TRANSVERSE WAVES IN CANALS.

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No case of transverse waves in canals of variable depth, whose section is a closed curve, has yet been exactly solved. When the free surface is at the level of the axis of a circular canal both Lamb[1] and Rayleigh[2] have obtained an approximate solution. But it is shown elsewhere[3] that no exact stable type can exist in a circular canal, and that the constrained type assumed by Lamb and Rayleigh to get an approximate result is unstable. In the present paper we propose to shew that transverse waves cannot be produced in a canal of closed section. In fact, one can anticipate this result from physical considerations. We shall also shew that in certain cases an approximate value of the frequency can be obtained, and that this value approaches $(g/\rho)^{\frac{1}{2}}$ as the depth becomes very small, ρ being the radius of curvature at the lowest point.

For simplicity we take the section of the canal symmetrical about the vertical, the origin at the highest point, and the axis of y vertically downwards (see Fig. 1).

The polar equation of a comprehensive class of closed curves symmetrical about $\theta = \frac{1}{2}\pi$ is given by

$$\Sigma\,(-1)^n\,a_{2n+1}\,\frac{\sin{(2n+1)}\,\theta}{r^{2n+1}} = 1. \tag{1}$$

We can take the highest point as $r = 0$, $\theta = 0$, and the lowest point as $r = c$, $\theta = \frac{1}{2}\pi$. If we further assume that all the a's are positive, $\theta = \frac{1}{2}\pi$ will only give one positive value of r.

Any function which is to represent the solution must be finite and continuous, together with its first partial derivatives, at least for all points of the section excepting the highest point. We can therefore assume the velocity potential ϕ and the stream function ψ to be given by either

$$\phi + i\psi = \Sigma\,K_n \sin k_n \left\{ \Sigma\,(-1)^n\,\frac{a_{2n+1}}{z^{2n+1}} + i \right\}, \tag{2.1}$$

[1] Lamb, *Hydrodynamics*, 1930, 5th edition, p. 420.

[2] Rayleigh, *Phil. Mag.*, 1899, **47**, 566.

[3] Seth, *ibid.*, 1937, **24**, 288-93.

or

$$\phi + i\psi = C + \Sigma K_n \cos k_n \left\{ \Sigma \, (-1)^n \, \frac{a_{2n+1}}{z^{2n+1}} + i \right\} \qquad (2\cdot2)$$

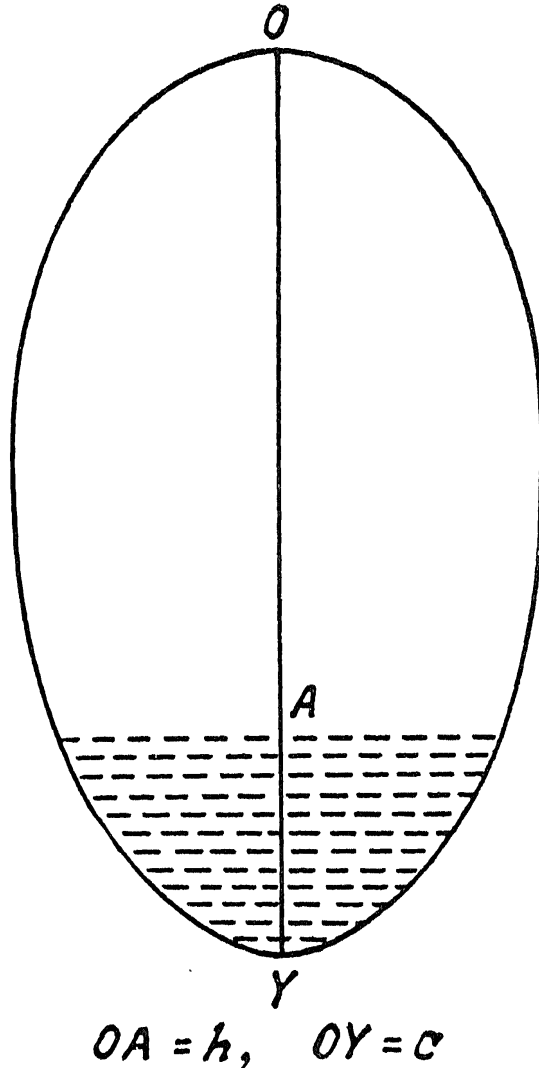both of which make (1) the stream line $\psi = 0$, all the K's, k's and c being real constants.

$$OA = h, \quad OY = c$$

Fig. 1.

The asymmetrical modes are given by $(2\cdot1)$, and the symmetrical by $(2\cdot2)$.

If we take $(2\cdot1)$, we get

$$\phi = \Sigma A_n \, \frac{\sin \left(n\theta + \frac{1}{2} p \pi \right)}{r^n}, \qquad (3\cdot1)$$

where p is 1 or 0 according as n is odd or even, A's being all real constants, as yet undetermined. Rewriting we have

$$\phi = x \, \Sigma \, \frac{1}{r^{2n+2}} \Bigg[A_{2n+1} + 2n \, A_{2n} \, y +$$

$$\sum_{2r+1=2n-1}^{n-2} (-1)^{r-n} A_{2r+1} y^{2n-2r} \frac{(\overline{2r+1}^2 - 1^2)(\overline{2r+1}^2 - 3^2) \cdots (\overline{2r+1}^2 - \overline{2n-2r+1}^2)}{(2n-2r)\,!}$$

$$+ \sum_{2r=2n-2}^{n+1} 2r \, (-1)^{r-n} \, A_{2r} \, y^{2n-2r+1} \frac{(\overline{2r}^2 - 2^2)(\overline{2r}^2 - 4^2) \cdots (\overline{2r}^2 - \overline{2n-2r}^2)}{(2n-2r+1)\,!} \Bigg] \qquad (3\cdot2)$$

$$= x \left[\frac{A_1}{r^2} + \frac{1}{r^4} (A_3 + 2A_2 y) + \frac{1}{r^6} (A_5 + 4A_4 y - 4A_3 y^2) \right.$$

$$+ \frac{1}{r^8} (A_7 + 6A_6 y - 12A_5 y^2 - 8A_4 y^3)$$

$$+ \frac{1}{r^{10}} (A_9 + 8A_8 y - 24 A_7 y^2 - 32 A_6 y^3 + 16 A_5 y^4)$$

$$+ \frac{1}{r^{12}} (A_{11} + 10 A_{10} y - 40 A_9 y^2 - 80 A_8 y^3 + 80 A_7 y^4 + 64 A_6 y^5)$$

$$\left. + \cdots \right]. \tag{3.3}$$

The condition at the free surface, $y = h$, which is now of the form

$$\sigma^2 \phi = - g \frac{\partial \phi}{\partial y} \tag{4}$$

gives $A_1 = 0$, and

$$2 A_2 (lh + 1) + lA_3 = 0,$$

$$8 A_2 h^2 + 4 A_3 h (lh + 3) - 4 A_4 (lh + 1) - lA_5 = 0,$$

$$24 A_3 h^3 - 8 A_4 h^2 (lh + 6) - 6 A_5 h (2 lh + 5) + 6 A_6 (lh + 1) + lA_7 = 0,$$

$$64 A_4 h^4 + 16 A_5 h^3 (lh + 10) - 16 A_6 h^2 (2 lh + 9) - 8 A_7 h (3 lh + 7)$$
$$+ 8 A_8 (lh + 1) + lA_9 = 0,$$

$$160 A_5 h^5 - 64 A_6 h^4 (lh + 10) - 80 A_7 h^3 (lh + 7) + 80 A_8 h^2 (lh + 4)$$
$$+ 10 A_9 h (4 lh + 9) - 10 A_{10} (lh + 1) - lA_{11} = 0,$$

where $l = \sigma^2/g$.

The elimination of A's from the above equations gives an infinite determinant to dertermine σ. If we proceed to evaluate σ by successive approximations we get equations of the following type :

$$lh + 1 = 0, \tag{5.1}$$

$$l^2 h^2 + 3 lh + 3 = 0, \tag{5.2}$$

$$l^3 h^3 + 6 l^2 h^2 + 15 lh + 15 = 0, \tag{5.3}$$

$$l^4 h^4 + 10 l^3 h^3 + 45 l^2 h^2 + 105 lh + 105 = 0, \tag{5.4}$$

. . . .

Thus at whatever stage we stop we cannot expect to get any positive value for lh, and hence no asymmetrical stable type is possible.

If (2.2) is used we again get equations of the same form as (5) to determine lh. There exists therefore no symmetrical stable type as well.

If we take

$$\phi = K \Sigma (-1)^n a_{2n+1} \frac{\cos (2n + 1) \theta}{r^{2n+1}}, \tag{6.1}$$

$$\psi = - K \Sigma \left[(-1)^n a_{2n+1} \frac{\sin (2n + 1) \theta}{r^{2n+1}} - 1 \right], \tag{6.2}$$

the boundary condition is satisfied.

The surface condition gives

$$l\left[\frac{a_1}{h^2}\left(1+\frac{x^2}{h^2}\right)^{-1} - \frac{a_3}{h^4}\left(1+\frac{x^2}{h^2}\right)^{-2} + \frac{4a_3}{h^4}\left(1+\frac{x^2}{h^2}\right)^{-3}\cdots\right]$$

$$= \frac{2a_1}{h^3}\left(1+\frac{x^2}{h^2}\right)^{-2} - \frac{12a_3}{h^5}\left(1+\frac{x^2}{h^2}\right)^{-3} + \frac{24a_3}{h^5}\left(1+\frac{x^2}{h^2}\right)^{-4}\cdots\right]. \quad (7)$$

If, therefore, x^2/h^4 and higher powers of x^2/h^2 can be neglected, we get

$$l\left[\frac{a_1}{h^2} + \frac{3a_3}{h^4} + \frac{5a_5}{h^6} + \cdots\right] = 2\left[\frac{a_1}{h^3} + \frac{6a_3}{h^5} + \cdots\right],$$

which will always give a positive value of l if all the a's are positive. In such a case we can take (6) as the constrained type to get an approximate value for the frequency of the gravest mode by using Rayleigh's method for cases in which the normal types cannot be accurately determined. The condition in this case to be satisfied is that the breadth of the free surface is very small compared with its depth below the highest point.

If $w = \phi + i\psi$, we can write

$$\sigma^2 = g \cdot \frac{\iint \left|\frac{dw}{dz}\right|^2 \cdot r\,d\theta\,dr}{\int \phi^2\,dx} \quad (7)$$

the double integral being taken over the area occupied by the liquid, and the single over its free surface.

For very small depths the section may be replaced by its circle of curvature at the lowest point. In such cases we know that[4]

$$\sigma = \left(\frac{g}{\rho}\right)^{\frac{1}{2}}\left[1 + \frac{H}{10\rho}\right], \quad (8)$$

H being the depth of the liquid, and ρ the radius of curvature at the lowest point. This value obviously approaches $(g/\rho)^{\frac{1}{2}}$ as $H \to 0$.

The last result is perfectly general, and may be verified by taking the exact solutions for a few hyperbolic sections. We know that for a rectangular hyperbola and for one in which the eccentricity is 2 the value of σ is given by[5]

$$\sigma = \left(\frac{g}{h}\right)^{\frac{1}{2}}, \quad (9)$$

h being the height of the free surface above the centre of the hyperbolas. In the limit h obviously approaches ρ.

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[4] Seth, *loc. cit.*, p. 392.

[5] Seth, *loc. cit.*, 1937, 23, 113–14.