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Quantum Mechanics in the de Sitter Spacetime and Inflationary Scenario

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Abstract. We consider the effect of quantizing the homogeneous mode of a scalar field on inflation. It is shown that any semiclassical description of the scalar field is bound to lead to density inhomogeneities which are unacceptably large.

Key words: gravitation, quantum fluctuations—universe, inflationary

1. Inflationary physics

The idea of an inflationary epoch as a cure for various cosmological 'problems' has caught the fancy of the physicists in recent years. As one started looking deeper into these scenarios new problems seem to surface. To begin with, the most (and only!) natural scenario suggested by Guth (1981) led to an extremely inhomogeneous universe (Guth & Weinberg 1983). By adopting a special kind of dynamical symmetry breaking scheme, 'new' inflationary scenario solves this problem (Linde 1982; Albrecht & Steinhardt 1982). However, quantum fluctuations of the scalar field in this model leads to large density inhomogeneities. It is necessary to fine tune the parameters in the potential in a rather arbitrary manner to arrive at 'correct' answers. Since the basic motivation for inflation stems from a desire to avoid fine tuning, it is not entirely clear whether we are any better of in the end.

More fundamental problems have come up recently regarding the 'new' inflation. Doubts have been cast on the validity of the semiclassical analysis which is resorted to, and also on the nature of the initial state of the scalar field prior to slow 'roll over'. (Evans & McCarthy 1985; Mazenko, Unruh & Wald 1985).

To provide complete answer to these questions, it is necessary to go beyond the semiclassical approximation. We must construct a quantum theory for the *interacting* scalar field in a Robertson-Walker background, and, couple suitable expectation value of the energy-momentum tensor of the scalar field to the background geometry as a source term. In this paper, we tackle a much less ambitious project: We treat the homogeneous mode of the scalar field as a quantum variable and describe the self-consistent dynamics of the coupled system. The final result is once again negative: any reasonable description for the initial state of the field leads to too much of inhomogeneities.

We wish to emphasize that the work described here must be considered as a 'toy model'. Taking into account the spatial degrees of freedom may change the nature of the result. Such a possibility is under investigation.

2. Quantum mechanics in de Sitter spacetime

Consider the action for a scalar field φ with a potential $V(\varphi)$:

$$\mathscr{A} = \int \sqrt{-g}\, \mathrm{d}^4 x \{\tfrac{1}{2}\varphi_i \varphi^i - V(\varphi)\}. \tag{1}$$

We shall take the spacetime to be a $k = +1$ universe with the line element:

$$\mathrm{d}s^2 = \mathrm{d}t^2 - S^2(t)\left[\frac{\mathrm{d}r^2}{1-r^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\varphi^2)\right]. \tag{2}$$

By assuming that $\varphi(\mathrm{x}, t) = \varphi(t)$ we shall reduce the quantum field theory problem in Equation (1) to a quantum mechanical problem. The action in (1) becomes,

$$\mathscr{A} = 2\pi^2 \int \mathrm{d}t\, S^3(t) \{\tfrac{1}{2}\dot{\varphi}^2 - V(\varphi)\} \tag{3}$$

$$= 2\pi^2 \int \mathrm{d}T \left\{\frac{1}{2}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}T}\right)^2 - S^6(T) V(\varphi)\right\}, \tag{4}$$

where,

$$T = \int \frac{\mathrm{d}t}{S^3(t)}. \tag{5}$$

Corresponding to the action in Equation (4) we have the Schrödinger equation:

$$i\frac{\partial \psi}{\partial T} = -\frac{1}{4\pi^2}\frac{\partial^2 \psi}{\partial \varphi^2} + 2\pi^2 V(\varphi) S^6 \psi. \tag{6}$$

(We are using units with $c = h = 1$ such that, φ^{-1}, S, t, $T^{-1/2}$ and $|\psi|^2$ have the dimensions of length). In a given background geometry, Equation (6) determines the probability functional $\psi[\varphi, t]$. To complete the dynamics, we should use the expectation values of T^i_k as the source of Einstein's equations. From Equations (l) and (3) it follows that,

$$T^0_0 = \tfrac{1}{2}\dot{\varphi}^2 + V(\varphi) = \frac{\hat{p}^2}{8\pi^4 S^6} + V = -\frac{1}{8\pi^4 S^6}\frac{\partial^2}{\partial \varphi^2} + V \tag{7}$$

$$T^1_1 = -\tfrac{1}{2}\dot{\phi}^2 + V(\varphi) = -\frac{\hat{p}^2}{8\pi^4 S^6} + V = \frac{1}{8\pi^4 S^6}\frac{\partial^2}{\partial \varphi^2} + V. \tag{8}$$

(We have used the fact that the canonical momentum $\hat{p}$ corresponding to φ is $2\pi^2 S^3\dot{\phi}$). The expectation values of T^0_0 and T^1_1 in a state $\psi(\varphi)$ are,

$$\langle T^0_0 \rangle \equiv \rho = \frac{a}{S^6} + b; \qquad \langle T^1_1 \rangle \equiv -p = -\frac{a}{S^6} + b \tag{9}$$

with,

$$a = -\frac{1}{8\pi^4}\int_{-\infty}^{+\infty} \psi^*(\varphi)\frac{\partial^2 \psi}{\partial \varphi^2}\mathrm{d}\varphi = a(t) \tag{10}$$

and,

$$b = \int_{-\infty}^{+\infty} |\psi|^2\, V(\varphi)\, \mathrm{d}\varphi = b(t). \tag{11}$$

Einstein's equation with T^i as the source are equivalent to a set of two equations:

$$\frac{\dot{S}^2+1}{S^2}=\frac{8\pi G}{3}\langle T_0^0\rangle=\frac{8\pi G}{3}\left[\frac{a}{S^6}+b\right], \tag{12}$$

$$\frac{\mathrm{d}a}{\mathrm{d}t}=-S^6\frac{\mathrm{d}b}{\mathrm{d}t}. \tag{13}$$

It can be easily verified that Equation (13) is equivalent to the equation of motion for the scalar field in the classical limit. In writing (12) and (13) we have ignored all other source terms except the scalar field. This is justifiable because we shall be mainly concerned with the inflationary phase in which $S(t) \propto \exp(Ht)$; during that epoch, radiation (S^{-4}) and matter (S^{-3}) terms cease to be relevant. Similarly the S^{-6} and S^{-2} terms in Equation (12) can also be ignored.

The complete dynamics is determined by Equations (6), (12) and (13). In order to produce an analytic solution to these equations, we shall approximate the potential $V(\varphi)$ by a constant V_0 for $\varphi < \varphi_f$ and by zero for $\varphi_f < \varphi < \varphi_b$. We assume infinite potential barrier at $\varphi = \varphi_b$. Such an idealization of the Coleman–Weinberg type potential turns out to be adequate for our purposes.

To solve the Schrödinger equation in this potential, it is necessary to know the initial wave function, $\psi[\varphi, 0]$. We shall assume the initial wave function to be a gaussian:

$$\psi[\varphi,0]=\left(\frac{1}{2\pi\sigma_0^2}\right)^{1/4}\exp\left\{-\frac{(\varphi-\varphi_i)^2}{4\sigma^2}+iL\varphi\right\}. \tag{14}$$

We have chosen Equation (14) such that $\langle\varphi\rangle = \varphi_t$, $(\Delta\varphi) = \sigma^2$ and

$$\left\langle\left(\frac{\mathrm{d}\varphi}{\mathrm{d}T}\right)\right\rangle=\frac{L}{2\pi^2} \tag{15}$$

represents the 'rolling down' velocity along the flat region.

The addition of a pure phase term in (14) which ensures a non-vanishing $\langle(\mathrm{d}\varphi/\mathrm{d}T)\rangle$, is absolutely essential. The actual potential has a gentle slope towards larger φ thereby inducing a 'roll over' velocity. Since we have idealized the potential by a constant V_0, it is necessary to put this term by hand.

It should be noted that the choice in Equation (14) is different from the usual choice made in inflationary models. It is usually *assumed* that the wave functional $\psi[\varphi(\mathrm{x}), t]$ desribing the state of $\varphi(\mathrm{x})$ is symmetric under $(\varphi \to -\varphi)$. (In other words, φ is as likely to 'roll' towards the positive side as towards the negative side). Such an assumption, however, suffers from the following difficulties:

(i) At a very basic level, one simply does not know whether such an assumption was realized in the early universe or not. Granted this uncertainty, it is worthwhile examining the sensitivity of the results to changes in this particular assumption. Note that the symmetry of the hamiltonian under $(\varphi \to -\varphi)$ does not guarantee the same symmetry for the initial state.

(ii) Once we force the initial state to have the symmetry, $\langle\varphi\rangle$ will vanish. This, in turn, forces us to consider the classical part and the quantum fluctuations in very different manner. Both in quantum mechanics, as well as in the quantum field theory based on Schrödinger (functional) equation, it is conventional to identify $\langle\varphi\rangle$ as the classical limit. Since we can no longer do this, (because $\langle\varphi\rangle$ vanishes, while φ_{class} has to evolve) it is necessary to take φ_{class} as the solution to classical equations and quantize *perturbations of φ around φ_{class}* by a suitable procedure. Crudely speaking, the φ_{class} 'rolls' down 'carrying' with it the fluctuations $\langle\varphi^2\rangle^{1/2}$. Such an approach, to say the least, is unusual. It is definitely worthwhile to see whether an initial state can be chosen so that $\langle\varphi\rangle$ represents the classical evolution and $\langle [\varphi - \langle\varphi\rangle]^2 \rangle^{1/2}$ characterizes the fluctuations.

(iii) As was noted repeatedly in the literature, inflation can proceed from any local region of space which satisfies the requisite conditions. If the initial state is thermal or chaotic, there will be a probability distribution for the values taken by φ (x, 0). It is interesting to examine the consequences if inflation proceeds from a region with nonzero value for L, *i.e.* a region where φ has a preference to roll along a specific direction.

These are our reasons to deviate from the usual assumptions and take Equation (14) with $L \neq 0$. We shall interpret $\langle\varphi\rangle$ as the classical limit of the field.

The general solution to the Schrödinger equation (6) for constant V can be easily obtained to be,

$$\psi[\varphi, t] = \int_{-\infty}^{+\infty} \frac{\mathrm{d}l}{\sqrt{2\pi}} C(l) \exp il\varphi \exp\left[-\frac{il^2}{2\pi^2}\int\frac{\mathrm{d}t}{S^3} - 2\pi^2 V_0 \int S^3\,\mathrm{d}t\right] \tag{16}$$

where the function $C(l)$ has to be determined from the initial condition (14). Calculating $C(l)$ and substituting in (16) we get the probability distribution to be,

$$P[\varphi, t] = |\psi[\varphi, t]|^2 = \left[\frac{1}{2\pi\sigma^2(t)}\right]^{1/2} \exp\left\{-\frac{\left(\varphi - \varphi_i - \frac{L}{\pi^2}T\right)^2}{4\sigma^2(t)}\right\} \tag{17}$$

with,

$$\sigma^2(t) = \sigma_0^2\left[1 + \frac{1}{4\pi^4\sigma_0^4}\left(\int\frac{\mathrm{d}t}{S^3}\right)^2\right]. \tag{18}$$

Straightforward use of Equations (16) and (17) will also yield the following expectation values:

$$\langle\varphi\rangle = \varphi_i + \frac{L}{\pi^2}\int\frac{\mathrm{d}t}{S^3},$$

$$\langle p\rangle = L. \tag{20}$$

When $V(\varphi)$ is a constant, $\hat{p}^2$ commutes with the Hamiltonian. Therefore, $\langle p^2\rangle$ is independent of time. Equation (13) is thus identically satisfied. In order to obtain a self-consistent description we only have to solve (12). Obviously, for $(Ht) \gtrsim 1$ the exponential solution

$$S(t) \simeq \frac{1}{H}\exp Ht \tag{21}$$

exists, leading to the usual inflationary scenario. As described before, only the b-term dominates (12) in the limit of $(Ht) \gtrsim 1$. We shall now consider the various constraints on this evolution.

3. Constraints on parameters

We shall take for φ_f and V_0, values similar to that in Coleman-Weinberg potential:

$$\varphi_f \simeq 1.2 \times 10^{15}\ \text{GeV}; \qquad V_0 \simeq (10^{15}\ \text{GeV})^4 \tag{22}$$

leading to

$$H \simeq 7 \times 10^9\ \text{GeV}. \tag{23}$$

From Equations (19) and (21) it follows that the 'centre' of the wave packet has the trajectory:

$$\langle \varphi \rangle = \varphi_i + \frac{(LH)H}{3\pi^2}(1 - \exp(-3Ht)). \tag{24}$$

In other words the expectation value of the scalar field starts 'rolling over' with constant velocity (. for $Ht \ll 1$, ($\langle\varphi\rangle - \varphi_i) \propto t$, but very soon slows down. It approaches the asymptotic value of

$$\langle \varphi \rangle_\infty = \varphi_i + (LH)\frac{H}{3\pi^2}. \tag{25}$$

For successful implementation of reheating we need $\langle\varphi\rangle$ to have 'fallen down' the well as $Ht \to \infty$. In other words,

$$\varphi_i + \frac{LH}{3\pi^2} H \gtrsim \varphi_f. \tag{26}$$

Since $\varphi_f \gg \varphi_I$ we may take this condition to be

$$\left(\frac{LH}{3\pi^2}\right) H \gtrsim \varphi_f. \tag{27}$$

Using Equations (22) and (23) in (27) we get,

$$LH \gtrsim 3 \times 10^6. \tag{28}$$

By taking $\varphi_f \lesssim \langle\varphi\rangle_\infty$, one can easily prolong the roll-ver phase as much as one wants; thus there is no difficulty in achieving sufficient inflation.

The initial state was assumed to be well localized near the origin and definitely far away from φ_f This implies that,

$$\sigma_0 \ll \varphi_f. \tag{28}$$

Equations (28) and (29) constrain the choice of parameters in the initial state.

In order to compute the inhomogeneities produced during the inflation, it is necessary to discuss the spatial degrees of freedom of φ. However, an order of magnitude estimate can be made along the following lines.

It is known that the value of the density contrast ($\delta\rho/\rho$) is given by (see *e.g* Starobinski 1982; Hawking 1982; Guth & Pi 1982; Bardeen *et al.* 1983)

$$\left(\frac{\delta\rho}{\rho}\right) = \varepsilon \left\langle \frac{H\Delta\varphi}{\dot\varphi} \right\rangle_{t=t_1}. \tag{30}$$

Here ε is a number of the order unity, $\Delta\varphi$ is the quantum spread in the scalar field, and $\dot\varphi$ is the roll-over velocity. The right-hand side should be evaluated at the time when galactic size perturbations 'freeze out' of the horizon. Physically one may interpret

$$\Delta\tau \equiv \frac{\Delta\varphi}{\dot\varphi} \tag{31}$$

as the time lag between the 'leading edge' and 'trailing edge' of the wave packet. Because of this time lag the scalar field 'falls over' the potential at different places at different times, leading to inhomogeneities.

Since we know the evolution of the wave packet the time lag $\Delta\tau$ can be computed directly as,

$$\Delta\tau = t_+ - t_- \tag{32}$$

where $t_\pm$ are the roots of the equation,

$$\langle \varphi \rangle_t \pm \sigma(t) = \varphi_f . \tag{33}$$

Straightforward algebra gives (talking $\varphi_f \gg \varphi_0$; $\sigma_0 \quad \varphi_f$),

$$\Delta\tau \simeq \frac{2\sqrt{2}}{3H}\left(1+\frac{1}{4L^2\sigma_0^2}\right)^{-1}\left(1+\frac{1}{8L^2\sigma_0^2}\right)^{1/2} \exp(3Ht_1) \tag{34}$$

so that,

$$\frac{\delta\rho}{\rho} = \frac{2\sqrt{2}}{3}\varepsilon\left(1+\frac{1}{4L^2\sigma_0^2}\right)^{-1}\left(1+\frac{1}{8L^2\sigma_0^2}\right)^{1/2} \exp(3Ht_1). \tag{35}$$

The disturbing exponential factor is due to the fact that $\langle \varphi \rangle$ is much smaller than the constant velocity assumed in the conventional roll-over, scenario. Note that,

$$\langle \dot{\varphi} \rangle = \left\langle \frac{d\varphi}{dt} \right\rangle = \frac{1}{S^3}\left\langle \frac{d\varphi}{dT} \right\rangle = \exp(-3Ht)\left(\frac{L}{\pi^2}\right). \tag{36}$$

While $(d\varphi/dT)$ remains constant, $(d\varphi/dt)$ keeps on decreasing. Clearly, the exponential in Equation (35) makes matters much worse than usual. One way to get out of this trouble will be to assume that the 'freeze-out time t_1' for the relevant length scale is of the order of H^{-1}. (Turner has pointed out that galactic size perturbation crosses the horizon at about $50H^{-1}$ before the end of inflation; *i.e.,* $t_1 \simeq 10H^{-1}$ (Turner 1983). Taking $t_1 \simeq 2H^{-1}$, therefore, can be a drastic approximation.) With this understanding, we write,

$$\frac{\delta\rho}{\rho} \gtrsim \frac{\beta}{\sqrt{2}}\left(1+\frac{1}{4L^2\sigma_0^2}\right)^{-1}\left(1+\frac{1}{8L^2\sigma_0^2}\right)^{1/2} \tag{36}$$

where β is probably a numerical factor of the order of (10–100). On the other hand, we want $(\delta\rho/\rho)$ to be about 10^{-4}. Clearly, it is necessary to have $L^2\sigma_0^2 \ll 1$. In this limit, we get,

$$\frac{\delta\rho}{\rho} = \beta(L\sigma_0) \tag{37}$$

which gives the further constraint,

$$L\sigma_0 \lesssim \beta^{-1}\, 10^{-4}. \tag{38}$$

Altogether we have arrived at the following constraints on the parameters (*cf.* Equations (28), (29) and (38))

$$LH \gtrsim 3\times 10^6; \qquad \sigma_0 \ll (10^{15}\ \text{GeV}); \qquad L\sigma_0 \lesssim \beta^{-1}\, 10^{-4}. \tag{39}$$

If these are the only constraints on the system, then they can be easily satisfied. For

Example, one can choose (with $\beta \simeq 10^2$).

$$L \simeq 5 \times 10^{-4}\,(\text{GeV})^{-1}; \qquad \sigma_0 \simeq 2 \times 10^{-3}\,(\text{GeV}). \tag{40}$$

Unfortunately, these are not the only accepted constraints. It is usually assumed that a scalar field in a de Sitter spacetime has fluctuations which are at least of the order of $(H/2\pi)$. This result corresponds to the usual 'temperature' $(H/2\pi)$ associated with the de Sitter spacetime. If we assume this result to be valid in an inflationary scenario, then it is necessary to satisfy the additonal constraint,

$$\sigma_0 \gtrsim \frac{H}{2\pi} \simeq 10^9\ \text{GeV}. \tag{41}$$

From the first constraint in (39), it follows that,

$$L \gtrsim 3 \times 10^6\, H^{-1} \tag{42}$$

Combining (42) and (41),

$$L\sigma_0 \gtrsim 5 \times 10^5, \tag{43}$$

which is grossly inconsistent with (38). In other words, the simplest version of quantum mechanical inflation leads to density inhomogeneities which are large by a huge factor.

4. Discussion

Treating the scalar field as a quantum mechanical object does not offer any relief from the disturbing conclusions already known in literature. The analysis however brings out two features: (i) The cause of the problems in new inflation is *not* the semiclassical approximation made in the usual analysis, (ii) The condition (41) plays a crucial role in producing too much of inhomogeneities. But for this constraint, one can arrive at acceptable values of density inhomogeneities.

It is possible that Equation (41) is not really as sacred as it is taken to be. It has been argued in literature that thermal effects due to event horizons do not restore a spontaneously broken symmetry (Hill 1985). Possibly, thermal effects do not contribute to the dynamics of de Sitter space either. In this connection it should be remembered that the spacetime is never truly de Sitter; it only approaches the de Sitter spacetime asymptotically.

The discussion presented in the paper needs to be generalized in three different aspects: (i) inclusion of spatial dependence, (ii) examination of other initial conditions, (iii) amore realistic description of the potential $V(\varphi)$. We hope to present such a detailed analysis in a future paper.

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