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Solutions of scalar and electromagnetic wave equations in the metric of gravitational and electromagnetic waves

T PADMANABHAN
Department of Physics, University College, Kerala University, Trivandrum 695 001
Address for Correspondence: T.C. 38/130, First Puthan Street, Manacad, Trivandrum 695 009

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Abstract. The wave equation for a scalar field φ and vector potential A^μ are solved in the background metric of a gravitational wave. The corresponding solutions when the metric is generated by a plane electromagnetic wave, is obtained from these solutions. The solution for the scalar wave is discussed in detail. It is found that because of the interaction, two new waves are generated in the lower order approximations. One of them has the same phase dependence as the original wave while the other shows a transient character. There is no interaction when the waves are along the same direction.

Keywords. Gravitational wave; scalar wave; electromagnetic wave; photon-photon interaction; photon graviton interaction.

1. Introduction

In this paper the scalar and the electromagnetic wave equations, in a spacetime curved by the presence of a gravitational or electromagnetic wave are solved. From the solutions one can discuss possible interactions between them.

Section 2 introduces the metric of the gravitational and electromagnetic waves, and obtains the necessary formulae needed later. In section 3 the scalar wave equation in the gravitational wave metric is solved and the solution discussed. Even though the solution obtained is exact (except for the effect of back reaction on the metric which can be made arbitrarily small by assuming the second wave to be much weaker) the discussion is confined to the first two orders of approximation, for the sake of simplicity. The solution in the metric of electromagnetic wave is obtained from the other solution and the similarities are pointed out. In section 4 we solve the vector wave equation for the vector potential of the electromagnetic wave in the gravitational wave metric and in the metric of electromagnetic wave. A detailed discussion of this solution will be published later.

Our metric has the signature $(+++-)$. Natural units with $G=C=1$ is used except when mentioned otherwise. We take our gravitational and electromagnetic waves to be propagating along the z-axis and work throughout with the retarded and advanced co-ordinates,

$$u=t-z \text{ and } v=t+z. \tag{1}$$

A prime denotes differentiation with respect to u. In order to avoid different letters, the letters L, a, k, etc. are used in the coming sections to denote two different entities.

2. Form of the metric

2.1. *Metric of the gravitational wave*

For the metric of the plane gravitational wave we take the form (given in Misner *et al* 1973)

$$ds^2=L^2(u)[e^{2\beta}dx^2+e^{-2\beta}dy^2]-du\,dv \tag{2}$$

$$\beta=\beta(u) \text{ is arbitrary.} \tag{3}$$

The Einstein's equation reduces to the single equation,

$$L''+(\beta')^2L=0 \tag{4}$$

which connects the background effect $L(u)$ with the 'ripples', $\beta(u)$.

To avoid the difficult problems of interpretation arising due to the non-Euclidian nature of the space time, we concentrate here on a form of $\beta(u)$ which is confined spatially. We take it to have the form,

$$\beta(u)=ae^{-k|u|} \tag{5}$$

so that, at any fixed point z, $\beta(u)$ rises with increasing t, till it reaches a maximum value of a at $t=0$ and thereafter decreases back to zero at $t=\infty$. In other words this represents an exponentially decreasing wave pulse travelling along, positive z axis. A more natural choice would have been a Gaussian wave packet. But this keeps the analysis simple. For this choice eq. (4) becomes,

$$L''+k^2a^2e^{-2k|u|}\,L=0. \tag{6}$$

The equation can be exactly solved by a series solution. However, a representative form of the $L(u)$ will be needed later. For this, the 1st order correction to flat space value of $L=1$ can be easily found by approximating,

$$L(u)=1+L_1(u)+O(a^4) \tag{7}$$

so that $\quad L''_1=-k^2a^2e^{-2k|u|}$

giving $\quad L_1=-\tfrac{1}{4}a^2e^{-2k|u|}$. $\quad$ (8)

Thus the solution to this order is,

$$L(u)=1-\tfrac{1}{4}a^2e^{-2k|u|}+O(a^4). \tag{9}$$

Notice that the deviations from flat space in $L(u)$ comes in the second order in a.

2.2. *Metric of the electromagnetic wave*

Consider an electromagnetic wave described by a vector potential

$$A^{\mu}=A(u) \text{ for } \mu=x$$
$$=0 \text{ for } \mu\neq x. \tag{10}$$

The spacetime will be curved by the presence of the energy density of this electromagnetic field. It can be shown that (Misner *et al* 1973 op. cit) the metric thus generated has the form given by

$$ds^2=L^2(u)[dx^2+dy^2]-dudv \tag{11}$$

where

$$LL''+\frac{G}{C^4}(A')^2=0 \tag{12}$$

(The G and C factors are introduced here to show the order of magnitude).

Quite clearly, the form of the electromagnetic wave metric is same as that of gravitational wave except that $\beta=0$. Thus by solving the equations for the gravitational wave with arbitrary L and β and by putting $\beta\equiv 0$, one can get the solutions in the metric of electromagnetic wave. This is the procedure for the scalar wave adopted in this paper. For the vector wave equation, however, there will be a slight modification.

For a concrete example we take a pulse of electromagnetic wave with the same form as eq. (5). Let

$$A(u)=ae^{-k|u|}. \tag{13}$$

Substituting this into eq. (12) and taking a first order approximation, we get the solution as,

$$L(u)=1-\frac{Ga^2}{4C^4}e^{-2k|u|}+O(a^4). \tag{14}$$

Now we will pass on to the solution of the wave equations in this metric.

3. Scalar wave equation

3.1. *Scalar wave equation in gravitational wave metric*

The scalar wave equation in general relativity has the form,

$$\varphi_{;\alpha}^{;\alpha}=\frac{1}{\sqrt{-g}}(\sqrt{-g}\varphi^{;\alpha})_{,\alpha}=0 \tag{15}$$

that is,

$$(\sqrt{-g}\, g^{\mu\nu}\, \varphi_{,\mu})_{,\nu} = 0. \tag{16}$$

For our case the metric has the components,

$$g_{uv} = g_{vu} = -\tfrac{1}{2} \quad g_{xx} = L^2\, e^{2\beta} \quad g_{yy} = L^2\, e^{-2\beta} \tag{17}$$

$$g^{uv} = g^{vu} = -2 \quad g^{xx} = L^{-2}\, e^{-2\beta} \quad g^{yy} = L^{-2}\, e^{2\beta} \tag{18}$$

$$\text{and } \sqrt{-g} = \tfrac{1}{2} L^2. \tag{19}$$

A direct calculation substituting these values leads to

$$-4L^2 \frac{\partial^2 \varphi}{\partial u \partial v} - 4LL' \frac{\partial \varphi}{\partial v} - e^{-2\beta} \frac{\partial^2 \varphi}{\partial x^2} + e^{2\beta} \frac{\partial^2 \varphi}{\partial y^2} = 0. \tag{20}$$

This equation, which determines the scalar field can be solved by method of separation. We assume a solution of the form,

$$\varphi(x, y, u, v) = X(x)\, Y(y)\, A(u, v). \tag{21}$$

Substitution and separation of x and y variables lead to the equations,

$$\frac{d^2 X}{dx^2} = -k_x^2 X \qquad \frac{d^2 y}{dy^2} = -k_y^2 Y \tag{22}$$

$$-\frac{4L^2}{A} \frac{\partial^2 A}{\partial u \partial v} - \frac{4LL'}{A} \frac{\partial A}{\partial v} = k_y^2\, e^{2\beta} + k_x^2\, e^{-2\beta} \tag{23}$$

where k_x and k_y are the (real) separation constants. Their form is chosen so that in the absence of gravitational wave, the form of φ reduces to a wave solution. Notice that eq. (23) implies the fact,

$$\text{If } \frac{\partial A}{\partial v} = \frac{\partial \varphi}{\partial v} = 0 \text{ then } k_x^2 = k_y^2 = 0 \text{ so that } \varphi = \varphi(u), \tag{24}$$

$$\text{If } \frac{\partial A}{\partial u} = 0 \text{ then there is no solution of wave nature.} \tag{25}$$

If these cases do not occur, we can further separate the equation with respect to u and v variables. Then we get the general solution, in a straightforward manner as,

$$\varphi(x, y, u, v) = CL^{-1}(u) \exp i\, [k_x x + k_y y + k_v v + I(u) u] \tag{26}$$

where

$$I(u) = \frac{1}{u} \int^{u} \left(\frac{k_x^2\, e^{-2\beta} + k_y^2\, e^{2\beta}}{4 k_v\, L^2} \right) du. \tag{27}$$

here k_v is the final separation constant and C is the final integration constant.

Now we discuss the important features of this solution. We begin with the flat space approximation which can be obtained trivially by putting $\beta=0$ and $L=1$. We get

$$\varphi(x, y, u, v) = C \exp i \left[k_x x + k_y y + k_v v + \frac{k_x^2+k_y^2}{4k_v} u\right] \tag{28}$$

$$= C \exp i \left[k_x x + k_y y + \left(k_v - \frac{k_x^2+k_y^2}{4k_v}\right) z + \left(k_v + \frac{k_x^2+k_y^2}{4k_v}\right) t\right] \tag{29}$$

which is a plane monochromatic wave before the arrival of gravitational wave, with

$$k_z = k_v - \frac{k_x^2+k_y^2}{4k_v} \tag{30}$$

$$\omega = k_v + \frac{k_x^2+k_y^2}{4k_v} \tag{31}$$

Direct computation will show that $k^2 = \omega^2$ as it should.

The next simplest case is that of linear approximation to the first order in a. Since $L=1+O(a^2)$ it does not contribute in this order. So the effects arise only through $I(u)$ terms. This becomes, to first order in

$$I(u) = \frac{k_x^2+k_y^2}{4k_v} + \frac{1}{u}\left(\frac{k_y^2-k_x^2}{2k_v}\right) \int^u \beta du + O(a^2). \tag{32}$$

So the solution becomes,

$$\varphi(x, y, u, v) = C \exp i \left[k_x x + k_y y + k_z z + \omega t + \frac{k_y^2-k_x^2}{2k_v} \int \beta du\right]$$

$$= \varphi_{\text{Flat}} \left[1 + i \frac{k_y^2-k_x^2}{2k_v} \int_{-\infty}^{u} \beta du + O(a^2)\right]$$

$$= \varphi_{\text{Flat}} + \varphi_{\text{New}} \tag{33}$$

where

$$\varphi_{\text{New}} = \varphi_{\text{Flat}} \left[i \cdot \frac{k_y^2-k_x^2}{2k_v} \int_{-\infty}^{u} \beta(u) du\right]. \tag{34}$$

Thus we see that because of the interaction a new wave is produced. The direction of the new wave is the same as that of the original wave. Only that its amplitude is different, and in fact very small. This effect will persist even after the passage of the wave ('the gravitational wave leaves its marks'). If we consider any particular point

z we know that at $t=-\infty$ there was a scalar wave of the form $\varphi=C\exp(ik_\mu x^\mu)$. Now if we observe it a long time after at $t\to+\infty$ we find that the wave there, was modified to the form

$$\varphi_{\text{Final}}=\varphi_{\text{Initial}}\left[1+i\,\frac{k_y^2-k_x^2}{2k_v}\int_{-\infty}^{+\infty}\beta(u)du\right] \tag{35}$$

now using (5)

$$\varphi_{\text{Final}}=\varphi_{\text{Initial}}\left[1+i\,\frac{k_y^2-k_x^2}{2k_v}\int_{-\infty}^{+\infty}a\,e^{-k|u|}\,du\right]$$

$$=\varphi_{\text{Initial}}\left[1+i\cdot\frac{a}{k}\cdot\frac{k_y^2-k_x^2}{k_v}\right]. \tag{36}$$

So that

$$\frac{\varphi_{\text{Final}}-\varphi_{\text{Initial}}}{\varphi_{\text{Initial}}}=i\left(\frac{k_y^2-k_x^2}{k_v}\right)\frac{a}{k}=i\left(\frac{k_y^2-k_x^2}{2k_v}\right)\tau a. \tag{37}$$

Notice that $\tau=2/k$ denotes the characteristic time for which the gravitational wave is nonzero. It is quite reasonable that the change of amplitude is proportional to this time element; and to the amplitude of the gravitational wave a.

Another interesting feature of this result is that no new wave is generated (to this order of approximation), when the initial φ-wave travels equally inclined to the x and y axes. That is when $k_x=k_y$. Qualitatively speaking, one can say that in the case g_{xx} and g_{yy} terms 'neutralize' each other to the order of approximation considered.

So far the effect of $L(u)$ did not arise. To understand its effects we have to take one more order into consideration. We write, then

$$\varphi(x,y,u,v)=C\left(1-\tfrac{1}{4}a^2e^{-2k|u|}\right)^{-1}\exp i\,[k_x\,x+k_y\,y+k_v\,v+I(u)u]$$

$$=C\left(1+\tfrac{1}{4}a^2e^{-2k|u|}\right)\exp(ik_\mu x^\mu)\cdot f(u). \tag{38}$$

Here we have denoted by $f(u)$, the expression

$$f(u)=\left\{\exp i\left[I(u)u-\frac{k_x^2+k_y^2}{4k_v}u\right]\right\}_{\text{Corrected}} \tag{39}$$

corrected properly up to the required order. From previous discussion we know that it will have a form

$$f(u)=1+i\left(\frac{k_y^2-k_x^2}{2k_v}\right)\int_{-\infty}^{u}\beta(u)\,du+p(u) \tag{40}$$

where p denotes corrections of the order of β^2 which are pertinent but whose detailed form does not concern us. Thus the solution becomes

$$\varphi=C\left(1+i\,\frac{k_y^2-k_x^2}{2k_v}\int^{u}\beta du+p(u)+\tfrac{1}{4}\,a^2\,e^{-2k\,|u|}\right)\exp\,(ik_\mu\,x^\mu).$$

We write

$$\varphi=\varphi_{\text{Flat}}\,(1+A+B) \tag{41}$$

Where

$$A(u)=i\left(\frac{k_y^2-k_x^2}{2k_v}\right)\int\beta du+p(u) \tag{42}$$

$$B(u)=\tfrac{1}{4}\,a^2\,e^{-2k\,|u|}. \tag{43}$$

Thus the direct effect of the introduction of L into the picture is the generation of a new ' modulated ' wave for the form,

$$\varphi_M=\tfrac{1}{4}\,a^2\,e^{-2k\,|u|}\,\varphi_{\text{Flat}}.$$

Notice that this ' modulated ' wave has the properties of both the free scalar wave and the gravitational wave. It represents a wave propagating with a damped simple harmonic contour. Of course, if we concentrate on a particular point z, this scattered term will die out after a long time.

The other contribution arises mainly from β factor (even though there *is* an L term in $I(u)$ and quite similar to the one analysed previously in linear approximation. The only difference now is that we calculate it for one higher order by including p. This will give nonzero contribution when we take the limit $t\to\infty$, unlike the terms arising from L factor.

Thus, to this order, the scattered term has two effects it generates a damped harmonic wave (which is transient at a point) propagating along z axis and the other of the same nature of the initial wave with a different amplitude.

It is the $L(u)$ term that creates modulated waves with an exponential damping which propagates along z-axis. It can be easily seen that higher order terms in $L(u)$ varies as $e^{-4k\,|u|}$, $e^{-6k\,|u|}$, etc. Thus after a sufficiently long time the first order result will have importance. On the contrary the β term ' leaves the mark on the wave '. Our solution eq. (26) is exact and, in principle can be used with any metric of given β.

So far we were discussing the most general solution. However a special case given by eq. (24) is also of interest. This gives the result that there is absolutely no interaction between the two waves when they propagate along the same direction. This is an exact result and we see that this behaviour is also exhibited in other cases.

3.2. *Scalar wave in the metric of electromagnetic wave*

The equation for the scalar wave has the same form as eq. (15) even when the spacetime is not a vacuum. Thus all our discussion in section (3.1) can be taken over here by putting $\beta\equiv 0$. Thus the solution becomes

$$\varphi(x, y, u, v) = CL^{-1} \exp i\,[k_x x + k_y y + k_v v + I^*(u)u] \tag{45}$$

$$I^*(u) = \frac{1}{u}\left(\frac{k_x^2+k_y^2}{4k_v}\right)\int^{u} \frac{du}{L^2}. \tag{46}$$

The flat space approximation, of course, is a plane wave. Notice that $L=1+O(A^2)$, so that, to first order in vector potential, there is no interaction. (In contrast, there was an interaction with gravitational wave to first order in β). This is to be expected because it is the gravitational field produced by the electromagnetic wave that causes the interaction, which is a second order effect. In the second order we have, (G, C introduced for showing order of magnitude)

$$L(u) = 1 - \frac{Ga^2}{4C^4}\, e^{-2k\,|u|} + O(a^4) \tag{47}$$

$$I^*(u) = \frac{1}{u}\left(\frac{k_x^2+k_y^2}{4k_v}\right)\int_{-\infty}^{u}\left(1 + \frac{Ga^2}{2C^4}\, e^{-2k\,|u|}\right) du$$

$$\cong \left(\frac{k_x^2+k_y^2}{4k_v}\right) + \frac{1}{u}\left(\frac{k_x^2+k_y^2}{4k_v}\right)\left(\frac{Ga^2}{2C^4}\right)\left(\frac{1}{2k}\right)\left[2 - e^{-2k\,|u|}\right] (\text{for } u>0) \tag{48}$$

so that,

$$\varphi = C\left[1 + \frac{Ga^2}{4C^4}\, e^{-2k\,|u|} + \left(\frac{Ga^2}{2C^4}\right)\left(\frac{i}{2k}\right)\left(\frac{k_x^2+k_y^2}{4k_v}\right)\left(2 - e^{-2k\,|u|}\right)\right]$$
$$\exp\,(ik_\mu x^\mu)$$

(correct to 2nd order in a) (for $u>0$) (49)

Thus the effect of the interaction arises in two ways. Writing

$$\varphi = \varphi_{\text{Flat}}\,(1 + A + B) \tag{50}$$

with

$$A = \frac{Ga^2}{4C^4}\, e^{-2k\,|u|}\left(1 - i\left(\frac{k_x^2+k_y^2}{2k_v}\right)\tau\right) \tag{51}$$

$$B = i\,\frac{Ga^2}{C^4}\left(\frac{k_x^2+k_y^2}{4k_v}\right)\tau\,;\ \text{with } \tau = \frac{1}{2k}. \tag{52}$$

We see that A will produce a propagating wave with damped harmonic contour and B generates an oscillatory wave of the same nature as φ_{Flat} with a changed amplitude. However in a fixed point the effect of A is transient and only B ' leaves the mark'. Also there is no interaction between the waves when propagating along the same direction. The effect will be similar even when the exact case is considered. There will be two new waves. One will have a modulated time dependence, while the other will have the same phase as the original wave with different amplitude.

Before we conclude this section it is to be noticed that eq. (51) can be written as

$$A=\frac{Ga^2}{4C^4}e^{-2k|u|}\left(1+\left(\frac{k_x^2+k_y^2}{2k_v}\right)^2\tau^2\right)^{1/2}e^{-i\triangle};\ \tan\triangle=\left(\frac{k_x^2+k_y^2}{2k_v}\right)\tau \quad (53)$$

so that this term adds a constant phase difference to the original wave, in addition to modulating the amplitude.

That completes our analysis of the scalar wave.

4. Solution of electromagnetic wave equation

4.1. *Electromagnetic wave in gravitational wave metric*

The equation to the vector potential A^μ, which describes the electromagnetic field, (in Lorentz gauge) satisfies the equation (see Misner *et al* 1973)

$$A^{\alpha;\beta}_{\beta}+\mathbf{R}^\alpha{}_\beta A^\beta=0 \quad (54)$$

which reduces to

$$A_\beta{}^{\alpha;\beta}=0 \quad (55)$$

since for the gravitational wave $\mathbf{R}_{\alpha\beta}=0$. The Christoffel symbols for our metric are,

$$\Gamma^x{}_{xu}=\Gamma^x{}_{ux}=\frac{L'}{L}+\beta';\ \Gamma^y{}_{yu}=\Gamma^y{}_{yu}=\frac{L'}{L}-\beta'$$

$$\Gamma^v{}_{xx}=L^2e^{2\beta}\left(\frac{L'}{L}+\beta'\right);\ \Gamma^v{}_{yy}=L^2e^{-2\beta}\left(\frac{L'}{L}-\beta'\right). \quad (56)$$

Substituting them into eq. (55) leads, after a straightforward but tedious, computation to the equations,

$$\Box^2A^u-\frac{2L'}{L}A^u{}_{,v}=0 \quad (57)$$

$$\Box^2A^v+2\left(\frac{L'}{L}+\beta'\right)A^x{}_{,x}+2\left(\frac{L'}{L}-\beta'\right)A^y{}_{,y}-\frac{2L'}{L}A^v{}_{,v}=0 \quad (58)$$

$$\Box^2A^x-\left(\frac{6L'}{L}+4\beta'\right)A^x{}_{,v}+2L^{-2}e^{-2\beta}\left(\frac{L'}{L}+\beta'\right)A^u{}_{,x}=0 \quad (59)$$

$$\Box^2 A^y - \left(\frac{6L'}{L} - 4B'\right) A^y{}_{,v} + 2L^{-2}e^{+2\beta}\left(\frac{L'}{L} - \beta'\right) A^u{}_{,y} = 0 \tag{60}$$

Where $\Box^2$ denotes the generalised D'Alembertian,

$$\Box^2 = g^{\mu\nu} \frac{\partial}{\partial x^\mu} \cdot \frac{\partial}{\partial x^\nu} = -4\frac{\partial^2}{\partial u \partial v} + L^{-2} e^{-2\beta} \frac{\partial^2}{\partial x^2} + L^{-2} e^{+2\beta} \frac{\partial^2}{\partial y^2} \tag{61}$$

and a comma denotes ordinary derivative with respect to the corresponding co-ordinate

$$A^k{}_{,\rho} = \frac{\partial A^k}{\partial x^\rho}. \tag{62}$$

The equations can be solved in a step by step manner. We start with eq. (57). This equation has the same structure as the equation for the scalar field and thus following the same route, we get the solutions,

$$A^u = f(u) \text{ arbitrary} \quad \text{(i)}$$

$$A^u = C_1 L^{-\frac{1}{2}} e^{i\theta} e^{iG(u)} \quad \text{(ii)} \tag{63}$$

where we have introduced the notations

$$\theta = k_x x + k_y y + k_v v \tag{64}$$

$$G(u) = \int_{-\infty}^{u} \frac{k_x^2 e^{-2\beta} + k_y^2 e^{+2\beta}}{4k_v L^2} \, du. \tag{65}$$

Now we substitute these solutions into (59) and (60) and solve for A^x and A^y. Then substituting these into eq. (58) we finally solve for A^v. The solution (63i) leads to no difficulty and gives the nearly trivial result,

$$A^\mu = f^\mu(u) \text{ for all } \mu, \text{ functions arbitrary} \tag{66}$$

indicating once again that a wave travelling along the same direction does not interact.

The other solution (63ii) leads to a more general set. The procedure can be somewhat simplified by assuming in the very beginning the solutions to be of the form,

$$A^u = \varphi_1(u) e^{i\theta}\,;\ \phi_1 = C_1 L^{-\frac{1}{2}} e^{iG}$$

$$A^v = \varphi_2(u) e^{i\theta}$$

$$A^x = \varphi_3(u) e^{i\theta}$$

$$A^y = \varphi_4(u) e^{i\theta} \tag{67}$$

where φ_j with $j=2, 3, 4$ has to be determined. Substituting in our equations gives the equation for ϕ_3 as

$$\varphi'_3+p(u)\phi_3=Q(u) \tag{68}$$

with

$$P(u)=-i\left(\frac{k_x{}^2e^{-2\beta}+k_y{}^2e^{+2\beta}}{4k_vL^2}\right)+\frac{3}{2}\frac{L'}{L}+\beta' \tag{69}$$

$$Q(u)=C_1\left(\frac{K_x}{2k_v}\right)L^{-5/2}e^{-\beta}e^{iG}\left(\frac{L'}{L}+\beta'\right). \tag{70}$$

Thus eq. (68) can be solved by elementary methods to give,

$$\varphi_3(u)=L^{-3/2}e^{-\beta}e^{iG}\left[\frac{C_1k_x}{2k_v}\int^u\frac{e^{-\beta}}{L}\left(\frac{L'}{L}+\beta'\right)du+C_3\right] \tag{71}$$

where C_3 is the integration constant.

Similarly we get, for φ_4 the solution (the equations for φ_3 and φ_4 and, hence the solutions differ only in sign of $\beta(u)$; and replacement of x by y)

$$\varphi_4(u)=L^{-3/2}e^{+\beta}e^{iG}\left[\frac{C_1k_y}{2k_v}\int^u\frac{e^{\beta}}{L}\left(\frac{L'}{L}-\beta'\right)du+C_4\right]. \tag{72}$$

Substituting the values for φ_1, φ_3, φ_4, in the equation for $\varphi_2(u)$ we reduce it to the form,

$$\varphi'_2(u)+R(u)\varphi_2=S(u) \tag{73}$$

where

$$R(u)=\frac{k_x{}^2e^{-2\beta}+k_y{}^2e^{2\beta}}{4iK_vL^2}+\frac{L'}{2L} \tag{74}$$

$$S(u)=\frac{k_x}{2k_v}\left(\frac{L'}{L}+\beta'\right)\varphi_3+\frac{k_y}{2k_v}\left(\frac{L'}{L}-\beta'\right)\varphi_4 \tag{75}$$

Solving eq. (73), we get the solution,

$$\varphi_2(u)=\frac{L^{-\frac{1}{2}}e^{iG}}{2kv}\int F(u)du+C_2L^{-\frac{1}{2}}e^{iG}$$

$$F(u) = \frac{1}{L}\left[k_x e^{-\beta}\left(\frac{L'}{L}+\beta'\right)\left(\frac{C_1 k_x}{2kv} f_1(u)+C_3\right)\right.$$

$$\left. + k_y e^{\beta}\left(\frac{L'}{L}-\beta'\right)\left(\frac{C_1 k_y}{2kv} f_2(u)+C_4\right)\right] \tag{76}$$

where

$$f_1(u) = \int \frac{e^{-\beta}}{L}\left(\frac{L'}{L}+\beta'\right) du \tag{77}$$

$$f_2(u) = \int \frac{e^{\beta}}{L}\left(\frac{L'}{L}-\beta'\right) du \tag{78}$$

Thus we have solved the equations. Equations (67, 71, 72, 76) express the vector potential A^{μ} in terms of the metric functions L and B. In the process we have introduced four integration constants C_i (with $i=1, 2, 3, 4$). To find their physical significance we take the flat space limit (initial state at $t=-\infty$) putting $L=1$, $\beta=0$ which will give,

$$A^u=C_1 e^{ik_\mu x^\mu};\ A^v=C_2 e^{ik_\mu x^\mu};\ A^x=C_3 e^{ik_\mu x^\mu};\ A^y=C_4 e^{ik_\mu x^\mu}. \tag{79}$$

Thus the constants have the simple physical interpretation that they are the amplitudes of the vector potential in the flat space limit. Suppose that we want to find the effect of the gravitational wave on a vector potential of specified form. We can then choose the constants C_i so that in the flat space limit it corresponds to the wave under consideration. Then from our general solution, substituting these values for C_i, we can understand the nature and effect of the interaction.

Qualitatively, one expects the solution to behave in a similar fashion to that of a scalar wave. With two types of newly generated waves one wave with a modulated amplitude and another with same time dependence but different amplitude. But the main complication arises from the fact that, because of the mixing of the components, components absent in the initial wave may be generated. This will give rise to the effects such as rotation of the plane of polarization, bending, etc. A detailed discussion of these effects is postponed to a subsequent paper.

4.2. *Electromagnetic wave in the electromagnetic wave metric*

The solution in this situation cannot be obtained directly by putting $\beta=0$ in the previous solution. This is because, in the presence of the electromagnetic wave, $\mathbf{R}_{\alpha\beta}$ is no longer zero and so one has to deal with the full equation

$$A_{\mu}{}^{\alpha;\mu}+\mathbf{R}_{\beta}{}^{\alpha}A^{\beta}=0. \tag{80}$$

However because of the simple form of the metric we are using the only nonzero component of $\mathbf{R}_{\beta}{}^{\alpha}$ is,

$$\mathbf{R}_u{}^v=\frac{4L''}{L}. \tag{81}$$

So that the equations become,

$$A_{\mu}{}^{\alpha;\mu}=0 \text{ for } \alpha=x, y, u \tag{82}$$

$$A_{\mu}{}^{v;\mu}=-\frac{4L''}{L}A^{u}. \tag{83}$$

From eq. (82) we see that for u, x, y components the solutions obtained in the last section can be adopted, by just putting $\beta\equiv0$. Thus we get assuming the solutions to be

$$A^{u}=\varphi_1(u)e^{i\theta}$$

$$A^{v}=\varphi_2(u)e^{i\theta}$$

$$A^{x}=\varphi_3(u)e^{i\theta}$$

$$A^{y}=\varphi_4(u)e^{i\theta} \tag{84}$$

We can write for φ_1, φ_3, φ_4

$$\varphi_1=C_1L^{-1/2}e^{i\alpha} \tag{85}$$

$$\varphi_3=L^{-3/2}e^{i\alpha}\left[C_3-C_1\frac{k_x}{2k_v}\cdot\frac{1}{L}\right] \tag{86}$$

$$\varphi_4=L^{-3/2}e^{i\alpha}\left[C_4-C_1\frac{k_y}{2k_vL}\right] \tag{87}$$

where $\alpha=G$ (with $\beta=0$)$=\dfrac{k_x{}^2+k_y{}^2}{4k_v}\displaystyle\int\frac{du}{L^2}$. (88)

The equation for A^v now reads

$$\Box^2A^v+\frac{2L'}{L}A^x{}_{,x}+\frac{2L'}{L}A^y{}_{,y}-\frac{2L'}{L}A^v{}_{,v}=-\frac{4L''}{L}A^u \tag{89}$$

with $$\Box^2=-4\frac{\partial^2}{\partial u\partial v}+L^{-2}\frac{\partial^2}{\partial x^2}+L^{-2}\frac{\partial^2}{\partial y^2} \tag{90}$$

Substituting φ_1, φ_3, φ_4 gives,

$$\varphi_2'(u)+R(u)\varphi_2=S(u) \tag{91}$$

with

$$R(u)=\frac{k_x{}^2+k_y{}^2}{4ik_vL^2}+\frac{L'}{2L} \tag{92}$$

$$S(u)=\frac{k_x}{2k_v}\cdot\frac{L'}{L}\varphi_3+\frac{k_y}{2k_v}\cdot\frac{L'}{L}\varphi_4+\frac{4L''}{L}\varphi_1 \tag{93}$$

which leads to the solution of eq. (91) as,

$$\varphi_2(u)=\frac{L^{-1/2}e^{i\alpha}}{2k_v}\int K(u)du+C_2L^{-1/2}e^{i\alpha}$$

$$K(u)=\frac{1}{L}\left[\frac{L'}{L}k_x\left(C_3-\frac{C_1k_x}{2k_vL}\right)+\frac{L'}{L}k_y\left(C_4-\frac{C_1k_y}{2k_vL}\right)+4C_1L''\right]. \tag{94}$$

That solves our problem. Equations (84, 85, 86, 87 and 94) determine the vector potential A^μ in terms of the metric function $L(u)$.

Taking the flat space limit one sees that the constants C_i has the same interpretation as in the gravitational wave case. Detailed discussion of these results, will be presented in a later paper.

It should be noted that these interactions discussed classically correspond to photon-graviton and photon-photon (via graviton exchange) interaction in the quantised level.

Reference

Misner C W, Thorne K S and Wheeler J A 1973 *Gravitation* (San Francisco: Freeman & Co.)