CONSISTENCY EQUATIONS OF FINITE STRAIN

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The components of finite strain referred to a point (x, y, z) in the strained state of an elastic body are known to be given by relations of the type.[1]

$$S_x = \frac{\partial u}{\partial x} - \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right]. \tag{1·1}$$

$$\sigma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} - \left[\frac{\partial u}{\partial y}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial w}{\partial z}\right]. \tag{1·2}$$

As there are six relations between the three components of displacement (u, v, w), there should exist six identical relations between S_x, S_y, S_z, σ_{yz}, σ_{zx}, σ_{xy}. These are called the consistency equations. In the classical small strain theory they are known to be of the form.[2]

$$\frac{\partial^2 S_x}{\partial y^2} + \frac{\partial^2 S_y}{\partial x^2} - \frac{\partial^2 \sigma_{xy}}{\partial x \partial y} = 0, \tag{2·1}$$

$$\frac{\partial^2 S_x}{\partial y \partial z} - \frac{1}{2}\frac{\partial}{\partial x}\left(-\frac{\partial \sigma_{yz}}{\partial x} + \frac{\partial \sigma_{zx}}{\partial y} + \frac{\sigma_{xy}}{\partial z}\right) = 0. \tag{2·2}$$

It is proposed to investigate their forms when finite components of strain are used.

Let (a, b, c) be the unstrained position of a particle whose position after strain is given by (x, y, z). An element of length in the unstrained state is given by

$$(ds_0)^2 = (da)^2 + (db)^2 + (dc)^2 \tag{3·1}$$

In the strained state we know that it takes the form

$$(ds)^2 = (1 - 2S_x)\, dx^2 + (1 - 2S_y)\, dy^2 + (1 - 2S_z)\, dz^2 - 2\sigma_{yz}\, dy\, dz - 2\sigma_{zx}\, dxdz - 2\sigma_{xy}\, dxdy. \tag{3·2}$$

Putting

$$g_{11} = 1 - 2S_x,\ g_{22} = 1 - 2S_y,\ g_{33} = 1 - 2S_z,$$

$$g_{23} = -\sigma_{yz},\ g_{31} = -\sigma_{zx},\ g_{12} = -\sigma_{xy},$$

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[1] Seth, *Phil. Trans. Roy. Soc.*, A, 1935, **234**, 231.

[2] Love, *Theory of Elasticity*, 4th ed., p. 49.

we can write (3·2) in the notation of the tensor calculus as

$$(ds)^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta. \tag{4}$$

That (4) can be reduced to (3) means that it is possible to construct a Galilean frame of reference. In other words the condition for flat space-time is satisfied. But we know that the necessary and sufficient condition for flat space-time is the vanishing of the Riemann-Christoffel tensor, which is given by

$$B^{\epsilon}_{\mu\nu\sigma} = \{\mu\,\sigma,\, \alpha\}\{\alpha\,\nu,\, \epsilon\} - \{\mu\,\nu,\, \alpha\}\{\alpha\,\sigma,\, \epsilon\} - \frac{\partial}{\partial x_\nu}\{\mu\,\sigma,\, \epsilon\} - \frac{\partial}{\partial x_\sigma}\{\mu\,\nu,\, \epsilon\}, \tag{5}$$

where

$$\{\mu\,\nu,\, \sigma\} = \tfrac{1}{2} g^{\sigma\lambda}\left(\frac{\partial g_{\mu\lambda}}{\partial x_\nu} + \frac{\partial g_{\nu\lambda}}{\partial x_\mu} - \frac{\partial g_{\mu\nu}}{\partial x_\lambda}\right), \tag{6·1}$$

$$[\mu\,\nu,\, \sigma] = \tfrac{1}{2}\left(\frac{\partial g_{\mu\sigma}}{\partial x_\nu} + \frac{\partial g_{\nu\sigma}}{\partial x_\mu} - \frac{\partial g_{\mu\nu}}{\partial x_\sigma}\right). \tag{6·2}$$

In three dimensions (5) has only the six components (1212), (1213), (1214), (1223), (1313), (1323). The vanishing of these constitute the six consistency equations. Three of them are of the type

$$\frac{\partial^2 g_{11}}{\partial y^2} + \frac{\partial^2 g_{22}}{\partial x^2} - 2\frac{\partial^2 g_{12}}{\partial x \partial y} = 2\{12,\, \alpha\}[12,\, \alpha] - 2\{11,\, \alpha\}[22,\, \alpha], \tag{7·1}$$

and three of the type

$$\frac{\partial^2 g_{11}}{\partial y \partial z} + \frac{\partial}{\partial x}\left(\frac{\partial g_{23}}{\partial x} - \frac{\partial g_{31}}{\partial y} - \frac{\partial g_{12}}{\partial z}\right) = 2\{31,\, \alpha\}[12,\, \alpha] - 2\{23,\, \alpha\}[11,\, \alpha]. \tag{7·2}$$

The right-hand side of (7·1) is found to be

$$\frac{2}{g}\begin{vmatrix} g_{11} & g_{12} & g_{13} & [12,\,1] \\ g_{21} & g_{22} & g_{23} & [12,\,2] \\ g_{31} & g_{23} & g_{33} & [12,\,3] \\ [12,\,1] & [12,\,2] & [12,\,3] & 0 \end{vmatrix} - \frac{2}{g}\begin{vmatrix} g_{11} & g_{12} & g_{13} & [22,\,1] \\ g_{21} & g_{22} & g_{23} & [22,\,2] \\ g_{31} & g_{32} & g_{33} & [22,\,3] \\ [11,\,1] & [11,\,2] & [11,\,3] & 0 \end{vmatrix}$$

with a similar expression for the right-hand side of (7·2) g is the determinant $(g_{11}\, g_{22}\, g_{33})$.

If, as has been done in some of the applications of finite strain, we neglect terms of an order higher than the second, (7·1) and (7·2) take the form

$$\frac{\partial^2 g_{11}}{\partial y^2} + \frac{\partial^2 g_{22}}{\partial x^2} - 2\frac{\partial^2 g_{12}}{\partial x \partial y} = 2\big([12,\,1]^2 + [12,\,2]^2 + [12,\,3]^2 - [11,\,1][22,\,1] - [11,\,2][22,\,2] - [11,\,3][22,\,3]\big)$$

$$= \tfrac{1}{2}\left(\frac{\partial g_{11}}{\partial y}\right)^2 + \tfrac{1}{2}\left(\frac{\partial g_{22}}{\partial x}\right)^2 + \tfrac{1}{2}\left(\frac{\partial g_{13}}{\partial y} - \frac{\partial g_{23}}{\partial x} - \frac{\partial g_{12}}{\partial z}\right)^2$$

$$- \tfrac{1}{2}\frac{\partial g_{11}}{\partial x}\left(2\frac{\partial g_{21}}{\partial y} - \frac{\partial g_{22}}{\partial x}\right) - \tfrac{1}{2}\frac{\partial g_{22}}{\partial y}\left(2\frac{\partial g_{12}}{\partial x} - \frac{\partial g_{11}}{\partial y}\right)$$

$$- \tfrac{1}{2}\left(2\frac{\partial g_{13}}{\partial x} - \frac{\partial g_{11}}{\partial y}\right)\left(2\frac{\partial g_{23}}{\partial y} - \frac{\partial g_{22}}{\partial z}\right), \qquad ((8\cdot 1)$$

and

$$\frac{\partial^2 g_{11}}{\partial y\,\partial z} + \frac{\partial}{\partial x}\left(\frac{\partial g_{23}}{\partial x} - \frac{\partial g_{31}}{\partial y} - \frac{\partial g_{12}}{\partial z}\right)$$

$$= 2\big([31, 1]\,[12, 1] + [31, 2]\,[12, 2] + [31, 3]\,[12, 3]$$

$$- [23, 1]\,[11, 1] - [23, 2]\,[11, 2] - [23, 3]\,[11, 3]\big)$$

$$= \tfrac{1}{2}\frac{\partial g_{11}}{\partial y}\frac{\partial g_{11}}{\partial z} + \tfrac{1}{2}\frac{\partial g_{22}}{\partial x}\left(\frac{\partial g_{23}}{\partial x} - \frac{\partial g_{31}}{\partial y} + \frac{\partial g_{12}}{\partial z}\right)$$

$$+ \tfrac{1}{2}\frac{\partial g_{33}}{\partial x}\left(\frac{\partial g_{23}}{\partial x} + \frac{\partial g_{31}}{\partial y} - \frac{\partial g_{12}}{\partial z}\right) - \tfrac{1}{2}\frac{\partial g_{11}}{\partial x}\left(-\frac{\partial g_{23}}{\partial x} + \frac{\partial g_{31}}{\partial y} + \frac{\partial g_{12}}{\partial z}\right)$$

$$- \tfrac{1}{2}\frac{\partial g_{22}}{\partial z}\left(2\frac{g_{12}}{\partial x} - \frac{\partial g_{11}}{\partial y}\right) - \tfrac{1}{2}\frac{\partial g_{33}}{\partial y}\left(2\frac{\partial g_{31}}{\partial x} - \frac{\partial g_{11}}{\partial z}\right). \qquad (8\cdot 2)$$

In terms of the strain components these may be written as

$$\frac{\partial^2 S_x}{\partial y^2} + \frac{\partial^2 S_y}{\partial x^2} - \frac{\partial^2 \sigma_{xy}}{\partial x\,\partial y} = \left(\frac{\partial S_x}{\partial y}\right)^2 + \left(\frac{\partial S_y}{\partial x}\right)^2 + \tfrac{1}{4}\left(\frac{\partial \sigma_{yz}}{\partial x} + \frac{\partial \sigma_{zx}}{\partial y} - \frac{\partial \sigma_{xy}}{\partial z}\right)^2$$

$$- \frac{\partial S_x}{\partial x}\left(\frac{\partial \sigma_{xy}}{\partial y} - \frac{\partial S_y}{\partial x}\right) - \frac{\partial S_y}{\partial y}\left(\frac{\partial \sigma_{xy}}{\partial x} - \frac{\partial S_x}{\partial y}\right)$$

$$- \left(\frac{\partial \sigma_{xz}}{\partial x} - \frac{\partial S_x}{\partial z}\right)\left(\frac{\partial \sigma_{yz}}{\partial y} - \frac{\partial S_y}{\partial z}\right), \qquad (9\cdot 1)$$

$$\frac{\partial^2 S_x}{\partial y \partial z} - \tfrac{1}{2}\frac{\partial}{\partial x}\left(-\frac{\partial \sigma_{yz}}{\partial x} + \frac{\partial \sigma_{zx}}{\partial y} + \frac{\partial \sigma_{xy}}{\partial z}\right)$$

$$= \tfrac{1}{2}\frac{\partial S_x}{\partial x}\left(-\frac{\partial \sigma_{yz}}{\partial x} + \frac{\partial \sigma_{zx}}{\partial y} + \frac{\partial \sigma_{xy}}{\partial z}\right) + \frac{\partial S_y}{\partial z}\left(\frac{\partial \sigma_{xy}}{\partial x} - \frac{\partial S_x}{\partial y}\right)$$

$$+ \frac{\partial S_z}{\partial y}\left(\frac{\partial \sigma_{zx}}{\partial x} - \frac{\partial S_x}{\partial z}\right) - \frac{\partial S_x}{\partial y} - \frac{\partial S_x}{\partial z} - \tfrac{1}{2}\frac{\partial S_y}{\partial x}\left(\frac{\partial \sigma_{yz}}{\partial x} - \frac{\partial \sigma_{zx}}{\partial y} + \frac{\partial \sigma_{xy}}{\partial z}\right)$$

$$- \tfrac{1}{2}\frac{\partial S_z}{\partial x}\left(\frac{\partial \sigma_{yz}}{\partial x} + \frac{\partial \sigma_{zx}}{\partial y} - \frac{\partial \sigma_{xy}}{\partial z}\right) \qquad (9\cdot 2)$$

It is understood that terms of the first degree are only to be substituted on the right-hand sides of (8) and (9). If we do that, we find that both sides of (9 · 1) reduce to

$$\left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 + \left(\frac{\partial^2 v}{\partial x \partial y}\right)^2 + \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 v}{\partial x^2}\frac{\partial^2 v}{\partial y^2} - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2},$$

and both sides of (9·2) to

$$\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y\partial z}+\frac{\partial^2 v}{\partial x^2}\frac{\partial^2 v}{\partial y\partial z}+\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y\partial z}-\frac{\partial^2 u}{\partial x\partial y}\frac{\partial^2 u}{\partial x\partial z}$$

$$-\frac{\partial^2 v}{\partial x\partial y}\frac{\partial^2 v}{\partial x\partial z}-\frac{\partial^2 w}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial z}.$$

To the second order of approximation the relations between the stresses and strains, if only two elastic constants are used, are found to be given by equations of the type[3]

$$\widehat{xx}=(\lambda\delta+2\mu S_x)(1-\delta-2S_x)-\mu(\sigma_{xy}^2+\sigma_{xz}^2), \tag{10·1}$$

$$\widehat{yz}=\mu\,\sigma_{yz}(1-\delta-2S_y)-\mu\,\sigma_{xy}\sigma_{xz}-\sigma_{yz}(\lambda\delta+2\mu S_z), \tag{10·2}$$

where $\delta=S_x+S_y+S_z$.

The consistency equations in stresses may be obtained from (9) with the help of (10).

Summary

The general form of the consistency equations satisfied by the components of finite strain of an elastic body are determined, and the particular case, when terms of an order higher than the second are neglected, is considered. The method of deriving consistency equations in stresses to second order of approximation has also been indicated.

[3] Murnaghan. *Amer. J. of Math.*, 1937, 59, 251.