PHYSICS AT PLANCK LENGTH

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ABSTRACT

Quantum gravitational effects alter the nature of spacetime at length scales of the order of Planck length ($\sim 10^{-33}$ cm). These effects can be investigated by concentrating one's attention on the conformal degree of freedom of the metric. It can be shown that physically measurable proper interval between any two events, in any spacetime, is bounded from below by Planck length. Quantum gravity shows promise in providing a universal ultraviolet cut-off.

GRAVITY—CLASSICAL AND QUANTUM

All known gravitational phenomena are described by Einstein's general theory of relativity, which interprets gravitation as an effect due to curvature of specetime manifold. Accordingly, the central quantity in the theory is the spacetime interval between two infinitesimally separated events:

$$ds^2 = g_{ik}(x)dx^i dx^k. \qquad (1)$$

The metric tensor $g_{ik}(x)$ which appears in (1) is determined by the distribution of matter through Einstein's field equations:

$$R^i_k - \tfrac{1}{2}\delta^i_k R = -8\pi G T^i_k. \qquad (2)$$

The description (1) and (2) is purely classical. We assume that by setting up co-ordinate systems (involving rods and clocks) in a suitable manner one can determine the value of g_{ik} at any point to arbitrarily high degree of accuracy. Such a tacit assumption, always made in a classical theory has the following immediate consequence: As dx^i goes to zero, ds^2 will go to zero. In particular consider two events $(t, \mathbf{x})$ and $(t, \mathbf{y})$ in a given spacelike hypersurface labelled by a time coordinate t. As $\mathbf{x}$ approaches $\mathbf{y}$ the proper distance between these two events will approach zero. This trivial fact can be stated formally as,

$$\lim_{x^i \to y^i} (ds^2 = g_{ik}dx^i dx^k) = 0, \qquad (3)$$

where $y^i = x^i + dx^i$. Note that classically measured $g_{ik}(x)$ does not affect the limiting process in any way.

Classical gravity, however, does not exist except as an approximation. It is necessary to take into account the quantum nature of the world by quantising gravity. Such an endeavour is fraught with difficulties because of two reasons: (i) Einstein's gravity is a nonlinear theory and leads to serious mathematical complexities when quantised (ii) The close laison between gravity and spacetime creates new conceptual difficulties which have no parallel in any other field theory.

Fortunately, quantum gravitational effects are not relevent except at very small length scales. One may estimate this scale by comparing the Einstein action,

$$\mathscr{A} = \frac{1}{16\pi G}\int R\sqrt{-g}\, d^4x \qquad (4)$$

with the quantum of action $\hbar$. It is easy to see that quantum gravitational effects are important at length scales of the order of "Planck length", L_p:

$$L_p \equiv \left(\frac{4\pi G\hbar}{c^3}\right)^{1/2} \approx 10^{-32}\,\text{cm}. \qquad (5)$$

In other words, when the proper distance between two events is less than L_p, the classical description presented above will break down. Quantum fluctuations in the metric tensor will be enormous and one can no longer treat $g_{ik}(x)$ as classically specified function. How can we describe physics under such bizzare conditions?

A partial—but extremely useful—answer to the above question came from a class of quantum cosmological models[1-3]. These models, in which the conformal degree of freedom of the metric was quantised, showed that the universe did not proceed below Planck size in its proper dimensions. We shall now examine this situation in detail, to see the relevence of quantum conformal fluctuations (QCF) to the physics at Planck length.

QUANTUM FLUCTUATIONS AND GROUND STATE

An electron located within a sphere of radius Δr around a proton will have an energy of the order of,

$$E \approx \frac{(\Delta p)^2}{2m} - \frac{q^2}{(\Delta r)} = \frac{\hbar}{2m(\Delta r)^2} - \frac{q^2}{(\Delta r)}. \quad (6)$$

The first term, which is entirely due to quantum fluctuations, prevents Δr from becoming zero. The minimum energy (ground) state occurs at the Bohr radius,

$$\Delta r = r_0 \equiv \frac{\hbar}{mq^2}. \quad (7)$$

The physics of Robertson-Walker universe is similar. A radiation filled, $k = 1$, classical universe is described by the line element,

$$ds^2 = a^2 \sin^2 \omega t \left[dt^2 - \frac{dr^2}{1 - kr^2} - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right]. \quad (8)$$

The quantum universe, on the other hand, is not described by such a singular, deterministic line element; it is represented by a series of (quantum) stationary geometries[4-6] (QSG, for short) parametrized by an integer n: In the nth stationary geometry the line element has the form:

$$ds^2_{(n)} = L_p^2 (n + \tfrac{1}{2}) \left[dt^2 - \frac{dr^2}{1 - r^2} - r^2 (d\theta^2 + \sin^2 \theta \, d\varphi^2) \right] \quad (9)$$

(for more details, see ref. 4–6). In particular, the "ground state" for the quantum universe has the dimension $\sim L_p$ and proper volume $\sim L_p^3$. The basic reason for the existence of such a "ground state" is again quantum fluctuations in the scale factor. It can be shown that[4-6] if we write the averaged scale factor for the quantum universe as,

$$Q^2(t) = Q_0^2(t)[1 + \Delta^2(t)]$$

(where $Q_0^2(t)$ is the classical scale factor) then,

$$\Delta^2(t) \sim \frac{L_p^2}{Q_0^2(t)}, \qquad \text{as } Q_0 \to 0. \quad (10)$$

In other words the fluctuations $\Delta^2(t)$, which diverge at the classical singularity ($Q_0(t) = 0$), provide us with a stable ground state for the universe, *via* the limit:

$$\lim_{Q_0 \to 0} Q^2(t) \simeq L_p^2. \quad (11)$$

Clearly it is meaningless to talk about length scales less than L_p in such a universe. (It would be analogous to talking about energies below $\frac{1}{2}\hbar\omega$ for an oscillator). Planck length appears as a physical lowerbound to proper length.

Interestingly enough, such a result can be proved in broad generality in any specetime. We shall now examine this generalization.

VACCUM FLUCTUATIONS OF QUANTISED GRAVITY

In the absence of gravitational field, the spacetime would be considered flat in the classical limit. Such a flat spacetime should be more properly treated as quantum gravitational vacuum. The omni-present vacuum fluctuations will now induce fluctuations in the classical value of the metric tensor,

$$g^0_{ik} = \eta_{ik} = \text{dia}(1, -1, -1, -1). \quad (12)$$

What are the physical effects of such metric fluctuation?

To begin with, consider the proper length between two events $(t, \mathbf{x})$ and $(t, \mathbf{y})$. In the absence of metric fluctuations, the proper length is just,

$$l_0(\mathbf{x}, \mathbf{y}) \equiv |\mathbf{x} - \mathbf{y}|. \quad (13)$$

When the quantum conformal fluctuations of the metric are taken into account, we have to deal with all the metrics of the form,

$$g_{ik} = (1+\phi(x))^2 \eta_{ik}. \quad (14)$$

Since the proper distance between $(t, \mathbf{x})$ and $(t, \mathbf{y})$ depends on the value of the quantum variable $\phi(\mathbf{x})$, we can no longer assign a *unique* proper length between $(t, \mathbf{x})$ and $(t, \mathbf{y})$. Instead, we should ask for the *probability* $P(l)$ for the proper length to have a particular value l. This probability, in turn, depends on the probability for a fluctuation of size $\phi(x)$ to occur, and can be computed by vacuum functional techniques. One obtains the distribution[7]

$$P[\phi(\mathbf{x})] = N \exp\left\{-\frac{1}{4\pi^2 L_p^2} \int d^3\mathbf{x}\, d^3\mathbf{y} \frac{\nabla\phi(\mathbf{x})\cdot\nabla\phi(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^2}\right\} \quad (15)$$

using which one can compute $P(l)$. The final answer is, (for details see ref. 8),

$$P[l]\,dl = \left(\frac{1}{2\pi\sigma^2}\right)^{1/2} \exp\left\{-\frac{(l-l_0)^2}{2\sigma^2}\right\} dl; l \geqslant 0, \quad (16)$$

with,

$$l_0 = |\mathbf{x}-\mathbf{y}|; \qquad \sigma^2 = \frac{l_0^2}{L^2}\left(\frac{L_p}{2\pi}\right)^2. \quad (17)$$

In (17), L denotes the resolution limit of the apparatus used to measure the proper length between $\mathbf{x}$ and $\mathbf{y}$.

We note that $P(l)$ is a Gaussian-peaked at the classical value l_0, as expected. As long as we consider measurements which are 'coarse-grained' over many Planck lengths (*i.e.* resolution limit $L \gg L_p$), the quantum spread in the Gaussian is small:

$$\frac{\sigma^2}{l_0^2} = \frac{1}{4\pi^2}\left(\frac{L_p}{L}\right)^2 \ll 1. \quad (18)$$

In this limit (which is valid even at the highest man-made accelerators), $P(l)$ is adequately approximated as a delta function $\delta(l-l_0)$ and one may neglect quantum gravitational effects. However as the resolution improves (*i.e.* $L \to L_p$), σ approaches l_0, and we lose the concept of a definite length between the events.

The mean square value of l^2 for the Gaussian distribution is given by,

$$\langle l^2 \rangle = l_0^2\left[1 + \frac{L_p^2}{4\pi^2 L^2}\right]. \quad (19)$$

From the definition of L as resolution limit, it follows that $l_0 \gtrsim L$. Taking the limits in proper order (maintaining $l_0 > L$), we get,

$$\lim_{L\to 0}\left\{\lim_{l_0 \to L} \langle l^2 \rangle\right\} = \left(\frac{L_p}{2\pi}\right)^2. \quad (20)$$

In other words, the mean square proper length is bounded from below at $(L_p/2\pi)^2$. The above discussion once again illustrates how fluctuations can lead to a stable lower bound in length scales.

Even though the intermediate steps involved the resolution length L, the final bound did not. This makes one suspect that it may be possible to obtain the lower bound without bothering about the measurement process. We shall now see how this can be done.

THE 'ZERO-POINT LENGTH' OF SPACETIME

Taking into account the conformal fluctuations of the metric, one can try to define the mean value of the spacetime interval at the coincidence limit by the following relation:

$$\lim_{x\to y} \langle ds^2 \rangle = \lim_{x\to y} \langle [1+\phi(x)]^2 \rangle \eta_{ik} dx^i dx^k. \quad (21)$$

Such a definition however runs into two difficulties immediately: (i) Mathematically, expressions like $\langle \phi^2(x) \rangle$ are divergent[9] and need to be "regularized" (ii) Physically, line interval depends on two events, x^i and $y^i \equiv x^i + dx^i$ and it is not meaningful to consider ϕ at a signal event x^i.

Fortunately, both these difficulties can be surmounted by a small modification of our definition. We shall consider $\langle ds^2 \rangle$ to be defined as the limit,

$$\lim_{x\to y} \langle ds^2 \rangle = \lim_{x\to y} \{1 + \langle \phi(x)\phi(y) \rangle\} \eta_{ik} dx^i dx^k. \quad (22)$$

Here and elsewhere the 'mean values' $\langle\ \rangle$ are defined *via* functional integration. For example,

$$\langle \phi(x)\phi(y) \rangle \equiv \frac{\int \mathscr{D}\phi\, \phi(x)\phi(y) \exp i\mathscr{A}/\hbar}{\int \mathscr{D}\phi \exp i\mathscr{A}/\hbar} \quad (23)$$

where $\mathscr{A}$, given by (4), becomes a functional of $\phi(x)$: (See, e.g. ref. 3).

$$\mathscr{A} = -\frac{1}{2L_p^2} \int \phi^i \phi_i \mathrm{d}^4 x. \quad (24)$$

Using (24), (23) and (22), we get, (with $l_0^2\ (x, y) \equiv \eta_{ik} \mathrm{d}x^i \mathrm{d}x^k$),

$$\lim_{x \to y} \langle \mathrm{d}s^2 \rangle = \lim_{x \to y} \langle \phi(x)\phi(y) \rangle l_0^2(x, y)$$
$$= \lim_{x \to y} \left(\frac{L_p}{2\pi}\right)^2 \frac{1}{l_0^2} l_0^2$$
$$= \left(\frac{L_p}{2\pi}\right)^2. \quad (25)$$

In other words, the mean square value of the line interval is bounded from below at $(L_p/2\pi)^2$. The discussion in the previous section illustrates how measurements respect this fact. In analogy with the zero-point energy of the harmonic oscillator we may attribute a "zero point length" between any two events in the spacetime. This result can also be looked upon as an 'uncertainly relation' between proper length and the conformal factor. We can interpret σ in (16) as the uncertainty Δl in the measurement of proper length. If $\Delta\phi$ denotes the uncertainty in the conformal factor then (17) can be stated equivalently as,

$$\Delta\phi\, \Delta l \gtrsim \left(\frac{L_p}{2\pi}\right). \quad (26)$$

All the previous results, especially those in quantum cosmology, are consistent with this principle.

The result in (25) is valid in any spacetime. Though the expectation value $\langle \phi(x)\phi(y) \rangle$ is a complicated function of (x, y) in a general spacetime[10], the coincidence limit of $\langle \phi(x)\phi(y) \rangle$ is dominated by the flat space behaviour and diverges as l_0^{-2}. Clearly, this feature is enough to reproduce (25).

We shall now examine the physical consequences of this result.

GRAVITY AND ULTRAVIOLET DIVERGENCES

It is well known that quantum field theory is bedevilled by divergences when straightforward perturbative techniques are used. Because of this situation it is impossible to construct a quantum theory from an arbitrary classical theory. Such a helplessness has forced the physicist to religiously adhere to a small subset of all possible theories, in which the divergences can be systematically removed.

It has been repeatedly suggested in literature (see ref. 11, 12) that gravity might provide a universal cut-off required to remove the ultraviolet divergences. Since a lower bound on length scale is equivalent to an ultraviolet cut-off, our result in the previous sections has important bearings to field theory.

Consider, for example, the usual definition for the two point Green's function [for a sialar field $\eta(x)$] *via* path integral

$$G_0(x,y) = \frac{\int \mathscr{D}\eta(x)\, \eta(x)\eta(y) \exp i\mathscr{A}[\eta]}{\int \mathscr{D}\eta \exp i\mathscr{A}[\eta]}, \quad (27)$$

with,

$$\mathscr{A}[\eta] = \tfrac{1}{2} \int \eta_i \eta^i \mathrm{d}^4 x. \quad (28)$$

The above equations presuppose that the spacetime is flat (or, at least, g_{ik} is fixed at some values). Since such an assumption is incorrect at high energies we should modify $G_0(x, y)$ in the following way: Let $G_0(x, y; g_{ik})$ denote the Green's function when the spacetime metric is g_{ik}. Then the correct Green's function is obtained by averaging over various metrics with the proper weightage $\exp i\, \mathscr{A}(g_{ik})$:

$$G_{\text{true}}(x, y) = \int \mathscr{D}g_{ik}\, G_0(x, y; g_{ik}) \exp i\mathscr{A}[g_{ik}]. \quad (29)$$

Performing this calculation (see ref. 9 for details) we get the final answer as,

$$G_{\text{true}}(x, y) = \frac{1}{4\pi^2 i} \left\{ \frac{1}{(x-y)^2 + \left(\frac{L_p}{2\pi}\right)^2 + i\varepsilon} \right\}. \quad (30)$$

Note that $G_{\text{true}}(x, y)$ has finite coincidence limit as $x \to y$, in sharp contrast with $G_0(x, y)$. Field theory calculations using G_{true} will be free from coincidence limit divergences. For example, it has been shown (in ref. 13) that the one-loop effective action is finite when gravitational effects are taken into account.

CONCLUSION

It was always hoped that Planck length will play a crucial role in quantum gravity. Analysing the conformal degree of freedom, one is led to envisage a far more fundamental role to Planck length. It provides a universal "lattice spacing" for the spacetime.

2 May 1985

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