

# ARE CRITICAL EXPONENTS MULTIVALUED OR UNIQUE ?

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Received September 8, 1972

(Communicated by Prof. K. P. Sinha, F.A.Sc.)

## ABSTRACT

The scaling law theories of the critical point phenomena, together with possible stability conditions at the critical point, allow a whole range of critical exponents in a 2-dimensional system but only two sets of exponents in a 3-dimensional system. In the 3-dimensional case the two sets explain the differences between the critical exponents in magnetic and ferroelectric transitions. It also seems possible to correlate the small differences between the exponents of liquid/gas and magnetic transitions. No set of exponents is allowed in a 1-dimensional case, corresponding to the absence of phase transitions in a 1-dimensional system with short range interactions.

## 1. INTRODUCTION

RECENTLY, Baxter<sup>1</sup> has announced a solution of the eight vertex problem, which includes as special cases the square lattice Ising, dimer, ice and Slater KDP models. One of the unexpected features of the solution of the 2-dimensional problem is that the form of the singularity in the configurational free energy depends upon the nature and the magnitude of the interactions, which implies that the critical exponents could depend upon the strength of the interactions. This is surprising because the experiments appear to give the same exponents in many different physical systems: for instance the configurational specific heat has a nearly logarithmic or weak power law  $\alpha \sim 1/8$  behaviour in almost all the experimental studies.<sup>2-4</sup> It is the presence of such common features which is responsible for the concepts of scaling and universality in critical phenomena.

It is interesting to point out that the scaling laws, taken in conjunction with the conditions for the stability of the system, admit the possibility of an infinite number of sets of critical exponents for a 2-dimensional system but only two distinct sets of exponents in 3 dimensions. Furthermore, the 3-dimensional result seems to clear up the anomalous differences between

the critical exponents observed in magnetic, liquid-gas or alloy ordering transitions on one hand and those observed in dielectric transitions on the other.<sup>2, 4, 5</sup> There is now clear evidence that in ferroelectric transitions  $\beta = 1/2$ ,  $\gamma = 1$  while in magnetic or liquid-gas critical points  $\beta \approx 1/3$ ,  $\gamma \approx 1\frac{1}{4}$ , at the same time all these transitions yielding  $\alpha \sim 0$  or  $1/8$ .

## 2. INEQUALITY ARISING FROM SCALING LAWS AND STABILITY CONDITIONS

The scaling law equation of state in the vicinity of the critical point is<sup>6</sup>

$$P = \rho \Phi(t, \rho^{1/\beta}) = \rho^\delta \Phi(t/\rho^{1/\beta}); \quad \delta = 1 + (\gamma/\beta). \quad (1)$$

Since the equivalent quantities in the magnetic, dielectric and other systems ( $P \leftrightarrow H \leftrightarrow E$ ;  $\rho_L - \rho_G \leftrightarrow M \leftrightarrow P_s$ , etc.) are well known, there is no loss of generality in considering the case of fluids. The thermal stability of the system requires that the first non-vanishing derivative of the free energy be an even, say  $2n$ -th, derivative. In terms of the state variable  $P$ , the stability condition at  $T_c$  is accordingly

$$\left. \begin{aligned} \frac{\partial P}{\partial \rho} = \frac{\partial^2 P}{\partial \rho^2} = \dots = \text{to } 2n - 2 \text{ terms} = 0 \\ \frac{\partial^{2n-1} P}{\partial \rho^{2n-1}} \neq 0 \end{aligned} \right\}. \quad (2)$$

This condition applied to the equation of state (1) yields

$$\delta = 2n - 1. \quad (3)$$

In deriving eq. (3), it is further assumed that the  $2n$ -th derivative of the free energy, besides being non-zero, is also definite. This assumption is equivalent to a well-behaved critical isotherm  $T = T_c$  not only in the 1 phase region but also at the critical point. The co-existence boundary is usually taken to be a branch cut in the phase plane, but the above assumption and the consequent integral value of  $\delta$  are not new.<sup>7</sup> Experiments give a value of  $\delta$  which is near but not necessarily equal to an integer and in this respect the present analysis is suggestive but not rigorous.

Now the scaling law hypothesis gives the following relations among the critical exponents:<sup>3</sup>

$$2 \frac{d\gamma}{d\eta} = 2 - \alpha = \gamma + 2\beta, \quad (4)$$

where  $d$  is the dimensionality of the system. The exponent  $\eta$  is further restricted<sup>8</sup> as

$$0 \leq \eta \leq 1, \quad (5)$$

and this yields

$$0 \leq \eta = \frac{d + (2 - d)n}{n} \leq 1. \quad (6)$$

The restriction on  $\eta$  would normally appear to be  $0 \leq \eta \leq 2$  as could be inferred from the definition<sup>9</sup> of the exponent  $\eta$  on the basis of the pair correlation function  $G(k) \sim 1/k^{2-\eta}$  as  $k \rightarrow 0$ . However, as Fisher<sup>9</sup> himself points out, the stronger restriction  $0 \leq \eta \leq 1$  seems to be called for. In any case, this point does not alter the subsequent discussions. In such a case the value  $n = 1$ , corresponding to the situation near an ordinary 1st order phase transition, would also be allowed. Since the present analysis is concerned with the singular behaviour of compressibility, etc., the results are not changed.

#### DISCUSSION OF THE INEQUALITY

Consider now a 2-dimensional system. Eq. (6) poses no restriction on the value of  $n \geq 2$  and an infinite range of values for the exponents is allowed. In the case of a 1-dimensional system  $0 \leq (1 + n)/n \leq 1$  allows no value of  $n$ , which is a reflection of the absence of critical phenomena in a 1-dimensional system. Only one value  $n = 2$  is allowed in a 4-dimensional system and if  $\alpha = 0$ , then one has  $\beta = \frac{1}{2}$ ,  $\gamma = 1$ ,  $\delta = 3$ , etc., a set of values identical with the most recent estimates<sup>10</sup>.

The situation for a 3-dimensional system,  $0 \leq (3 - n)/n \leq 1$ , is interesting and allows only two values of  $n$ . The values of the other exponents are given by

$$\left. \begin{array}{l} n = 2, \quad \eta = \frac{1}{2}, \quad \delta = 3, \quad \gamma = 2\beta; \quad \alpha = 2 - 4\beta \\ n = 3, \quad \eta = 0, \quad \delta = 5, \quad \gamma = 4\beta; \quad \alpha = 2 - 6\beta \end{array} \right\}. \quad (7)$$

In almost all the investigations, the specific heat seems to have a logarithmic or weak power law singularity  $\alpha \sim 0$ . Then the two sets of critical exponents are

$$\left. \begin{array}{l} \alpha = 0 \text{ (log)}, \quad \beta = 1/2, \quad \gamma = 1, \quad \delta = 3 \\ \alpha = 0, \quad \beta = 1/3, \quad \gamma = 4/3, \quad \delta = 5 \end{array} \right\}. \quad (8)$$

This seems to explain a striking result observed in many experiments. Magnetic transitions appear to follow the second set  $\beta \sim 1/3$ ,  $\gamma \sim 4/3$ , whereas dielectric transitions give the critical exponents  $\beta \sim 1/2$ ,  $\gamma \sim 1$  and yet both groups of transitions show the same singularity in specific heat  $\alpha \sim 0$ . The specific heat divergence near the liquid-gas critical point seems to be a weak power law  $\alpha = 1/8$  singularity.<sup>11</sup> If so, one would have  $\beta = 5/16$ ,  $\gamma = 5/4$ . There is some evidence as yet inconclusive that the  $\gamma$  exponents are slightly different in liquid-gas transitions, being nearer this set  $\gamma = 5/4$ , etc., than the set  $\gamma = 4/3$ , etc. obeyed at magnetic transitions. So in the absence of exact solutions of 3-dimensional models, it is difficult to comment on the applicability of the interesting exact results in 2 and 4 dimensions to a study of 3-dimensional systems.

#### 4. ACKNOWLEDGEMENT

Thanks are due to Drs. K. Govindarajan and B. Viswanathan for discussions and to the DAE for financial support.

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- 3755/72. Published by B. S. Venkatachar, Editor, "Proceedings of the Indian Academy of Sciences", Bangalore and printed by V. J. F. Jesudason at the Bangalore Press, Bangalore-18.