AN APPLICATION OF THE THEORY OF FINITE STRAIN

[illegible]

[illegible] problems has been developed on the [illegible] of [illegible] and [illegible] have been [illegible] Many [illegible] The [illegible] of the [illegible] the amplitude of oscillation having [illegible] that the limits of perfect elasticity are [illegible] the string remains [illegible] throughout. [illegible] well known [illegible] is not necessarily independent of the [illegible] oscillation.

[illegible] a light elastic string whose other end [illegible] of the string [illegible], and the increase in its [illegible] L. Some writers use the initial [illegible] length of the string in the Hooke's [illegible] when the stretch is small, but not when [illegible] Again, the generalized Hooke's Law [illegible] the tension and stretch are connected by means of [illegible]

$$T = \tfrac{1}{2}E\left[1 - \frac{1}{(1+s)^2}\right] \qquad (1)$$

$$s = \frac{\text{increase in length}}{\text{unstretched length}},$$

[illegible]

[illegible] when s^2 and higher powers of s can be [illegible]

[illegible] 244, A, [illegible]

[illegible] 1936, 156, A, [illegible]

If x is measured downwards from the fixed point, the equation of motion is

$$\frac{m\, d^2x}{dt^2} = mg - \tfrac{1}{2}\,\mathrm{E}\left(1 - \frac{l^2}{x^2}\right). \tag{2}$$

Putting $y = x/l$, we have on integration

$$\tfrac{1}{2}\, ml\left(\frac{dv}{dt}\right)^2 = (mg - \tfrac{1}{2}\,\mathrm{E})\, y - \tfrac{1}{2}\,\frac{\mathrm{E}}{y} + \mathrm{C}, \tag{3}$$

C being a constant of integration.

Now the velocity of the particle vanishes at two points in its line of motion. Let these be given by $y = \alpha$ and $y = \beta$ $(\alpha > \beta)$. We can re-write (3) as

$$\tfrac{1}{2}\, ml\left(\frac{dy}{dt}\right)^2 = (\tfrac{1}{2}\,\mathrm{E} - mg)\,\frac{(\alpha - y)\,(y - \beta)}{y}, \tag{4}$$

where
$$\alpha\,\beta = \frac{\mathrm{E}}{\mathrm{E} - 2\,mg}. \tag{5}$$

$\mathrm{T} = mg = \tfrac{1}{2}\,\mathrm{E}$ gives infinite elongation, and hence corresponds to the yield point. Therefore, $mg < \tfrac{1}{2}\,\mathrm{E}$.

From (4) we see that t and y are connected by means of the elliptic integral of the second kind, *viz.*,

$$t = \sqrt{\frac{ml}{\mathrm{E} - 2mg}} \int_y^\alpha \frac{y\, dy}{\sqrt{y\,(\alpha - y)\,(y - \beta)}} \tag{6}$$

Putting $y = \alpha\,(1 - k^2 \sin^2\phi)$, where $k^2\alpha = (\alpha - \beta)$, we get

$$t = 2\,\sqrt{\alpha}\cdot\sqrt{\frac{ml}{\mathrm{E} - 2\,mg}} \int_0^\phi \sqrt{1 - k^2\sin^2\phi}\; d\phi$$

$$= 2\,\sqrt{\alpha}\cdot\sqrt{\frac{ml}{\mathrm{E} - 2mg}}\;\mathrm{E}\,(k, \phi). \tag{7}$$

But
$$mg = \tfrac{1}{2}\,\mathrm{E}\left[1 - \frac{1}{(1 + \mathrm{L}/l)^2}\right],$$

or

$$\sqrt{\frac{\mathrm{E}}{\mathrm{E} - 2mg}} = 1 + \frac{\mathrm{L}}{l}, \tag{8·1}$$

so that
$$\alpha\,\beta = \left(1 + \frac{\mathrm{L}}{l}\right)^2. \tag{8·2}$$

Putting
$$k^2 = 1 - \frac{\beta}{\alpha} = \sin^2\theta,$$

we get,

$$\alpha = \left(1 + \frac{\mathrm{L}}{l}\right)\sec\theta = \frac{l'}{l}\sec\theta, \tag{8·3}$$

and $$\text{Amplitude} = l\,(\alpha - \beta) = l\left(1 + \frac{L}{l}\right)\sin\theta\tan\theta$$
$$= l'\sin\theta\tan\theta, \tag{8·4}$$
where l' is the stretched length in the position of equilibrium.

We can now rewrite (7) as

$$t = 2\sqrt{\frac{ml}{E}}\left(\frac{l'}{l}\right)^{\frac{3}{2}}\sqrt{\sec\theta}\,E\,(k,\phi). \tag{9}$$

The period of a complete oscillation is, therefore, given by

$$T = 4\sqrt{\frac{ml}{E}}\left(\frac{l'}{l}\right)^{\frac{3}{2}}\sqrt{\sec\theta}\,E_0, \tag{10}$$

E_0 being the complete elliptic integral of the second kind.

If Hooke's Law is adopted, the period is known to be

$$T_0 = 2\pi\sqrt{\frac{ml}{E}}. \tag{11}$$

Hence,

$$\frac{T}{T_0} = \frac{2}{\pi}E_0\left(\frac{l'}{l}\right)^{\frac{3}{2}}\sqrt{\sec\theta}. \tag{12}$$

As the amplitude increases with both θ and l', the ratio (T/T_0) increases with the amplitude.

The variation of $(2/\pi)\,E_0\,\sqrt{\sec\theta}$ and $\sin\theta\tan\theta$ with respect to θ is given in the following table:

TABLE

θ	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
$(2/\pi)\,E_0\,\sqrt{\sec\theta}$	1	1·000	1·001	1·004	1·014	1·037	1·090	1·217	1·589	∞
Sin θ tan θ	0	0·031	0·124	0·289	0·539	0·913	1·500	2·582	5·585	∞

It is clear from the table that the period does not vary very much with the amplitude for small values of θ. But this variation cannot be neglected when θ exceeds 40°.

For a given value of θ we see from (12) that

$$(T/T_0)^2 \propto l'^3,$$

which shews that the variation of (T/T_0) with l' follows the law of the semi-cubical parabola.