EFFECT OF INTERNAL MOTIONS ON THE DECAY OF A MAGNETIC FIELD IN A FLUID CONDUCTOR*

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ABSTRACT

The decay of a magnetic field in a fluid conductor with internal motions is considered in case the magnetic and velocity fields have symmetry about an axis The underlying characteristic value problems are solved for certain simple velocity fields The method of solution is based on a new classification of the basic axisymmetric modes of the magnetic field in terms of Gegenbauer polynomials. The principal conclusion of the paper is that velocity fields of reasonable patterns and magnitudes can alter the time of decay that will obtain in the absence of motions by quite large factors. The bearing of this result on the problem of the origin of the earth's magnetic field is briefly discussed Certain other related questions are also considered.

1. INTRODUCTION

Current interest in the interaction between fluid motions and magnetic fields is in large measure due to the many astrophysical and geophysical phenomena which appear to be manifestations of this interaction. Indeed, the possibilities one might envisage by requiring a fluid to conform simultaneously to the laws of hydrodynamics and electrodynamics are so many that it might be profitable, in the first instance, to begin at the two ends. We might, for example, start from a well-understood hydrodynamic situation and see how the phenomenon is altered progressively as a magnetic field of increasing strength is impressed on the fluid. In recent years several such problems have been studied (cf. Chandrasekhar 1953); and in some cases the theoretical predictions have been confirmed by experiments (Lehnert 1952, 1955; Nakagawa 1955). Or we might start from a well-understood electrodynamic situation and see how the phenomenon is progressively altered as fluid motions of increasing strength are allowed to be present. It is somewhat surprising that no such problem has hitherto been analyzed. It is the object of this paper to consider one such problem. The problem we shall consider is the free decay of a magnetic field in a fluid conductor. If there are no internal motions, the equation governing the decay is

$$\nabla^2 \boldsymbol{H} = \frac{1}{4\pi\sigma} \frac{\partial \boldsymbol{H}}{\partial t}, \tag{1}$$

where σ denotes the electrical conductivity. The solution of this problem is well known: it is, in fact, one of the standard problems in electrodynamics (cf. Stratton 1941). We now ask: How is this decay affected if the fluid is in a state of internal motions? Apart from its theoretical interest, the problem appears to be relevant for discussions relating particularly to the origin of the earth's magnetic field. This context in which the problem first suggested itself is described in the next section.

2. THE PHYSICAL CONTEXT OF THE PROBLEM

The problem of the decay of a magnetic field in a fluid conductor is the first one to which one's attention is directed when one considers the origin of the earth's magnetic field. The reason is the following:

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According to the solution of equation (1), the mode which decays most slowly is characterized by a mean life (cf. eq. [31]):

$$\tau = \frac{4\sigma R^2}{\pi}. \tag{2}$$

It has been estimated by Bullard and Gellman (1954) that the electrical conductivity of the earth's core is 3×10^{-6} e.m.u.; this value of σ, together with the known radius ($=3.5 \times 10^8$ cm) of the core, leads to a mean life

$$\tau \text{ (earth's core) } = 14000 \text{ years }. \tag{3}$$

On the other hand, paleomagnetic studies have established that, apart from secular variations, the earth has retained its magnetic field at approximately its present strength throughout most of its history (except, possibly, for brief periods during reversals; cf. Runcorn 1955*a*, *b*, in this connection). Consequently, there must be some mechanism which maintains the field and prevents its decay.

It was first suggested by Larmor (1919) as a possible explanation that the earth's magnetic field might be maintained by fluid motions in the core in the manner of a self-exciting dynamo. This suggestion received a serious setback when Cowling (1934; see also Backus and Chandrasekhar 1956) proved that steady dynamo action is impossible if the magnetic field and the fluid motions have axial symmetry. The possibility, nevertheless, of dynamo action has been explored, intensively, in recent years by Bullard and Gellman (1954), by Parker (1955), and by Elsasser (1955, 1956).

Bullard avoids a direct conflict with Cowling's theorem by considering velocity and magnetic fields which are azimuth-dependent. More particularly, Bullard and Gellman (1954) have sought to solve the electromagnetic equations numerically by using a particular velocity distribution and adjusting its magnitude to give a steady field. The method of solution is based on expansion in orthogonal functions; and, as Bullard (1955) has himself expressed, "it is difficult to establish the existence of solutions by numerical methods."

In contrast to Bullard, Parker and Elsasser have sought to avoid a conflict with Cowling's theorem by giving up, also, the idea of a strictly steady dynamo. They devise a sequence of interactions—nonuniform rotation generating a toroidal magnetic field from an initial poloidal field; a succession of rising "cyclones" creating, out of the toroidal field, loops of flux in the meridional planes, these loops coalescing and generating a poloidal field—and try to establish by semiquantitative arguments the inherent plausibility of the scheme by examining each element of the sequence separately.

The foregoing brief account of the current investigations on the dynamo problem discloses one surprising lacuna. The lacuna becomes apparent when we restate the underlying arguments as follows: One knows that in the absence of internal motions the earth's magnetic field will decay with a mean life of the order of 14000 years; "therefore," one seeks to devise fluid motions of a pattern and intensity which will prevent the decay altogether and maintain a steady state. In so arguing, one has not sought to clarify to one's self the extent to which given internal motions can retard (or accelerate) the free decay of a magnetic field in a fluid conductor. For example, one can ask: Can the time of decay of 14000 years, in the absence of internal motions, be prolonged to 500000 years (say) by velocity fields of reasonable magnitudes and patterns? If this were possible, the problem of the origin of the earth's magnetic field would take on a different complexion. For, the earth's magnetic field appears to reverse itself at intervals of the order of a quarter- to a half-million years by passing through an intermediate transitory phase of nearly zero random field (cf. Hospers 1954; Runcorn 1955*b*). One can therefore ask whether any particular meaning can be attached to constructing—if one can!—dynamos which will maintain themselves for periods longer than between reversals. In all cases it is clear that the extent to which the decay of a magnetic field in a fluid conductor can be retarded or accelerated by internal motions of assigned patterns is an important theoretical question. It is to answering this question that this paper is principally devoted.

In concluding this section it may be simply stated that the problem considered in this paper is relevant in other astronomical connections, as, for example, the growth and decay of the magnetic fields of sunspots (cf. Cowling 1946); but in these other connections the physical problems are not as direct.

3. THE EQUATIONS OF THE PROBLEM

In considering the problem of the decay of a magnetic field in a fluid conductor with internal motions, we shall restrict ourselves to the case when the fluid is incompressible, the conductivity is constant, and the velocity and the magnetic fields have symmetry about an axis. The equations valid under these conditions have been derived in the preceding paper (Chandrasekhar 1956*b*; this paper will be referred to hereafter as "Paper I") They are (Paper I, eqs. [33] and [34]):

$$\Delta_5 P - \frac{\partial P}{\partial t} = \frac{1}{\varpi^3} \frac{\partial (\varpi^2 P, \varpi^2 U)}{\partial (z, \varpi)} \tag{4}$$

and

$$\Delta_5 T - \frac{\partial T}{\partial t} = \frac{1}{\varpi} \frac{\partial (T, \varpi^2 U)}{\partial (z, \varpi)} - \frac{1}{\varpi} \frac{\partial (V, \varpi^2 P)}{\partial (z, \varpi)}, \tag{5}$$

where it may be recalled that length, time, and velocity are measured in the units (Paper I, eq. [7]),

$$R\,(= 3.47 \times 10^8 \text{ cm}), \quad 4\pi\sigma R^2\,(= 1.44 \times 10^5 \text{ years}), \quad \text{and} \quad (4\pi\sigma R)^{-1}\,(= 7.65 \times 10^{-5} \text{ cm/sec}), \tag{6}$$

respectively (The values given in parentheses are those which are appropriate for the earth's core.) In equations (4) and (5) P and T are the scalars defining the poloidal and toroidal parts of the magnetic field, and U and V are the scalars defining the meridional and rotational motions. Thus (Paper I, eqs. [15] and [16])

$$\boldsymbol{h} = -\varpi \frac{\partial P}{\partial z} \mathbf{1}_\varpi + \varpi T \mathbf{1}_\varphi + \frac{1}{\varpi} \frac{\partial}{\partial \varpi} (\varpi^2 P)\, \mathbf{1}_z \tag{7}$$

and

$$\boldsymbol{v} = -\varpi \frac{\partial U}{\partial z} \mathbf{1}_\varpi + \varpi V \mathbf{1}_\varphi + \frac{1}{\varpi} \frac{\partial}{\partial \varpi} (\varpi^2 U)\, \mathbf{1}_z, \tag{8}$$

where $\boldsymbol{h} = \boldsymbol{H}/(4\pi\rho)^{1/2}$ is a quantity of the dimensions of a velocity.

In considering equations (4) and (5), we shall regard the velocity scalars U and V as given and seek solutions for the scalars P and T characterizing the magnetic field.

It will be observed that there is a fundamental difference in the equations governing the poloidal and toroidal fields: The decay of the poloidal field is unaffected by the presence of toroidal fields or rotational motions; it is affected only by the presence of meridional motions. On the other hand, the decay of the toroidal field is affected by the presence of a poloidal field, provided there is nonuniform rotation.

Considering, first, the equation satisfied by P, we can separate the time dependence by writing

$$P(z, \varpi)\, e^{-\lambda^2 t} \quad \text{in place of} \quad P(z, \varpi, t) \tag{9}$$

and obtain

$$\Delta_5 P + \lambda^2 P = \frac{1}{\varpi^3} \frac{\partial (\varpi^2 P, \varpi^2 U)}{\partial (z, \varpi)} . \tag{10}$$

This equation is valid only for $r \leq 1$. For $r > 1$, the equation governing P is

$$\Delta_5 P = 0 . \tag{11}$$

For a specified velocity field, U, we must seek solutions of equations (10) and (11) which have no singularity at the origin, vanish at infinity, and are such that

$$P \text{ and } \frac{\partial P}{\partial r} \quad \text{are continuous on} \quad r = 1 . \tag{12}$$

These boundary conditions clearly define a characteristic value problem for λ^2; and in the context of the physical problem discussed in § 2, greatest interest is attached to the dependence of the lowest characteristic value for λ^2 on the sign and amplitude of the specified velocity field.

Considering, next, the equation for T, we can express the general solution as a superposition of a solution of the homogeneous equation

$$\Delta_5 T - \frac{\partial T}{\partial t} = \frac{1}{\varpi} \frac{\partial (T, \varpi^2 U)}{\partial (z, \varpi)} , \tag{13}$$

together with a particular solution of the nonhomogeneous equation (5). Equation (13), like equation (10), can be solved by considering exponentially decaying solutions. Thus, writing

$$T (z, \varpi)\, e^{-\lambda^2 t} \quad \text{in place of} \quad T (z, \varpi, t) , \tag{14}$$

we obtain

$$\Delta_5 T + \lambda^2 T = \frac{1}{\varpi} \frac{\partial (T, \varpi^2 U)}{\partial (z, \varpi)} . \tag{15}$$

The boundary conditions with respect to which this equation must be solved are

$$T = 0 \quad \text{on} \quad r = 1 \quad \text{and nonsingular at} \quad r = 0 . \tag{16}$$

This is again a characteristic value problem for λ^2; and its solution will determine the "free decay" of a toroidal field in the absence of nonuniform rotation or poloidal fields. If a poloidal field should be present and the rotation is nonuniform, then there will be an additional "induced" term (proportional to P) in the time variation of T. In particular, if the poloidal field which is present is one of the fundamental modes determined by the characteristic value problem for P, we can find a particular solution for T which will decay exponentially with the same mean life as the inducing field. Thus the decay of a toroidal field will, in general, consist of two terms: a free decay and an induced decay. In this paper we shall not consider this latter term: we shall, in effect, suppose that V is constant.

4. THE SOLUTION OF THE DECAY PROBLEM IN THE ABSENCE OF MOTIONS

The fundamental modes of decay of a magnetic field in a spherical conductor have been the subject of extensive investigations and go back to Horace Lamb (1881). Their classification in terms of the solutions of the vector-wave equation are well known through Elsasser's (1946*a*, *b*, and 1947) systematization. Nevertheless, the particular classification of the axisymmetric modes which we shall need appears to be novel in this connection. We shall therefore begin with a description of this classification.

a) The Poloidal Modes

In the absence of motions the equations governing the poloidal scalar, P, are (cf. eqs. [10] and [11])

$$\Delta_5 P = -\alpha^2 P \qquad (r \leqslant 1) \tag{17}$$

and

$$\Delta_5 P = 0 \qquad (r > 1), \tag{18}$$

where, for later convenience, we have used α^2 in place of λ^2.

In spherical polar co-ordinates the axisymmetric five-dimensional Laplacian, Δ_5, is

$$\Delta_5 = \frac{\partial^2}{\partial r^2} + \frac{4}{r}\frac{\partial}{\partial r} + \frac{1-\mu^2}{r^2}\frac{\partial^2}{\partial \mu^2} - \frac{4\mu}{r^2}\frac{\partial}{\partial \mu}. \tag{19}$$

The fundamental solutions of equation (17) which are free of singularity at the origin are (cf. Chandrasekhar 1956*a*, eq. [17])

$$P_n = \frac{J_{n+3/2}(\alpha r)}{r^{3/2}} C_n^{3/2}(\mu) \qquad (r \leqslant 1), \tag{20}$$

where $C_n^{3/2}(\mu)$ is the Gegenbauer polynomial[1] and $J_{n+3/2}$ is the Bessel function of order $(n+\frac{3}{2})$. The corresponding solution of equation (18) is

$$P_n = \frac{A}{r^{n+3}} C_n^{3/2}(\mu) \qquad (r > 1), \tag{21}$$

where A is a constant to be determined.

The boundary conditions at $r = 1$ (eq. [12]) require

$$J_{n+3/2}(\alpha) = A \qquad (\text{continuity of } P \text{ at } r = 1) \tag{22}$$

and

$$\left\{\frac{d}{dr}[r^{1/2}J_{n+3/2}(\alpha r)]\right\}_{r=1} = \left[\frac{d}{dr}\left(\frac{A}{r^{n+1}}\right)\right]_{r=1} \qquad \left(\text{continuity of } \frac{d}{dr}(r^2 P) \text{ at } r = 1\right). \tag{23}$$

On simplifying equation (23) and eliminating A by means of equation (22), we find

$$\tfrac{1}{2}J_{n+3/2}(\alpha) + \alpha J'_{n+3/2}(\alpha) = -(n+1)A = -(n+1)J_{n+3/2}(\alpha) \tag{24}$$

or

$$\alpha J'_{n+3/2}(\alpha) + (n+\tfrac{3}{2})J_{n+3/2}(\alpha) = 0. \tag{25}$$

But the quantity on the left-hand side of this last equation is $J_{n+1/2}(\alpha)$. Hence the equation for determining α is

$$J_{n+1/2}(\alpha) = 0. \tag{26}$$

The characteristic values of α are, therefore, the roots of $J_{n+1/2}(x)$.

To distinguish the different zeros of the Bessel functions of different orders, we shall let

$$\alpha_{j,m} \text{ denote the } j\text{th zero of } J_{m+3/2}(x). \tag{27}$$

This notation will be used consistently in this paper.

[1] The notation is that of Watson (1944).

The fundamental solutions representing the various poloidal modes are, therefore,

$$P_{n,\,j}=\frac{J_{n+3/2}\left(\alpha_{j,\,n-1}r\right)}{r^{3/2}}\,C_n^{3/2}\left(\mu\right)\,. \tag{28}$$

We shall call this the (n, j) *poloidal mode*. In particular, the solution describing the zero-order modes is

$$P_{0,\,j}=\frac{J_{3/2}\left(\alpha_{j,\,-1}r\right)}{r^{3/2}}, \tag{29}$$

where $\alpha_{j,\,-1}$, being the jth zero of $J_{1/2}(x)$ (which is proportional to $\sin x/\sqrt{x}$), is given by

$$\alpha_{j,\,-1}=j\pi \qquad (j=1,\,2,\,\ldots)\,. \tag{30}$$

The corresponding decay times are (cf. eqs. [6] and [9])

$$\tau_j=\frac{4\pi\sigma R^2}{\alpha_{j,\,-1}^2}=\frac{4\sigma R^2}{j^2\pi} \qquad (j=1,\,2,\,\ldots)\,. \tag{31}$$

The functions $P_{n,\,j}$ provide a complete set of orthogonal functions in the unit sphere in five dimensions. The orthogonality relations are

$$\int_{-1}^{+1}C_n^{3/2}(\mu)\,C_m^{3/2}(\mu)\,\left(1-\mu^2\right)d\mu=\frac{2\,(n+1)\,(n+2)}{2n+3}\,\delta_{mn} \tag{32}$$

and

$$\int_0^1 rJ_{n+3/2}\left(\alpha_{j,\,n-1}r\right)J_{n+3/2}\left(\alpha_{k,\,n-1}r\right)dr=\tfrac{1}{2}\left[J_{n+3/2}\left(\alpha_{j,\,n-1}\right)\right]^2\delta_{jk}\,. \tag{33}$$

b) The Toroidal Modes

In the absence of motions, the toroidal modes of decay are given by the solutions of the equation

$$\Delta_5 T=-\alpha^2 T\,, \tag{34}$$

which are finite at the origin and which satisfy the boundary condition

$$T=0 \qquad \text{on} \qquad r=1\,. \tag{35}$$

Accordingly, the fundamental solutions describing the various toroidal modes are

$$T_{n,\,j}=\frac{J_{n+3/2}\left(\alpha_{j,\,n}r\right)}{r^{3/2}}\,C_n^{3/2}\left(\mu\right)\,. \tag{36}$$

We shall call this the (n, j) *toroidal mode*. In particular, the solution describing the zero-order modes is

$$T_{0,\,j}=\frac{J_{3/2}\left(\alpha_{j,\,0}r\right)}{r^{3/2}}, \tag{37}$$

where $\alpha_{j,\,0}$ is the jth zero of $J_{3/2}(x)$. For later reference we may note here that

$$\alpha_{1,\,0}^2=20.19\,. \tag{38}$$

The lowest toroidal mode, therefore, decays with a mean life which is about half that of the lowest poloidal mode.

The functions $T_{n,j}$, like the functions $P_{n,j}$, form a complete set of orthogonal functions in the unit sphere in five dimensions. The orthogonality relation (33) is now replaced by

$$\int_0^1 r J_{n+3/2}(\alpha_{j,n} r) J_{n+3/2}(\alpha_{k,n} r)\, dr = \tfrac{1}{2} [J'_{n+3/2}(\alpha_{j,n})]^2 \delta_{jk}. \tag{39}$$

c) Some Useful Formulae

We have seen that the fundamental solutions for the poloidal and toroidal modes are expressed in terms of the Gegenbauer polynomials. We shall find, when working with them, the need for the various recurrence relations which these polynomials satisfy. Accordingly, we shall list here, for later reference, the most important of these relations:

$$\mu C_n^{3/2} = \frac{1}{2n+3} [(n+1) C_{n+1}^{3/2} + (n+2) C_{n-1}^{3/2}] , \tag{40}$$

$$(1-\mu^2) \frac{d C_n^{3/2}}{d\mu} = \frac{1}{2n+3} [-n(n+1) C_{n+1}^{3/2} + (n+2)(n+3) C_{n-1}^{3/2}] , \tag{41}$$

$$\begin{aligned} \mu^2 C_n^{3/2} = {} & \frac{(n+1)(n+2)}{(2n+3)(2n+5)} C_{n+2}^{3/2} \\ & + \left[\frac{(n+1)(n+3)}{(2n+3)(2n+5)} + \frac{n(n+2)}{(2n+3)(2n+1)} \right] C_n^{3/2} + \frac{(n+1)(n+2)}{(2n+3)(2n+1)} C_{n-2}^{3/2} , \end{aligned} \tag{42}$$

$$\begin{aligned} \mu(1-\mu^2) \frac{d C_n^{3/2}}{d\mu} = {} & -\frac{n(n+1)(n+2)}{(2n+3)(2n+5)} C_{n+2}^{3/2} + \frac{3n(n+3)}{(2n+5)(2n+1)} C_n^{3/2} \\ & + \frac{(n+1)(n+2)(n+3)}{(2n+3)(2n+1)} C_{n-2}^{3/2} , \end{aligned} \tag{43}$$

$$\begin{aligned} \mu \frac{d}{d\mu} (1-\mu^2) C_n^{3/2} = {} & -\frac{(n+2)^2(n+1)}{(2n+3)(2n+5)} C_{n+2}^{3/2} - \frac{(n+2)(n+1)}{(2n+5)(2n+1)} C_n^{3/2} \\ & + \frac{(n+1)^2(n+2)}{(2n+3)(2n+1)} C_{n-2}^{3/2} . \end{aligned} \tag{44}$$

5. THE INFLUENCE OF FLUID MOTIONS ON THE DIFFERENT MODES OF THE MAGNETIC FIELD

To make the characteristic value problems formulated in § 3 definite, we must specify the velocity field. For this reason it is important that we have some initial idea as to the patterns of fluid motion to which the various modes of the magnetic field are most sensitive. In this section we shall try to obtain this preliminary information by considering the rate of change of an integral property of the field (such as its energy) in the presence of prescribed motions.

a) The Toroidal Modes

In the absence of nonuniform rotation, the equation satisfied by the toroidal scalar is (eq. [5])

$$\Delta_5 T - \frac{\partial T}{\partial t} = \frac{1}{\varpi} [T, \varpi^2 U] , \tag{45}$$

where, for the sake of brevity, we have introduced the notation (cf. Paper I, eq. [71])

$$[\phi, \psi] = \frac{\partial (\phi, \psi)}{\partial (z, \varpi)} . \tag{46}$$

Multiplying equation (45) by $\varpi^3 T$ and integrating over the range of the variables, we have

$$\int\int \varpi^3 T \Delta_5 T d\varpi d z - \frac{1}{2} \frac{d}{dt} \int\int \varpi^3 T^2 d\varpi d z = \int\int \varpi^2 T [T, \varpi^2 U] d\varpi d z . \tag{47}$$

Transforming the integral on the right-hand side of this equation by first using the lemma of Paper I, § 6 (eq. [82]), and then integrating by parts, we obtain

$$\begin{aligned}\int\int \varpi^2 T [T, \varpi^2 U] d\varpi d z &= - \int\int \varpi^2 U [T, \varpi^2 T] d\varpi d z \\ &= - \int\int \varpi^3 U \frac{\partial T^2}{\partial z} d\varpi d z \\ &= + \int\int \varpi^3 T^2 \frac{\partial U}{\partial z} d\varpi d z .\end{aligned} \tag{48}$$

Using this last result in equation (47), we have

$$-\frac{d\mathfrak{T}}{dt} = \int\int \varpi^3 \left(- T \Delta_5 T + T^2 \frac{\partial U}{\partial z} \right) d\varpi d z , \tag{49}$$

where

$$\mathfrak{T} = \tfrac{1}{2} \int\int \varpi^3 T^2 d\varpi d z , \tag{50}$$

is a measure of the energy in the field.

Since U must vanish on the boundary (cf. Paper I, eq. [70]), it is clear that the normal modes of the meridional motions are described by the same functions as the toroidal modes of the magnetic field.

We shall suppose, then, that

$$U = \beta T_{m, k} , \tag{51}$$

where β is a constant, and, further, that

$$T = T_{n\ j} \quad \text{at} \quad t = 0 . \tag{52}$$

For T and U thus prescribed, equation (49) gives

$$-\frac{d}{dt} \mathfrak{T} (n, j; m, k) = \int\int \varpi^3 T_{n, j}^2 \left(\alpha_{j, n}^2 + \beta \frac{\partial T_{m, k}}{\partial z} \right) d\varpi d z , \tag{53}$$

since

$$\Delta_5 T_{n, j} = - \alpha_{j, n}^2 T_{n, j} . \tag{54}$$

By using the orthogonality relations (32) and (39) we can rewrite equation (53) in the form

$$-\frac{d}{dt} \mathfrak{T} (n, j; m, k) = \frac{(n+1)(n+2)}{2n+3} [J'_{n+3/2} (\alpha_{j, n})]^2 \alpha_{j, n}^2 + \beta I (n, j; m, k; T) , \tag{55}$$

where

$$I (n, j; m, k; T) = \int\int \varpi^3 T_{n, j}^2 \frac{\partial T_{m, k}}{\partial z} d\varpi d z \tag{56}$$

is the matrix element which characterizes the interaction between the (n, j) toroidal mode and the (m, k) velocity mode.

In spherical polar co-ordinates, equation (56) takes the form

$$I(n, j; m, k; T) = \int_0^1 \int_{-1}^{+1} r^4 (1-\mu^2) T_{n,j}^2 \left(\mu \frac{\partial}{\partial r} + \frac{1-\mu^2}{r} \frac{\partial}{\partial \mu} \right) T_{m,k} \, dr \, d\mu . \tag{57}$$

Substituting for $T_{m,k}$ in accordance with equation (36) and making use of the recurrence relations (40) and (41), we find

$$\begin{aligned} I(n, j; m, k; T) = \frac{1}{2m+3} \int_0^1 \int_{-1}^{+1} dr \, d\mu \, r^4 (1-\mu^2) T_{n,j}^2 \\ \times \left\{ \frac{d}{dr} \left[\frac{J_{m+3/2}(\alpha_{k,m} r)}{r^{3/2}} \right] [(m+1) C_{m+1}^{3/2} + (m+2) C_{m-1}^{3/2}] \right. \\ \left. + \frac{J_{m+3/2}(\alpha_{k,m} r)}{r^{5/2}} [-m(m+1) C_{m+1}^{3/2} + (m+2)(m+3) C_{m-1}^{3/2}] \right\} . \end{aligned} \tag{58}$$

From equation (58) certain selection rules can be derived from which one can infer the different velocity modes which will interact with a given toroidal mode of the magnetic field and give rise to a nonvanishing matrix element, $I(n, j; m, k; T)$. In this paper we shall, however, restrict ourselves to the case

$$n = 0 \quad \text{when} \quad T = T_{0,j} = \frac{J_{3/2}(\alpha_{j,0} r)}{r^{3/2}} . \tag{59}$$

The general case has, meantime, been treated by Siciy Pao (1956).

Since $T_{0,j}^2$ is angle-independent, it is apparent from equation (58) that only the velocity modes $(1, k)$ can arrest (or accelerate) the initial decay of the magnetic energy in the $(0, j)$ toroidal mode. Thus

$$I(0, j; m, k; T) = 0 \quad \text{for } m \neq 1 , \tag{60}$$

and, when $m = 1$,

$$\begin{aligned} I(0, j; 1, k; T) \\ = \frac{4}{5} \int_0^1 r^4 \left[\frac{J_{3/2}(\alpha_{j,0} r)}{r^{3/2}} \right]^2 \left\{ \frac{d}{dr} \left[\frac{J_{5/2}(\alpha_{k,1} r)}{r^{3/2}} \right] + 4 \frac{J_{5/2}(\alpha_{k,1} r)}{r^{3/2}} \right\} dr . \end{aligned} \tag{61}$$

By making use of the recurrence relations satisfied by the Bessel functions, we can reduce the integral on the right-hand side to give

$$I(0, j; 1, k; T) = \tfrac{4}{5} \alpha_{k,1} \int_0^1 [J_{3/2}(\alpha_{j,0} r)]^2 J_{3/2}(\alpha_{k,1} r) \frac{dr}{\sqrt{r}} . \tag{62}$$

With $I(0, j; 1, k; T)$ given by equation (62), we have (cf. eq. [55])

$$-\frac{d}{dt} \mathfrak{T}(0, j; 1, k) = \tfrac{2}{3} [J'_{3/2}(\alpha_{j,0})]^2 \alpha_{j,0}^2 + \beta I(0, j; 1, k; T) . \tag{63}$$

We shall return to this equation in § 5, c, after we have derived a similar formula for the poloidal modes.

b) The Poloidal Modes

The equation satisfied by the poloidal scalar is (eq. [4])

$$\Delta_5 P - \frac{\partial P}{\partial t} = \frac{1}{\varpi^3} [\varpi^2 P, \varpi^2 U] . \tag{64}$$

Multiplying this equation by $\varpi^3 P$ and integrating over the range of the variables, we obtain, as in § 5, *a* (cf. eq. [49]),

$$-\frac{d\mathfrak{P}}{dt} = \int\int \varpi^3 \left(-P\Delta_5 P - P^2 \frac{\partial U}{\partial z} \right) d\varpi dz , \tag{65}$$

where

$$\mathfrak{P} = \tfrac{1}{2} \int\int \varpi^3 P^2 d\varpi dz . \tag{66}$$

We shall suppose that

$$P = P_{n,j} \qquad \text{at } t = 0 \tag{67}$$

and that the velocity field is the same as that considered in § 5, *a*, equation (51). Then equation (65) gives

$$-\frac{d}{dt} \mathfrak{P}(n, j; m, k) = \int\int \varpi^3 P_{n,j}^2 \left(a_{j,n-1}^2 - \beta \frac{\partial T_{m,k}}{\partial z} \right) d\varpi dz ; \tag{68}$$

and, by using the orthogonality relation (33), we obtain

$$-\frac{d}{dt} \mathfrak{P}(n, j; m, k) = \frac{(n+1)(n+2)}{2n+3} [J_{n+3/2}(a_{j,n-1})]^2 a_{j,n-1}^2 - \beta I(n, j; m, k; P) , \tag{69}$$

where

$$I(n, j; m, k; P) = \int\int \varpi^3 P_{n,j}^2 \frac{\partial T_{m,k}}{\partial z} d\varpi dz . \tag{70}$$

For the case $n = 0$, we find, as in § 5, *a*, that the matrix element

$$I(0, j; m, k; P) = 0 \qquad \text{if } m \neq 1 \tag{71}$$

and that, when $m = 1$ (cf. eq. [62]),

$$I(0, j; 1, k; P) = \tfrac{4}{5} a_{k,1} \int_0^1 [J_{3/2}(a_{j,-1} r)]^2 J_{3/2}(a_{k,1} r) \frac{dr}{\sqrt{r}} . \tag{72}$$

With $I(0, j; 1, k; P)$ given by equation (72), we have (cf. eq. [69])

$$-\frac{d}{dt} \mathfrak{P}(0, j; 1, k) = \tfrac{2}{3} [J_{3/2}(a_{j,-1})]^2 a_{j,-1}^2 - \beta I(0, j; 1, k; P) , \tag{73}$$

which is the present analogue of equation (63).

c) Numerical Results

The matrix elements $I(0, j; 1, k; T)$ and $I(0, j; 1, k; P)$ have been evaluated numerically for the cases $j = 1$ and $k = 1$, 2, and 3. The resulting numerical forms of equations (63) and (73) are

$$-\frac{d}{dt} \mathfrak{T}(0, 1; 1, k) = \begin{cases} 1.817 + 0.29660\beta & (k = 1) , \\ 1.817 + 0.05413\beta & (k = 2) , \\ 1.817 - 0.00746\beta & (k = 3) , \end{cases} \tag{74}$$

and

$$-\frac{d}{dt}\mathfrak{P}(0,\,1;\,1,\,k)=\begin{cases}\frac{4}{3}-0.14112\beta & (k=1)\,,\\ \frac{4}{3}+0.03446\beta & (k=2)\,,\\ \frac{4}{3}-0.01982\beta & (k=3)\,.\end{cases} \tag{75}$$

Remembering that the unit of velocity appropriate for the earth's core is 7.6×10^{-5} cm/sec and that velocities of the order of 10^{-2} cm/sec are believed to exist, we can conclude from equations (74) and (75) that the prevalence of motions of the order contemplated will profoundly alter the rate of decay of the earth's magnetic field such as will obtain in the total absence of motions.

Two further points which are worth noting are, first, that we can retard or accelerate the initial rate of decay by simply changing the sign of β (i.e., the sense of the meridional circulation) and, second, that the lowest velocity mode (1, 1), which interacts with both the (0, 1) poloidal and the (0, 1) toroidal modes, affects them in *opposite* ways—a fact which should be kept in mind if we should wish to select a velocity field which will retard the decay of both the poloidal and the toroidal fields.

6. THE EFFECT OF THE VELOCITY MODE $(1, i)$ ON THE DECAY OF TOROIDAL MAGNETIC FIELDS

We now return to the principal problem of the paper, namely, that of determining the effect of internal motions on the decay of a magnetic field.

In this section we shall consider the effect of meridional currents on the decay of toroidal magnetic fields. As we have seen in § 3, this leads us directly to the following characteristic value problem: To determine λ^2 so that the equation

$$\Delta_5 T+\lambda^2 T=\frac{1}{\varpi}\frac{\partial\,(T,\,\varpi^2 U)}{\partial\,(z,\,\varpi)} \tag{76}$$

allows solutions which are free of singularity at the origin and which vanish on the unit sphere.

In spherical polar co-ordinates, equation (76) has the form

$$\Delta_5 T+\lambda^2 T=-\frac{\partial T}{\partial r}\left[\frac{\partial}{\partial\mu}(1-\mu^2)\,U\right]+\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 U)\left[(1-\mu^2)\frac{\partial T}{\partial\mu}\right], \tag{77}$$

where Δ_5 is now given by equation (19).

Now, to make the characteristic value problem definite, we must specify a velocity field. Since the preliminary discussion in § 5 has shown that the lowest toroidal mode, (0, 1), is sensitive mostly to the presence of the velocity modes $(1, i)$, we shall suppose that this is the pattern of the velocity field which is present. In other words, we shall assume that

$$U=\beta\,\frac{J_{5/2}(\alpha_{i,1} r)}{r^{3/2}}\,\mu=\beta\,U_1(r)\,\mu \qquad \text{(say)}\,, \tag{78}$$

where β is a constant. For this chosen form of U, equation (77) becomes

$$\Delta_5 T+\lambda^2 T=\beta\left[(3\mu^2-1)\,U_1\frac{\partial T}{\partial r}+\mu(1-\mu^2)\frac{\partial T}{\partial\mu}\frac{1}{r^2}\frac{d}{dr}(r^2 U_1)\right]; \tag{79}$$

and in the solution of this equation we shall mostly be interested in the dependence of the lowest characteristic value of λ^2 on β.

The method of solution we shall adopt consists in expanding T in terms of the basic toroidal functions $T_{n,j}$. It is, however, convenient to carry out this expansion in two steps: First, we expand T in the Gegenbauer polynomials $C_n^{3/2}(\mu)$ in the manner

$$T=\sum_{n=0}^{\infty} T_n(r)\,C_n^{3/2}(\mu)\,, \tag{80}$$

and then expand the radial functions, $T_n(r)$, in terms of $J_{n+3/2}(\alpha_{j,n}r)$.

For T given by equation (80),

$$\Delta_5 T=\sum_{n=0}^{\infty} \mathfrak{D}_n T_n C_n^{3/2}(\mu)\,, \tag{81}$$

where

$$\mathfrak{D}_n=\frac{d^2}{dr^2}+\frac{4}{r}\frac{d}{dr}-\frac{n(n+3)}{r^2}. \tag{82}$$

Substituting the expansions (80) and (81) in equation (79), we get

$$\begin{aligned}\sum_{n=0}^{\infty} \mathfrak{D}_n T_n C_n^{3/2}+\lambda^2\sum_{n=0}^{\infty} T_n C_n^{3/2}&\\ =\beta\Big[U_1\sum_{n=0}^{\infty}\frac{dT_n}{dr}(3\mu^2-1)\,C_n^{3/2}&+\frac{1}{r^2}\frac{d}{dr}(r^2U_1)\sum_{n=0}^{\infty}T_n\mu(1-\mu^2)\frac{dC_n^{3/2}}{d\mu}\Big].\end{aligned} \tag{83}$$

Using the recurrence relations (42) and (43) and equating the terms in the different orders of the Gegenbauer polynomials (which we can, in view of their orthogonality), we find

$$\begin{aligned}\mathfrak{D}_n T_n+\lambda^2 T_n&\\ =\beta\Big\{&\frac{n(n-1)}{(2n-1)(2n+1)}\Big[3U_1\frac{dT_{n-2}}{dr}-(n-2)\,T_{n-2}\frac{1}{r^2}\frac{d}{dr}(r^2U_1)\Big]\\ &+\frac{(n+3)(n+4)}{(2n+3)(2n+7)}\Big[3U_1\frac{dT_{n+2}}{dr}+(n+5)\,T_{n+2}\frac{1}{r^2}\frac{d}{dr}(r^2U_1)\Big]\\ &+\Big[\frac{3(n+1)(n+3)}{(2n+3)(2n+5)}+\frac{3n(n+2)}{(2n+3)(2n+1)}-1\Big]U_1\frac{dT_n}{dr}\\ &+\frac{3n(n+3)}{(2n+1)(2n+5)}T_n\frac{1}{r^2}\frac{d}{dr}(r^2U_1)\Big\}.\end{aligned} \tag{84}$$

We now expand $T_n(r)$ in terms of $J_{n+3/2}(\alpha_{j,n}r)$ in the manner

$$T_n(r)=\sum_{j=1}^{\infty} A_{n,j}\frac{J_{n+3/2}(\alpha_{j,n}r)}{r^{3/2}}, \tag{85}$$

where $A_{n,j}$ are constants. For T_n given by equation (85),

$$\mathfrak{D}_n T_n=-\sum_{j=1}^{\infty}\alpha_{j,n}^2 A_{n,j}\frac{J_{n+3/2}(\alpha_{j,n}r)}{r^{3/2}} \tag{86}$$

and

$$r^{3/2}\frac{dT_n}{dr}=\frac{1}{2n+3}\sum_{j=1}^{\infty}A_{n,j}\alpha_{j,n}\left[nJ_{n+1/2}(\alpha_{j,n}r)-(n+3)J_{n+5/2}(\alpha_{j,n}r)\right] \tag{87}$$

$$=\frac{1}{2n+3}\sum_{j=1}^{\infty}A_{n,j}\alpha_{j,n}\Phi_{n,j} \qquad \text{(say)}.$$

Also it can be verified that

$$\frac{d}{dr}(r^2U_1)=\tfrac{1}{5}\alpha_{i,1}\left[3J_{3/2}(\alpha_{i,1}r)-2J_{7/2}(\alpha_{i,1}r)\right]\sqrt{r}=\mathfrak{U}(r)\sqrt{r}\quad\text{(say)}. \tag{88}$$

Inserting equations (86)–(88) in equation (84), we have

$$\begin{aligned}
&\sum_{j=1}^{\infty}(-\alpha_{j,n}^2+\lambda^2)A_{n,j}J_{n+3/2}(\alpha_{j,n}r)\\
&=\frac{\beta}{r^{3/2}}\Bigg[\!\!\Bigg[\frac{n(n-1)}{(2n-1)(2n+1)}\Bigg\{-(n-2)\mathfrak{U}(r)\sum_{k=1}^{\infty}A_{n-2,k}J_{n-1/2}(\alpha_{k,n-2}r)\\
&\qquad+\frac{3J_{5/2}(\alpha_{i,1}r)}{2n-1}\sum_{k=1}^{\infty}A_{n-2,k}\alpha_{k,n-2}\Phi_{n-2,k}\Bigg\}\\
&\quad+\frac{(n+3)(n+4)}{(2n+5)(2n+7)}\Bigg\{(n+5)\mathfrak{U}(r)\sum_{k=1}^{\infty}A_{n+2,k}J_{n+7/2}(\alpha_{k,n+2}r)\\
&\qquad+\frac{3J_{5/2}(\alpha_{i,1}r)}{2n+7}\sum_{k=1}^{\infty}A_{n+2,k}\alpha_{k,n+2}\Phi_{n+2,k}\Bigg\}\\
&\quad+\frac{3n(n+3)}{(2n+1)(2n+5)}\mathfrak{U}(r)\sum_{k=1}^{\infty}A_{n,k}J_{n+3/2}(\alpha_{k,n}r)\\
&+\left\{\frac{3(n+1)(n+3)}{(2n+3)(2n+5)}+\frac{3n(n+2)}{(2n+3)(2n+1)}-1\right\}\frac{J_{5/2}(\alpha_{i,1}r)}{2n+3}\sum_{k=1}^{\infty}A_{n,k}\alpha_{k,n}\Phi_{n,k}\Bigg]\!\!\Bigg].
\end{aligned} \tag{89}$$

Multiplying equation (89) by $rJ_{n+3/2}(\alpha_{j,n}r)$ and integrating from 0 to 1, we get

$$\begin{aligned}
&\tfrac{1}{2}(-\alpha_{j,n}^2+\lambda^2)\left[J'_{n+3/2}(\alpha_{j,n})\right]^2A_{n,j}\\
&=\beta\Bigg[\frac{n(n-1)}{(2n-1)(2n+1)}\sum_{k=1}^{\infty}A_{n-2,k}P(n-2,k;n,j)+\sum_{k=1}^{\infty}A_{n,k}P(n,k;n;k)\\
&\qquad+\frac{(n+3)(n+4)}{(2n+5)(2n+7)}\sum_{k=1}^{\infty}A_{n+2,k}P(n+2,k;n,j)\Bigg],
\end{aligned} \tag{90}$$

where the expressions for the various matrix elements on the right-hand side can be written down by comparison with equation (89).

Equation (90) clearly provides an infinite determinant for λ^2. It is apparent that the matrix is reducible if we include terms only in $n = 0$ and 1. For this reason, it would appear that we can get a good approximation to the lowest characteristic root by considering only the submatrix $(0, k; 0, j)$. In this approximation the characteristic determinant simplifies considerably and is given by (cf. eqs. [89] and [90])

$$\tfrac{1}{2}\left(-\alpha_{j,0}^{2}+\lambda^{2}\right)\left[J'_{3/2}(\alpha_{j,0})\right]^{2} A_{0,j}=\beta \sum_{k=1}^{\infty} A_{0,k} P(0, k; 0, j)\,, \tag{91}$$

where

$$P(0, k; 0, j)=\tfrac{2}{5} \alpha_{k,0} \int_{0}^{1} J_{5/2}(\alpha_{i,1} r) J_{5/2}(\alpha_{k,0} r) J_{3/2}(\alpha_{j\,0} r) \frac{d r}{\sqrt{r}}\,. \tag{92}$$

The characteristic determinant provided by equation (91) has been solved numerically for the cases $i = 1$ and 2. The results are summarized in Tables 1 and 2, where

TABLE 1

THE EFFECT OF THE VELOCITY MODE (1, 1) ON THE DECAY OF TOROIDAL MAGNETIC FIELDS

THE LOWEST CHARACTERISTIC ROOT λ^2 FOR VARIOUS VALUES OF THE AMPLITUDE, β, OF THE VELOCITY MODE

β	λ^2 2d Approx.	λ^2 3d Approx.	λ^2 4th Approx.	β	λ^2 2d Approx.	λ^2 3d Approx.	λ^2 4th Approx.
0*	20 19	20 19	20 19	0*	20 19	20 19	20 19
− 2	18 03	18 04	18 04	+ 5	25 92	25 90	25 90
− 4	15 96	15 98	15 98	+ 10	32 1	31 9	31 9
− 6	13 98	14 03	14 03	+ 15	38 6	38 0	38 0
− 8	12 11	12 22	12 22	+ 20	45 6	44 0	44 0
−10	10 35	10 54	10 50	+ 30	60 8	55 6	56 0
−12	8 73	9 02	8 95	+ 40	78 3	66 6	67 6
−14	7 28	7 67	7 53	+ 50	103	77 1	79 2
−16	6 06	6 50	6 24	+ 60		87 3	90 8
−18	5 13	5 52	5.10	+ 70		97 2	102
−20	4 70	4 72	4 07	+100		126	138
−22		4 09	3 15				
−24		3 58	2 32				
−26		3 14	1 55				
−28		2 69	0 84				
−30		2 14	0 17				

* The value for $\beta = 0$ is exact.

the different "approximations" refer to the number of rows and columns in the characteristic matrix which were included in the evaluation of the characteristic root. The convergence of the characteristic roots toward a limiting value as we go to higher approximations appears satisfactory, particularly in the case $i = 1$. In any case, it would appear that the general nature of the dependence of λ^2 on β (illustrated in Figs. 1 and 2) has been established. We shall return to a discussion of this dependence in § 8.

7. THE EFFECT OF THE VELOCITY MODE $(1, i)$ ON THE DECAY OF POLOIDAL MAGNETIC FIELDS

In this case the equation to be solved is

$$\Delta_5 P+\lambda^2 P=\frac{1}{\varpi^3} \frac{\partial(\varpi^2 P, \varpi^2 U)}{\partial(z, \varpi)}\,; \tag{93}$$

TABLE 2

THE EFFECT OF THE VELOCITY MODE (1, 2) ON THE DECAY OF TOROIDAL MAGNETIC FIELDS

THE LOWEST CHARACTERISTIC ROOT λ^2 FOR VARIOUS VALUES OF THE AMPLITUDE, β, OF THE VELOCITY MODE

β	λ^2 2d Approx.	λ^2 3d Approx.	λ^2 4th Approx.	β	λ^2 2d Approx.	λ^2 3d Approx.	λ^2 4th Approx.
0*	20 19	20 19	20 19	0*	20 19	20 19	20 19
−1	19 94	19 95	19 95	+5	20 34	20 49	20 48
−2	19 60	19 63	19 64	+10	19 46	19 86	19 88
−3	19 13	19 21	19 23	+15	18 1	18 7	18 9
−4	18 53	18 67	18 73	+20	16 4	17 3	17 7
−5	17 78	18 00	18 12	+25	14 6	15 6	16 4
−6	16 9	17 2	17 4	+30	12 7	13 8	15 1
−7	15 8	16 2	16 6	+35	10 8	11 8	13 7
−8	14 5	14 9	15 8	+40	8 8	9 8	12 4
−9	13 1	13 5	14 8	+45	6 7	7 7	11 2
−10	11 5	11 8	13 9	+50	4 7	5 6	9 9
−11	9 7	9 9	13 0	+55	2 6	3 4	8 7
−12	7 8	7 8	12 1	+60	0 5	1 2	7.6
−13	5 7	5 4	11 4				
−14	3 6	2 8	11 0				
−15	1 3	0 1	10 9				
−16	..	..	11.4				

* The value for $\beta = 0$ is exact.

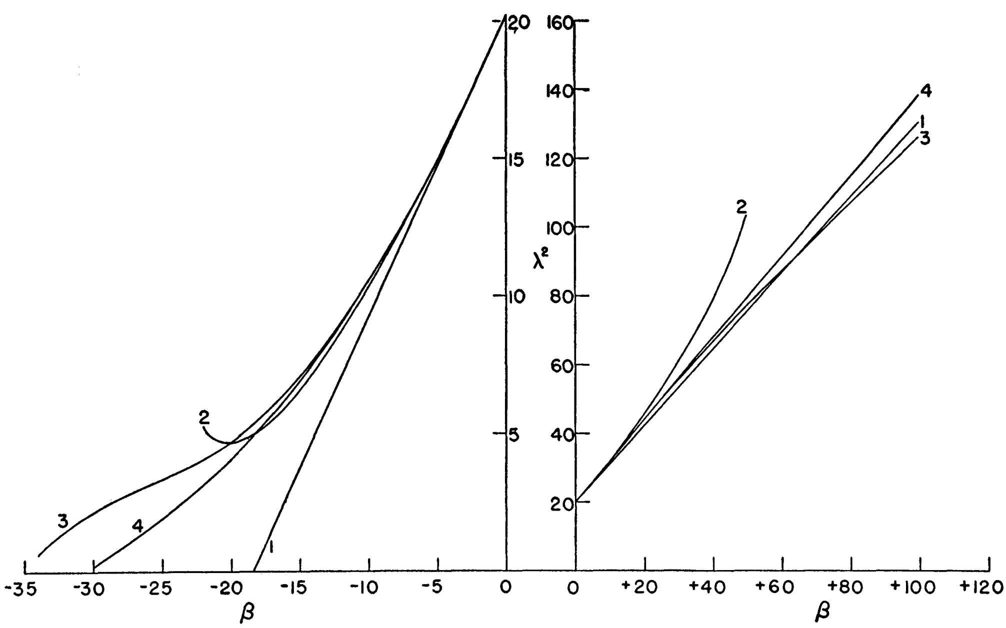

FIG. 1.—The effect of the velocity mode (1, 1) on the decay of toroidal magnetic fields. The lowest characteristic root λ^2 for various values of the amplitude, β, of the velocity mode. The curves are labeled by the order of the approximation on which they are derived.

and the boundary conditions are that the solution of this equation, which is finite at the origin, and its normal derivative both join continuously (on the unit sphere) a solution of the equation

$$\Delta_5 P = 0 , \tag{94}$$

which vanishes at infinity. In spherical polar co-ordinates, the equation is

$$\Delta_5 P + \lambda^2 P = -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 P)\left[\frac{\partial}{\partial \mu}(1-\mu^2)\,U\right] + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 U)\left[\frac{\partial}{\partial \mu}(1-\mu^2)P\right] ; \tag{95}$$

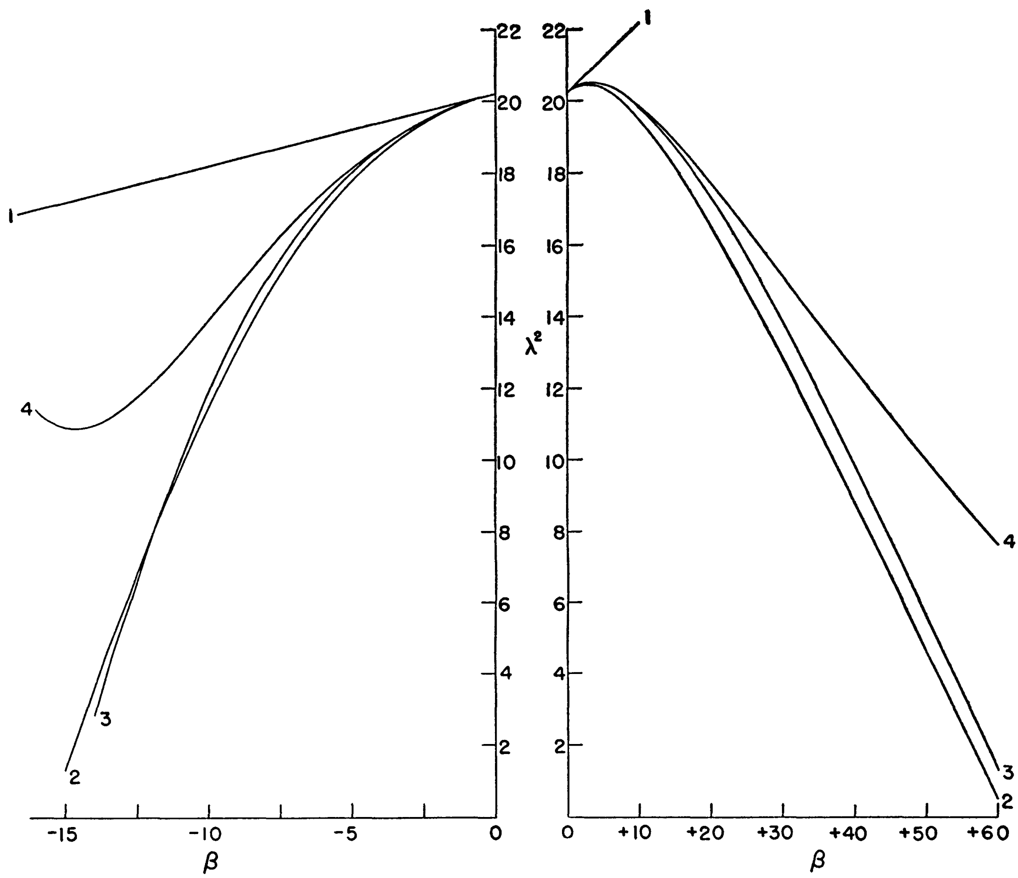

FIG. 2.—The effect of the velocity mode (1, 2) on the decay of toroidal magnetic fields. The lowest characteristic root λ^2 for various values of the amplitude, β, of the velocity mode. The curves are labeled by the order of the approximation on which they are derived

and with the same assumption regarding the velocity field as in § 6 (eq. [78]) we have

$$\Delta_5 P + \lambda^2 P = \beta\left\{(3\mu^2 - 1)\,U_1\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 P)\right] + \left[\frac{1}{r^2}\frac{d}{d r}(r^2 U_1)\right]\mu\frac{\partial}{\partial \mu}(1-\mu^2)P\right\} . \tag{96}$$

In solving equation (96), we shall expand P in terms of the basic poloidal functions $P_{n,j}$. Thus, letting

$$P = \sum_{n=0}^{\infty}\sum_{j=1}^{\infty} A_{n,j}\,\frac{J_{n+3/2}(a_{j,n-1}r)}{r^{3/2}}\,C_n^{3/2}(\mu) , \tag{97}$$

where the $A_{n,j}$'s are constants, we find, as in § 6, that we are led to the following infinite system of equations:

$$\sum_{j=1}^{\infty}(-\alpha_{j,n-1}^{2}+\lambda^{2})A_{n,j}J_{n+3/2}(\alpha_{j,n-1}r)$$
$$=\frac{\beta}{r^{3/2}}\left[\left[\frac{n(n-1)}{(2n-1)(2n+1)}\left\{-n\mathfrak{U}(r)\sum_{k=1}^{\infty}A_{n-2,k}J_{n-1/2}(\alpha_{k,n-3}r)\right.\right.\right.$$
$$\left.+\frac{3J_{5/2}(\alpha_{i,1}r)}{2n-1}\sum_{k=1}^{\infty}A_{n-2,k}\alpha_{k,n-3}\Psi_{n-2,k}\right\}$$
$$+\frac{(n+3)(n+4)}{(2n+5)(2n+7)}\left\{(n+3)\mathfrak{U}(r)\sum_{k=1}^{\infty}A_{n+2,k}J_{n+7/2}(\alpha_{k,n+1}r)\right. \tag{98}$$
$$\left.+\frac{3J_{5/2}(\alpha_{i,1}r)}{2n+7}\sum_{k=1}^{\infty}A_{n+2,k}\alpha_{k,n+1}\Psi_{n+2,k}\right\}$$
$$-\frac{(n+1)(n+2)}{(2n+1)(2n+5)}\mathfrak{U}(r)\sum_{k=1}^{\infty}A_{n,k}J_{n+3/2}(\alpha_{k,n-1}r)$$
$$\left.\left.+\left\{\frac{3(n+1)(n+3)}{(2n+3)(2n+5)}+\frac{3n(n+2)}{(2n+3)(2n+1)}-1\right\}\frac{J_{5/2}(\alpha_{i,1}r)}{2n+3}\sum_{k=1}^{\infty}A_{n,k}\alpha_{k,n-1}\Psi_{n,k}\right]\right]$$

where

$$\Psi_{n,k}=[(n+2)J_{n+1/2}(\alpha_{k,n-1}r)-(n+1)J_{n+5/2}(\alpha_{k,n-1}r)] \tag{98'}$$

and $\mathfrak{U}(r)$ has the same meaning as in equation (88).

Multiplying equation (98) by $rJ_{n+3/2}(\alpha_{j,n-1}r)$ and integrating from 0 to 1, we get

$$\tfrac{1}{2}(-\alpha_{j,n-1}^{2}+\lambda^{2})[J_{n+3/2}(\alpha_{j,n-1})]^{2}A_{n,j}$$
$$=\beta\left[\frac{n(n-1)}{(2n-1)(2n+1)}\sum_{k=1}^{\infty}A_{n-2,k}Q(n-2,k;n,j)+\sum_{k=1}^{\infty}A_{n,k}Q(n,k;n,j)\right. \tag{99}$$
$$\left.+\frac{(n+3)(n+4)}{(2n+5)(2n+7)}\sum_{k=1}^{\infty}A_{n+2,k}Q(n+2,k;n,j)\right],$$

where the expressions for the various matrix elements on the right-hand side can be written down by comparison with equation (98). Equation (99) provides the required characteristic determinant for λ^2.

Again, in view of the structure of the characteristic matrix given by equation(99), it appears that we can (as in § 6) obtain a satisfactory approximation to the lowest characteristic root by considering only the submatrix $(0, k; 0, j)$. In this approximation the characteristic determinant is given by

$$\tfrac{1}{2}(-\alpha_{j,-1}^{2}+\lambda^{2})[J_{3/2}(\alpha_{j,-1})]^{2}A_{0\,j}=\beta\sum_{k=1}^{\infty}A_{0,k}Q(0,k;0,j), \tag{100}$$

where (cf. eqs. [98] and [99])

$$Q(0,\, k;\, 0,\, j) =$$
$$-\frac{2\,\alpha_{k,-1}}{15}\int_0^1 J_{5/2}(\alpha_{i,1}r)\,[2J_{1/2}(\alpha_{k,-1}r)-J_{5/2}(\alpha_{k,-1}r)]\,J_{3/2}(\alpha_{j,-1}r)\frac{dr}{\sqrt{r}}$$
$$-\frac{2\,\alpha_{i,1}}{25}\int_0^1 [3J_{3/2}(\alpha_{i,1}r)-2J_{7/2}(\alpha_{i,1}r)]\,J_{3/2}(\alpha_{k,-1}r)\,J_{3/2}(\alpha_{j,-1}r)\frac{dr}{\sqrt{r}}. \tag{101}$$

The characteristic determinant provided by equation (100) has been solved numerically for the cases $i = 1$ and 2. The results are summarized in Tables 3 and 4 (and

TABLE 3

THE EFFECT OF THE VELOCITY MODE (1, 1) ON THE DECAY OF POLOIDAL MAGNETIC FIELDS

THE LOWEST CHARACTERISTIC ROOT λ^2 FOR VARIOUS VALUES OF THE AMPLITUDE, β, OF THE VELOCITY MODE

β	λ^2		β	λ^2	
	2d Approx	3d Approx.		2d Approx.	3d Approx.
0*	9 870	9 870	0*	9 870	9 870
− 2	10 549	10 552	+ 2	9 153	9 154
− 4	11 198	11 212	+ 4	8 391	8 387
− 6	11 82	11 86	+ 6	7 571	7 548
−10	13 01	13 15	+ 8	6 68	6 61
−15	14 4	14 8	+10	5 70	5 56
−20	15 8	16 5	+12	4 60	4 38
−30	18 4	20 0	+14	3 37	3 05
			+15	2 70	2 34
			+16	1 98	1 59
			+17	1 22	0 82
			+18	0 41	0 02

* The value for $\beta = 0$ is exact.

illustrated in Figs. 3 and 4), where the different "approximations" again refer to the number of rows and columns in the characteristic matrix which were included in the evaluation of the characteristic root.

8. SOME CONCLUSIONS AND SOME QUESTIONS

In considering the results summarized in Tables 1–4 and Figures 1–4, we should, of course, remember that they have been derived for particular velocity fields. The velocity modes assumed are the (1, 1) and the (1, 2) modes. These modes, in the usual treatments of thermal convection (Chandrasekhar 1952; Backus 1955), will be described as belonging to $l = 2$. And it is known that the first mode to be excited by thermal instability in a fluid sphere is $l = 1$ and $j = 1$; however, the Rayleigh numbers required to excite the modes $l = 2$ are not very much larger: thus, for a sphere with a rigid boundary, the Rayleigh numbers for the onset of instability are 8.04×10^3 for $l = 1$ and 1.04×10^4 and 1.33×10^4 for the two modes (1, 1) and (1, 2) belonging to $l = 2$. Therefore, the velocity modes assumed in §§ 6 and 7 can very well prevail.

The principal conclusion to be drawn from the results of §§ 6 and 7 is, then, that, with suitable amplitudes and signs, likely patterns of meridional currents can alter the times of decay of a magnetic field by quite large factors. It would, indeed, appear that factors of the order of 20 or more can be achieved by $|\beta| \sim 30$. Values of β of this order correspond to velocities which are very moderate. Thus in the earth's core (where the unit of velocity adopted has the value 7.7×10^{-5} cm/sec) $|\beta| \sim 30$ corresponds to velocities of the order of 2×10^{-4} cm/sec; and this is five or six times as small as the velocities which are believed to be present.

The conclusion, as it pertains to the earth, can be stated somewhat differently as follows: Velocities that are believed to prevail in the earth's core correspond to $|\beta| \sim 200$. From the results derived in §§ 6 and 7, it would certainly appear that velocities of

TABLE 4

THE EFFECT OF THE VELOCITY MODE (1, 2) ON THE DECAY OF POLOIDAL MAGNETIC FIELDS

THE LOWEST CHARACTERISTIC ROOT λ^2 FOR VARIOUS VALUES OF THE AMPLITUDE, β, OF THE VELOCITY MODE

β	λ^2		β	λ^2	
	2d Approx.	3d Approx.		2d Approx.	3d Approx.
0*	9 870	9 870	0*	9 870	9 870
− 3	9 584	9 577	+ 1	9 950	9 949
− 5	9 37	9 36	+ 2	10 02	10 01
−10	8 77	8 76	+ 3	10 08	10 06
−15	8 12	8 14	+ 4	10 13	10 08
−20	7.45	7.51	+ 5	10 16	10 06
−25	6 75	6.90	+ 6	10 17	9 98
−30	6 04	6 29	+ 7	10 15	9 83
−35	5 32	5 69	+ 8	10 1	9 5
−40	4 53	5 10	+ 9	10 0	9 0
−45	3 87	4 52	+10	9.9	8 2
−50	3 14	3 95	+11	9 6	7 0
−55	2 40	3 40	+12	9 3	5 3
−60	1 66	2 85	+13	8 9	
−65	0 92	2.30	+14	8 4	
−70	0 18	1 77	+15	7 7	
−75	...	1 24	+18	4 7	

* The value for $\beta = 0$ is exact.

this magnitude (if of proper pattern and sign) can prolong the times of decay by factors of the order of 50 or more. To the question raised in § 2: "Can the time of decay of 14000 years, in the absence of internal motions, be prolonged to 500000 years (say) by velocities of reasonable magnitudes and patterns?" the answer would seem to be "Yes"; and if the answer is "Yes," the subsequent remarks in § 2 relative to the origin of the earth's magnetic field should have substance.

While the foregoing represents the principal conclusion of this paper, the analysis has disclosed aspects of the decay of a magnetic field in the presence of fluid motions which are unexpected and which raise a number of further questions. We may comment on some of these.

First, we may notice that, in agreement with what was indicated in § 5, the velocity mode (1, 1) has opposite effects on the poloidal and the toroidal fields: thus meridional circulation of a sense which retards the decay of poloidal fields accelerates the decay of toroidal fields, and conversely. At first sight, this might appear an unfortunate fact, as one might have hoped that the principal velocity pattern which retards the decay of the poloidal field will also retard the decay of the toroidal field; but this does not appear to be the case. However, it should be remembered in this connection that, if the fluid is in a state of nonuniform rotation (as the earth's core most likely is), then, as has been pointed out in § 3, there will be in the decay of the toroidal field a term induced by the poloidal field and proportional to it. If the free decay is a strongly damped term in the prevailing meridional circulation, then it is the induced term which will be decisive.

A second fact which emerges from the results of §§ 6 and 7 is that it is not *always* true that a reversal of the sense of the meridional circulation is accompanied by a reversal of

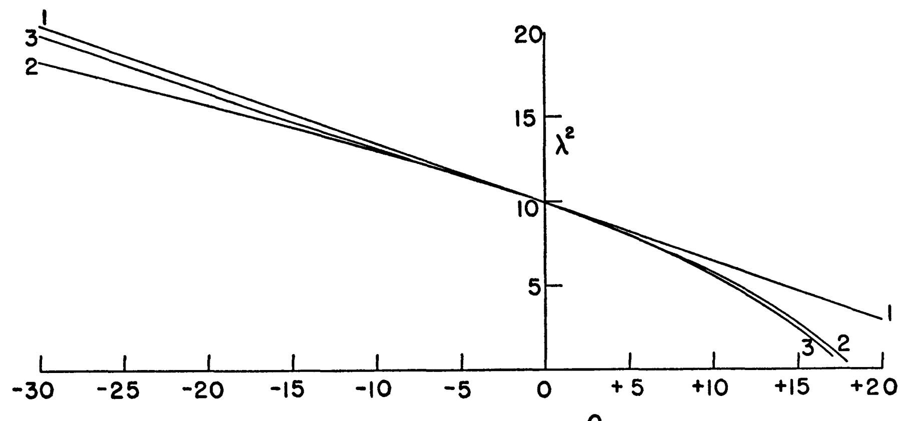

FIG. 3.—The effect of the velocity mode (1, 1) on the decay of poloidal magnetic fields The lowest characteristic root λ^2 for various values of the amplitude, β, of the velocity mode. The curves are labeled by the order of the approximation on which they are derived

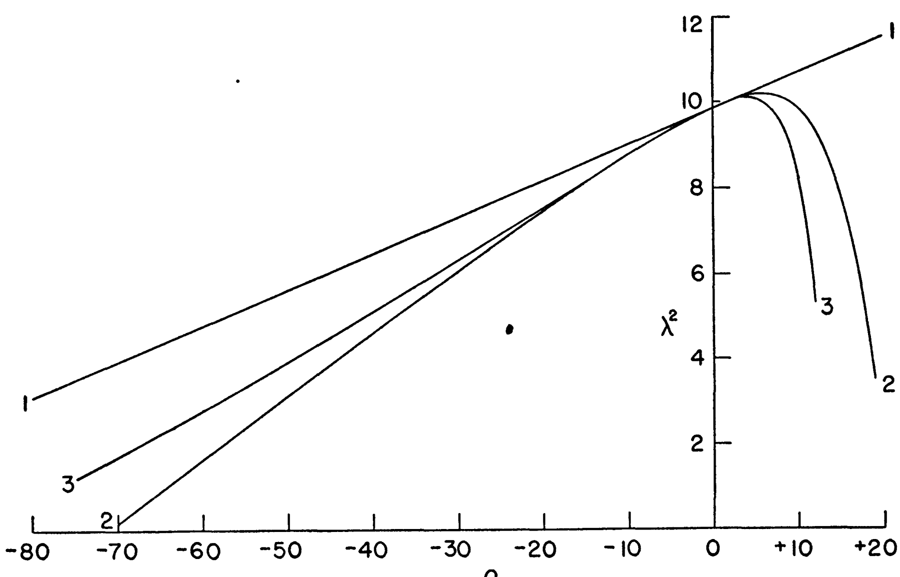

FIG. 4.—The effect of the velocity mode (1, 2) on the decay of poloidal magnetic fields The lowest characteristic root λ^2 for various values of the amplitude, β, of the velocity mode. The curves are labeled by the order of the approximation on which they are derived.

its effect on the decay of a magnetic field. The (1, 1) velocity mode does act in this way with respect to both the poloidal and the toroidal fields (see Figs. 1 and 3). But the (1, 2) mode acts differently. Thus a negative β results in a slowing-down of the decay of both toroidal and poloidal fields; although a slightly positive β does result in a slightly accelerated decay, the trend is soon reversed; and for increasing β the decay is retarded (as when β is negative) (see Figs. 2 and 4).

The analysis of §§ 6 and 7 discloses, therefore, aspects of the decay problem which require much detailed investigation. And several related questions also suggest themselves. Some of these are the following.

1. Equations (10) and (15) are not self-adjoint. Can one still assume that the solutions belonging to the different characteristic values provide a complete basis for the expansion of an arbitrary function which satisfies the boundary conditions of the problem? If not, how is one to express the decay of an arbitrary field?
2. Are the characteristic values of equations (10) and (15) necessarily positive? In particular, are complex roots excluded?
3. Why is it that sometimes the reversal of the sense of the meridional circulation is accompanied by a reversal of its effects on the magnetic field and sometimes not?
4. What is the asymptotic behavior of λ^2 as $\beta \rightarrow +\infty$ and as $\beta \rightarrow -\infty$?
5. What is the general nature of the induced term in the decay of a toroidal field?

Answers to the foregoing questions may be expected to throw some light on the interpretation of a number of astrophysical and geophysical phenomena in which magnetic fields play a role. One example may be given.

Cowling (1946) has pointed out that, for a typical sunspot ($R = 10^9$ cm; $\sigma = 10^{-8}$ e.m.u.), the time of decay may be estimated at 300 years. Nevertheless, the magnetic field of a sunspot grows to its full strength in a matter of days; similarly, the time it takes for the field to disappear, once it has started to decline, is also a matter of days. As Cowling has emphasized, these facts raise a number of very difficult questions. But there is one circumstance which would appear to be relevant to this problem and which does not seem to have been considered. It is this: The unit of velocity appropriate to the problem on hand is $(\sigma R)^{-1} = 0.1$ cm/sec; and it is known that systematic velocities of the order of 1 km/sec are present in sunspots. Consequently, $\beta \sim 10^6$, and one can ask: What does β of this magnitude imply for the decay and growth of magnetic fields? And can it be that a suitable pattern of velocity with $\beta \sim 10^6$ can alter the decay (and growing) time of magnetic fields by a factor of the order of 10^4–10^5? Answers to questions 3 and 4 are clearly very relevant to this problem.

It is possible that, to answer with some degree of completeness the various questions that have been raised (and others that will naturally occur), one may have to look for a simplified model to which exact analytical methods can be applied to a greater extent than appears possible with the ones considered in this paper.

The subject is being pursued along several directions.

In concluding this paper, it is again a pleasure to record my indebtedness to Miss Donna Elbert, who carried out the extensive numerical work which underlies the calculations included in this paper. My thanks are also due to Miss Siciy Pao for carefully checking the manuscript.

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