POTENTIAL SOLUTIONS NEAR AN ANGULAR POINT

By B. R. Seth

(From the Department of Mathematics, Hindu College, Delhi)

Received February 4, 1939

It is generally believed that the velocity (or stress) in the case of an irrotational potential solution is zero or infinite at an angular point according as the corresponding angle is less or greater than π. But no general analytical demonstration is available even for two dimensional cases excepting the one given by Pearson[1] for the torsion of prisms. In this paper we shall shew that if πa be the angle,

(i) the velocity is zero if $a < 1$;

(ii) the velocity is in general infinite if $a > 1$;

(iii) the velocity is zero even in (ii) if the physical problem and the section are both symmetrical about the bisector of the angle. This has been generally overlooked by writers. What happens is that the re-entrant angle behaves like two angles less than π and hence the velocity becomes zero and not infinite.

The velocity potential which is a solution of Laplace's equation generally satisfies a boundary condition of the form

$$\left[\frac{\partial\phi}{\partial x} - \sum_{s=0}^{n} f_s\right] \cos(x\nu) + \left[\frac{\partial\phi}{\partial y} + \sum_{s=0}^{n} \chi_s\right] \cos(y\nu) = 0, \tag{1}$$

where f_n and χ_n are homogeneous integral polynomials of the nth degree in x and y and ν denotes the direction of the normal drawn to the boundary.

We can re-write (1) as

$$\frac{\partial\psi}{\partial s} = \left[\sum_{s=0}^{n} f_s\right] \cos(x\nu) - \left[\sum_{s=0}^{n} \chi_s\right] \cos(y\nu), \tag{2}$$

where s is measured along the boundary.

For fixed boundaries all f's and χ's vanish; for a uniform motion of translation f_0 and χ_0 are only present; for a motion of rotation and for certain viscous motions f_1 and χ_1, and for Saint-Venant's flexure problem f_2 and χ_2 also have to be taken.

---

[1] Todhunter and Pearson, *History of Elasticity*, Vol. II, Part II, pp. 412–14.

Let us solve the corresponding problem for a circular sector given by $\theta = \pm \frac{1}{2}\pi a, r = c$.

Along $\theta = \pm \frac{1}{2}\pi a$ (2) gives

$$\frac{\partial\psi}{\partial r} = \sum_{s=0}^{n} (A_s \pm B_s) r^s, \tag{3}$$

A's and B's being known constants and s integral.

We can satisfy (3) if we take

$$\psi = \sum_{s=0}^{n} \frac{r^{s+1}}{s+1} \left[A_s \frac{\cos (s+1)\theta}{\cos \frac{1}{2}(s+1)\pi a} + B_s \frac{\sin (s+1)\theta}{\sin \frac{1}{2}(s+1)\pi a} \right] + \psi_1, \tag{4}$$

where ψ_1 is such a solution of Laplace's equation that

$$\frac{\partial\psi_1}{\partial r} = 0 \text{ over } \theta = \pm \tfrac{1}{2}\pi a.$$

Since $-\frac{1}{2}\pi a \leqslant \theta \leqslant \frac{1}{2}\pi a$ the most general form ψ_1 can have is

$$\psi_1 = \sum_{s=0}^{\infty} \left[F_s \left(\frac{r}{c}\right)^{2s/a} \sin \frac{2s\theta}{a} + E_s \left(\frac{r}{c}\right)^{(2s+1)/a} \cos \frac{(2s+1)\theta}{a} \right], \tag{5}$$

where F_s and E_s are as yet undetermined constants.

Along $r = c$ (2) gives

$\frac{\psi}{\theta}$ = a known function of θ which can be expanded in a Fourier's series.

$$= \sum_{s=0}^{n} c^{s+2} \left[A_s \frac{\sin (s+1)\theta}{\cos \frac{1}{2}(s+1)\pi a} - B_s \frac{\cos (s+1)\theta}{\sin \frac{1}{2}(s+1)\pi a} \right]$$

$$- \sum_{s=0}^{\infty} \left[\frac{2s}{a} F_s \cos \frac{2s\theta}{a} - \frac{(2s+1)}{a} E_s \sin \frac{(2s+1)\theta}{a} \right]. \tag{6}$$

Comparing the co-efficients on both sides we get the values of F_s and E_s. The corresponding value of $\phi + i\psi$ is now given by [2]

$$\phi + i\psi = \sum_{s=0}^{n} L_{s+1} z^{s+1} + \sum_{s=0}^{\infty} F_s z^{2s/a} + i \sum_{s=0}^{\infty} E_s z^{(2s+1)/a}, \tag{7}$$

which shows that in general the velocity

$$q = \left| \frac{d}{dz} (\phi + i\psi) \right|$$

[2] It can happen that an E_s or a F_s may become infinite for some value of a. In such a case a L_s must also become infinite. Combining these two terms we get a term of the type $z^s \log z$, s being a positive integer. The velocity corresponding to this term is always zero at $z = 0$.

will contain a term of the type $r^{1/a-1} f(\theta)$, and hence it will be zero or infinite when r approaches zero according as

$$a < \text{or} > 1.$$

When both the section and the physical conditions are symmetrical about the bisector of the angle ψ must be odd in θ, and hence A_s and E_s must both vanish. In such a case (7) shows that the velocity is always zero.

We may look upon the problem in the following manner as well. In the boundary condition (1) ν, the direction of the normal is defined at all points except at an angular point where it can be anything. Hence at an angular point the co-efficients of both cos $(x\nu)$ and cos $(y\nu)$ should separately either vanish or become infinite. But this does not give the criterion to distinguish between the two cases.