit $\omega \ll k/S$. The spread σ_k of the wave packet for the kth mode is given in equation (36) to be

$$\sigma_k(t)=\frac{H}{k^{3/2}\sqrt{2}}\left(1+\left(\frac{k}{HS}\right)^2\right)^{1/2}. \quad \text{(A1.6)}$$

On differentiating this with respect to time we can express $\dot{\sigma}_k$ in terms of the σ_k as

$$\dot{\sigma}_k=-H\sigma_k\frac{\theta_k^2}{1+\theta_k^2} \quad \text{(A1.7)}$$

where $\theta_k = k/(HS).\phi_0$ is the expectation value of the homogeneous mode of scalar field. At late times ϕ_0 goes as

$$\phi_0=\frac{1}{4}\frac{2p+3}{p}\phi_i e^{\frac{1}{2}(2p-3)Ht} \quad \text{(A1.8)}$$

where we have assumed $V_i = 0$. Differentiating with respect to time we get

$$\dot{\phi}_0=\tfrac{1}{2}(2p-3)H\phi_0(t) \quad \text{(A1.9)}$$

where

$$p=\frac{\lambda}{H}=\left(\frac{\omega^2}{H^2}+\frac{9}{4}\right)^{1/2}. \quad \text{(A1.10)}$$

For the inverted oscillator potential (Equation A1.5) we have

$$V' = -\omega^2 \phi. \tag{A1.11}$$

Hence from Equation (A1.4) we express ρ_k as

$$\rho_k(t) = \dot{\phi}_0 \dot{\sigma}_k - \omega^2 \phi_0 \sigma_k. \tag{A1.12}$$

Substituting in Equation (Al.12) the expressions for $\dot{\sigma}_k$ and $\dot{\phi}_0$ from Equations (A1.7) and (A1.9) respectively, we get,

$$\rho_k(t) = -\phi_0 \sigma_k \left(\frac{1}{2}(2p-3) H^2 \frac{\theta_k^2}{1+\theta_k^2} + \omega^2 \right). \tag{A1.13}$$

At the epoch of Hubble radius crossing (*i.e.*, at $t = t_k$), $\theta_k = 2\pi$. From equation(Al.13), $\rho_k(t_k)$ turns out to be,

$$\begin{aligned} \rho_k(t_k) &= -\phi_0(t_k)\sigma_k(t_k)\left(\frac{1}{2}(2p-3) H^2 \frac{4\pi^2}{1+4\pi^2} + \omega^2 \right) \\ &= -\phi_0(t_k)\sigma_k(t_k)\left(\frac{1}{2}(2p-3) H^2 + \omega^2 \right). \end{aligned} \tag{A1.14}$$

As we have repeatedly pointed out, we are interested in the case of $\omega < H$. In this limit,

$$2p - 3 \simeq \frac{2\omega^2}{3H^2} \tag{A1.15}$$

and we have

$$\rho_k(t_k) = -\phi_0(t_k)\sigma_k(t_k)\frac{4\omega^2}{3}. \tag{A1.16}$$

Had we neglected the term arising from the potential in ρ_k we would have got

$$\rho_k(t_k) = -\tfrac{1}{3}\phi_0(t_k)\sigma_k(t_k)\omega^2. \tag{A1.17}$$

So we see that at least for the case of the inverted oscillator potential, dropping the term arising out of the potential in ρ_k does not significantly change the result.

Let us now consider the case of a more general potential

$$V(\phi) = (\eta \phi_f)^4 \left[1 - \lambda \left(\frac{\phi}{\phi_f} \right)^n \right]. \tag{A1.18}$$

We assume that the potential is very flat. So the expression for ρ_k remains as in the previous case

$$\sigma_k = \frac{H}{k^{3/2}\sqrt{2}}(1+\theta_k^2)^{1/2} \tag{A1.19}$$

and

$$\dot{\sigma}_k = -H\sigma_k \frac{\theta_k^2}{1+\theta_k^2}. \tag{A1.20}$$

ϕ_0 will be governed by the classical equation of motion

$$\ddot{\phi}_0 + 3H\dot{\phi}_0 + \left.\frac{\partial V}{\partial \phi}\right|_{\phi=\phi_0} = 0. \tag{A1.21}$$

For the potential given in Equation (Al.18) we have

$$V'(\phi_0) = -(n\eta^4 \lambda \phi_f^{4-n})\phi_0^{n-1}. \tag{A1.22}$$

So the equation of motion for ϕ_0 goes as

$$\ddot{\phi}_0 + 3H\dot{\phi}_0 = (n\eta^4 \lambda \phi_f^{4-n})\phi_0^{n-1}. \tag{A1.23}$$

This, however, cannot be solved exactly. Hence we will use some approximations. We assume that the potential is very flat near $\phi_0 = 0$. In that case the acceleration term $\ddot{\phi}_0$ is much smaller compared to $3H\dot{\phi}_0$. So we will neglect $\ddot{\phi}_0$ in Equation (A1.23). This gives

$$\dot{\phi}_0 = \frac{n\eta^4\lambda}{3}\frac{\phi_f^{4-n}}{H}\phi_0^{n-1}. \tag{A1.24}$$

Using Equations (A1.4), (A1.20) and (A1.22) in (A1.24) we get

$$\rho_k(t) = -\sigma_k \phi_0^{n-1}\left(\frac{1}{3}\frac{\theta_k^2}{1+\theta_k^2} + 1\right)n\eta^4 \lambda \phi_f^{4-n}. \tag{A1.25}$$

At the epoch t_k when the perturbation leaves the Hubble radius we have

$$\rho_k(t_k) = -\sigma_k(t_k)\phi_0^{n-1}\left(\frac{1}{3}\frac{(2\pi)^2}{1+(2\pi)^2} + 1\right)n\eta^4 \lambda \phi_f^{4-n} \tag{A1.26}$$

$$\simeq -\frac{4}{3}\sigma_k(t_k)\phi_0^{n-1}(t_k)n\eta^4 \lambda \phi_f^{4-n}. \tag{A1.27}$$

Had we dropped the contribution from the term containing the potential, we would have got

$$\rho_k(t_k) \simeq -\frac{1}{3}\sigma_k(t_k)\phi_0^{n-1}(t_k)n\eta^4 \lambda \phi_f^{4-n}. \tag{A1.28}$$

Once again we observe that the result does not change significantly by dropping that term.

Appendix 2

In this appendix we will derive an expression for the expectation value of T_0^0 of the scalar field ϕ in the Gaussian state. The action for a free scalar field is given by

$$\mathscr{A} = \int\left(\frac{1}{2}\dot{\phi}^2 - \frac{1}{2S^2}(\bar{\nabla}\phi)^2\right)S^3 \mathrm{d}^3x\,\mathrm{d}t. \tag{A2.1}$$

We have ignored terms which arise from the potential. For our purpose it is sufficient to consider the case of a free field. The action can be expressed in terms of the Fourier modes $\hat{\phi}_k(t)$. To do this we first expand $\hat{\phi}_k(\mathbf{x}, t)$ as

$$\hat{\phi}(\mathbf{x}, t) = \int\frac{\mathrm{d}^3k}{(2\pi)^3}e^{i\mathbf{k}\cdot\mathbf{x}}\hat{\phi}_k(t). \tag{A2.2}$$

In terms of $\phi_k(t)$ the action can be written as

$$\mathscr{A} = \int \frac{d^3k}{(2\pi)^3} dt\, S^3 \left(\frac{1}{2} |\dot{\hat{\phi}}_k|^2 - \frac{1}{2} \left(\frac{k}{S} \right)^2 |\hat{\phi}_k|^2 \right). \tag{A2.3}$$

$\hat{\phi}_k$ are complex quantities. It is easier to work with real quantities instead of $\hat{\phi}_k$. So we express $\hat{\phi}_k(t)$ in terms of its real and imaginary parts

$$\hat{\phi}_k = R_k + i\, I_k. \tag{A2.4}$$

All R_ks and I_k s are not independent. Because of the fact that $\hat{\phi}_k$ is a real field, R_k for $k > 0$ are related to R_k for $k<0$. Similarly, the values of I_k for positive and negative k are related.

The action can now be expressed in terms of R_k and I_k as

$$\mathscr{A} = \int \frac{d^3k}{(2\pi)^3} dt\, S^3 \left\{ \frac{1}{2} (\dot{R}_k^2 + \dot{I}_k^2) - \frac{1}{2S^2} [(kR_k)^2 + (kI_k)^2] \right\}. \tag{A2.5}$$

From this action we can compute the momentum conjugate to R_k and I_k. We will denote them by P_{Rk} and P_{Ik}.

$$P_{R_k} = \left(\frac{S}{2\pi} \right)^3 \dot{R}_k \tag{A2.6}$$

and

$$P_{I_k} = \left(\frac{S}{2\pi} \right)^3 \dot{I}_k. \tag{A2.7}$$

Using the action in Equation (A2.5) we may now write the Schrödinger equation for the kth mode as

$$i \frac{\partial}{\partial t} \psi_k = -\frac{(2\pi)^3}{2S^3} \frac{\partial^2}{\partial R_k^2} \psi_k + \frac{1}{2} \left(\frac{k}{s} \right)^2 \frac{S^3}{(2\pi)^3} R_k^2 \psi_k \tag{A2.8}$$

and a similar equation for I_k.

We assume that ψ_k is a Gaussian of the form,

$$\psi_k = A_k \{ e^{-B_k[(R_k - f_k)^2 + (I_k - g_k)^2]} \}. \tag{A2.9}$$

The full wavefunction for the field is the product of the wavefunctions for all the modes.

The energy density T_0^0 is given by

$$T_0^0 = \frac{1}{2} \dot{\hat{\phi}}^2 + \frac{1}{2S^2} (\nabla \hat{\phi})^2. \tag{A2.10}$$

To evaluate the expectation value of T_0^0 we need to compute the terms $\langle \phi^2 \rangle$ and $\langle (\bar{\nabla} \phi)^2 \rangle$. We have already specified the quantum state of the kth mode. Hence we can compute the expectation value of $\hat{\phi}^2$ and $(\bar{\nabla} \hat{\phi})^2$ A straightforward computation gives us the following result:

$$\langle \dot{\hat{\phi}}^2 \rangle = (\text{term independent of } \mathbf{x}) + \dot{\bar{\phi}}^2, \tag{A2.11}$$

$$\langle (\nabla \hat{\phi})^2 \rangle = 2 \int \frac{d^3k}{(2\pi)^3} \frac{k^2}{S^2} \sigma_k^2 + (\nabla \bar{\phi})^2, \tag{A2.12}$$

where $\bar{\phi}$ denotes expectation value of $\hat{\phi}$. The interesting point in both these equations is that only the second term contributes to spatial dependence, This term depends only on the mean value of the field. So only the second term contributes to the density perturbations. Thus the expectation value of T^0_0 for our purpose can be taken to be

$$\langle T^0_0 \rangle = \frac{1}{2}\dot{\bar{\phi}}^2 + \frac{1}{2S^2}(\nabla\bar{\phi})^2 \quad \text{(A2.13)}$$

which is Equation (80).

References

Bardeen, J. M., Steinhardt, P. J., Turner, M. S. 1983, *Phys. Rev.*, **D28**, 679.

Brandenberger, R. 1985, *Rev.*, *Mod. Phys.*, **57**, 1.

Feynraan, R. P., Hibbs, A. R. 1965, *Quantum Mechanics and Path Integrals*, McGraw Hill, New York.

Guth, A. H. 1981, *Phys. Rev.*, **D23**. 347.

Guth, A. H., So-Y-Pi, 1982, *Phys. Rev. Lett.*, **49**, 1110.

Harrison, E. R. 1970, *Phys. Rev.*, **Dl**, 2726.

Hawking, S. 1982, *Phys. Lett.*, **115B**, 295.

Kazanas, D. 1980, *Astrophys. J.*, **241**, L59.

Landau, L. D., Lifshitz, E. M. 1985, *Quantum Mechanics Non-Relativistic Theory*, Pergamon Press.

Narlikar, J. V., Padmanabhan, T. 1986, *Gravity*, *Gauge Theories and Quantum Cosmology*, D. Reidel, Dordrecht.

Padmanabhan, T., Seshadri, T. R. 1986, *Phys. Rev.*, **D34**, 951.

Padmanabhan, T., Seshadri, T. R., Singh, T. P. 1989, *Phys. Rev.*, **D39**, 2100.

Sato, K. 1981, *Mon. Not. R. astr. Soc*, **195**, 467.

Seshadri, T. R., Padmanabhan, T. 1989, *J. Astrophys. Astr.*, **10**, 391.

Tomonaga, S., 1946, Prog. Th. Phys., **1**, 27.

Wheeler, J. A. 1962, *Geometrodynamics*, *Topics of Modern Physics*, Vol. I, Academic Press, New York.

Zeldovich, Ya, B. 1972, *Mon. Not. R. astr. Soc*, **160**, lp.