ON POLAR RECIPROCAL CONVEX DOMAINS

by R. P. BAMBAH, *Institute for Advanced Study, Princeton, N.J., U.S.A., and Punjab University College, Hoshiarpur.*

(Communicated by H. Gupta, F.N.I.)

(*Received September 4; read December 4, 1953.*)

1. Let k and K be two symmetrical convex domains in the plane with centres at the origin. Suppose k and K are polar reciprocal with respect to the unit circle C centred at the origin.

Let $\Delta(K)$ be the critical determinant* of K and $c(k)$ the covering constant of k, i.e. the upper bound of the determinants of the covering lattices for k, where a lattice Λ is called a covering lattice for k if every point in the plane lies in one at least of the bodies obtained from k by applying to it all possible lattice translations.

Mahler (1948) proved that

$$\tfrac{1}{2} \leq \Delta(K)\ \Delta(k) \leq \tfrac{3}{4}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \text{(A)}$$

and that both the inequalities are best possible.

It seems worth noticing that

$$2 \leq \Delta(K)\ c(k) \leq \tfrac{9}{4}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

and that both inequalities here again are best possible.

2. In this section we prove (1).

Write $H_s(K)$ for the area of the smallest symmetrical convex hexagon† circumscribed to K and $h_s(k)$ for the area of the largest symmetrical convex hexagon† inscribed in k. Then it is known that

$$\Delta(K) = \tfrac{1}{4}\ H_s(K), \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

and

$$c(k) = h_s(k); \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

(2) follows from Theorem 1 of Mahler (1948) and Theorem 3 of Dowker (1944) while (3) is lemma 8 of Bambah and Rogers (1952).

Let H be the symmetrical convex hexagon circumscribed to K with area $a(H) = H_s(K)$. Then H', the polar reciprocal of H, is a symmetrical convex hexagon inscribed in k so that

$$c(k) = h_s(k) \geq a(H'),$$

where $a(H')$ is the area of H'. Consequently

$$\Delta(K)\, c(k) = \tfrac{1}{4} H_s(K)\, h_s(k) \geq \tfrac{1}{4} a(H)\, a(H') \geq 2; \quad \ldots \quad \ldots \quad (4)$$

the last inequality following from inequalities A of Mahler (1948).

---

* For definition see e.g., K. Mahler, *Proc. Royal Soc.*, A, 187 (1946), 151.

† By a hexagon we mean a polygon with at most six sides.

Now suppose h is the convex symmetrical hexagon of area $a(h) = h_s(k)$ inscribed in k. Then h', its polar reciprocal, is a convex symmetrical hexagon circumscribed to K, so that

$$\Delta(K) = \tfrac{1}{4}H_s(K) \leq \tfrac{1}{4}a(h'),$$

where $a(h')$ is the area of h'. Therefore we have

$$\Delta(K)\,c(k) = \tfrac{1}{4}H_s(K)\,h_s(k) \leq \tfrac{1}{4}a(h')\,a(h) \leq \tfrac{9}{4}, \qquad \ldots \ldots \quad (5)$$

the last inequality following from inequalities A of Mahler (1948).

(4) and (5) together prove (1).

Let k and K be the squares:

$$k: |x| \leq 1;\ |y| \leq 1;\ K: |x| + |y| \leq 1.$$

Then

$$c(k) = 4,\ \Delta(K) = \tfrac{1}{2} \text{ and } c(k)\ \Delta(K) = 2. \ldots \quad \ldots \quad \ldots \quad (6)$$

If k and K become the unit circle or the regular hexagon inscribed in the unit circle and its polar reciprocal, then

$$c(k) = \frac{3\sqrt{3}}{2},\ \Delta(K) = \frac{\sqrt{3}}{2},$$

and

$$c(k)\ \Delta(K) = \tfrac{9}{4}, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (7)$$

which together with (6) shows that the inequalities (1) are best possible.

3. From the well-known results

$$3^{-n+1}\ V(K) \leq c(K) \leq V(K) \text{ (Rogers)},$$

$$2^{-n}\ V(k) \leq \Delta\ (k) \leq \frac{1}{a_n} V(k), \quad a_n \to 4{\cdot}921 \text{ as } n \to \infty$$

(Minkowski and Davenport-Rogers)

and

$$\frac{2^n J_n}{(n!\,n^n)^{\frac{1}{2}}} \leq V(k)V(K) \leq J_n^2$$

(Dvoretzky Rogers and Santalo)

(where J_n is the volume of the unit sphere in n dimensions), one can find analogous inequalities for $c(k)\ \Delta(K)$ and $c(k)\ c(K)$ for polar reciprocal symmetrical convex bodies k and K in n dimensions but they appear to be far from best possible.

References

Bambah, R. P. and Rogers, C. A. (1952). *Jour. London Math. Soc.*, **23**, 304–314.
Dowker, C. H. (1944). *Bull. Amer. Math. Soc.*, **50**, 120–122.
Mahler, K. (1948). *Proc. Kon. Ned. Akad V Weten., Amsterdam,* **51**, 482–485.

Issued February 20, 1954.