VORTEX MOTION IN RECTANGULAR CYLINDERS.

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The general problem of the motion of a liquid due to a rectilinear vortex in a cylinder whose cross-section is bounded by four arcs of mutually orthogonal curves can be readily solved with the help of conformal transformation. By considering the infinite series of images in the four bounding planes Greenhill[1] has solved two particular cases. From his analysis it appears that this method of images is bound to be rather long and inconvenient in the general case. Moreover it is not always free from preliminary limitations which may not be found essential when the solution of the problem has been obtained. For example, in the case of a rectangle bounded by two concentric arcs and two radii inclined at an angle $\frac{1}{n}\pi$ Greenhill has to make the supposition in the beginning that $\frac{1}{n}\pi$ is a sub-multiple of two right angles. As he remarks at the end of his paper his solution holds good whatever $\frac{1}{n}\pi$ may be.[2]

We take the plane of a cross-section as the plane of a complex variable z. Let a system of orthogonal curves in the z-plane be given by

$$\omega = \alpha + i\,\beta = f(z)\,;$$

so that the sides of the z-rectangle can be taken as $\alpha = 0$, $\alpha = 2\mathrm{K}$, $\beta = 0$, and $\beta = 2\mathrm{K}'$, K and K′ being two constants. The ω-rectangle can be mapped on a ζ-half plane by means of the relation

$$\omega = \mathrm{A}\int[(\zeta-\xi_1)\,(\zeta-\xi_2)\,(\zeta-\xi_3)\,(\zeta-\xi_4)]^{-\frac{1}{2}}\,d\zeta + \mathrm{B}, \qquad \text{(1)}$$

ξ_1, ξ_2, ξ_3, and ξ_4 being the points on the real axis in the ζ-plane that correspond to the angular points P, Q, R, and O in the ω-plane.

Putting $\xi_1 = k'/k$, $\xi_2 = \infty$, $\xi_3 = -k/k'$, and $\xi_4 = 0$, we get

$$\omega = \mathrm{A}\int[\zeta(\zeta + k/k')\,(\zeta - k'/k)]^{-\frac{1}{2}}\,d\zeta + \mathrm{B}.$$

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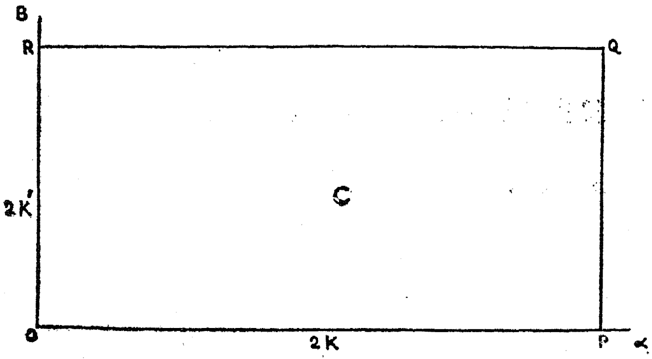
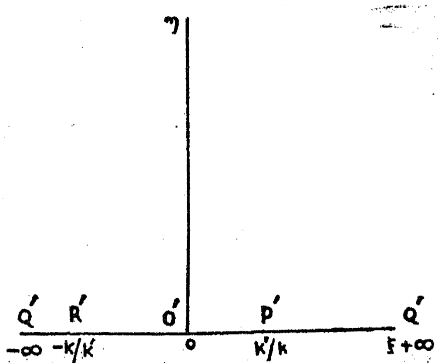
1 Greenhill, *Quart. J. of Math.*, 1877, **15**, 23-29.

2 This limitation in the method of images has been pointed out by Ramsey in his *Hydromechanics*, Part 2, Art. 189, (1913) edition.



ω-plane

FIG. 1.



ξ-plane

FIG. 2.

Adjusting the constants A and B we can write[3]

$$\zeta = -\, k/k' \, \mathrm{Cn}^2 \left(\tfrac{1}{2}\,\omega - \mathrm{K} - i\mathrm{K}'\right) = k'/k \; nc^2 \left(\tfrac{1}{2}\omega\right) \qquad \ldots \quad \ldots \quad (2\cdot 1)$$

$$= \frac{k}{k'} \, \frac{dn\,\omega - \mathrm{Cn}\,\omega}{1 - dn\,\omega}, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2\cdot 2)$$

k being the modulus, and 4K, $4i\mathrm{K}'$ the periods of the elliptic function. Referred to C, the centre of the ω-figure, as the origin this transformation becomes

$$\zeta = \frac{i\,(1 + k\,\mathrm{Sn}\,\omega)}{\mathrm{Cn}\,\omega - i\,k'\,\mathrm{Sn}\,\omega} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2\cdot 3)$$

Hence the relation

$$\zeta = \frac{i\,[1 + k\,\mathrm{Sn}\,\{f\,(z)\}]}{\mathrm{Cn}\,\{f\,(z)\} - i\,k'\,\mathrm{Sn}\,\{f\,(z)\}} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

transforms the z-rectangle into the upper half of the ζ-plane.

Let there be a vortex of strength m at a point z_0 within the z-rectangle, and let ω_0 and ζ_0 be the corresponding points of the ω-plane and ζ-plane. If ϕ is the velocity potential and ψ the stream function of the motion, we know that

$$\phi + i\,\psi = \frac{mi}{2\pi} \log \frac{\zeta - \zeta_0}{\zeta - \zeta_0{}'}, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

which, after using (2.1), becomes

$$\phi + i\,\psi = \frac{mi}{2\pi} \log \frac{\mathrm{C}n^2\,\tfrac{1}{2}\,\{f\,(z_0{}')\}\,[\mathrm{C}n^2\,\tfrac{1}{2}\,\{f\,(z)\} - \mathrm{C}n^2\,\tfrac{1}{2}\,\{f\,(z_0)\}]}{\mathrm{C}n^2\,\tfrac{1}{2}\,\{f\,(z_0)\}\,[\mathrm{C}n^2\,\tfrac{1}{2}\,\{f(z)\} - \mathrm{C}n^2\,\tfrac{1}{2}\,\{f\,(z_0{}')\}]}, \qquad (5)$$

where $\zeta_0{}' = \xi_0 - i\,\eta_0$ and $z_0{}' = x_0 - iy_0$. The stream function is, therefore, given by

$$\psi = \frac{m}{4\pi} \log \frac{[\mathrm{C}n^2\,\tfrac{1}{2}\,\{f(z)\} - \mathrm{C}n^2\,\tfrac{1}{2}\,\{f(z_0)\}]\,[\mathrm{C}n^2\,\tfrac{1}{2}\,\{f(z')\} - \mathrm{C}n^2\,\tfrac{1}{2}\,\{f(z_0{}')\}]}{[\mathrm{C}n^2\,\tfrac{1}{2}\,\{f(z)\} - \mathrm{C}n^2\,\tfrac{1}{2}\,\{f\,(z_0{}')\}]\,[\mathrm{C}n^2\,\tfrac{1}{2}\,\{f\,(z')\} - \mathrm{C}n^2\,\tfrac{1}{2}\,\{f(z_0)\}]} \qquad \ldots \quad (6)$$

[3] We have followed Dr. Glaisher in writing $ncu = 1/cnu$, with a similar notation for the other elliptic functions.

To determine the curve described by the vortex we apply the well-known result due to Routh,[4] *viz.*

$$\chi(x, y) = \chi(\xi, \eta) + \frac{m}{4\pi} \log \left| \frac{d\zeta}{dz} \right|, \quad \ldots \quad \ldots \quad \ldots \quad (7)$$

$\chi(x, y)$ being the stream function for the motion of the vortex in the z-plane and $\chi(\xi, \eta)$ of the corresponding vortex in the ζ-plane. From (4) we easily get

$$\chi(\xi, \eta) = -\frac{m}{4\pi} \log \eta. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

η and $| d\zeta/dz |$ can be determined either from (2.2) or from (3) according as O or C in Fig. 1 is taken as the origin. Taking (2. 2) we have

$$2 i \eta = \frac{k}{k'} \left[\frac{dn\ \omega - Cn\ \omega}{1 - dn\ \omega} - \frac{dn\ \omega' - Cn\ \omega'}{1 - dn\ \omega'} \right]$$

$$= \frac{k}{k'} \left[\frac{(dn\ \omega - dn\ \omega') - (Cn\ \omega - Cn\ \omega') + (Cn\ \omega\ dn\ \omega' - Cn\ \omega'\ dn\ \omega)}{(1 - dn\ \omega)(1 - dn\ \omega')} \right],$$

which, after using Jacob's addition formulæ[5] for the various terms in square brackets, becomes

$$2 i \eta = -\frac{k}{k'} \frac{Sn\ \alpha\ Sn\ i\beta\ (k'^2 + k^2\ Cn\ \alpha\ Cn\ i\beta - dn\ \alpha\ dn\ i\beta)}{(dn\ \alpha - dn\ i\beta)^2}$$

$$= -\frac{k}{k'} \cdot \frac{Sn\ \alpha\ Sn\ i\beta}{Cn\ \alpha + Cn\ i\beta} \cdot \frac{dn\ \alpha\ Cn\ i\beta - dn\ i\beta\ Cn\ \alpha}{dn\ \alpha - dn\ i\beta} \quad \ldots \quad (9)$$

Again we have

$$\left| \frac{d\zeta}{dz} \right|^2 = \left| \frac{d\zeta}{d\omega} \cdot \frac{d\omega}{dz} \right|^2 = \left| \frac{d\zeta}{d\omega} \right|^2 f'(z)\, f'(z'),$$

where $f'(z) = d\{f(z)\}/dz$. Using (2·1) we get

$$\left| \frac{d\zeta}{d\omega} \right|^2 = \frac{k'^2}{k^2} \cdot \frac{Sn\ \frac{1}{2}\omega\ Sn\ \frac{1}{2}\omega'\ dn\ \frac{1}{2}\omega\ dn\ \frac{1}{2}\omega'}{Cn^3\ \frac{1}{2}\omega\ Cn^3\ \frac{1}{2}\omega'}.$$

Since[6]

$$Sn\ \tfrac{1}{2}\omega\ Sn\ \tfrac{1}{2}\omega' = \frac{1}{k^2} \cdot \frac{dn\ i\beta - dn\ \alpha}{Cn\ i\beta - Cn\ \alpha},$$

$$Cn\ \tfrac{1}{2}\omega\ Cn\ \tfrac{1}{2}\omega' = \frac{k'^2}{k^2} \cdot \frac{dn\ i\beta - dn\ \alpha}{Cn\ i\beta\ dn\ \alpha - Cn\ \alpha\ dn\ i\beta},$$

$$dn\ \tfrac{1}{2}\omega\ dn\ \tfrac{1}{2}\omega' = k'^2 \frac{cn\ i\beta - cn\ \alpha}{cn\ i\beta\ dn\ \alpha - cn\ \alpha\ dn\ i\beta},$$

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[4] Routh, *Proc. London Math. Soc.*, 1881, 12, 82-84; also Ramsey *Hydromechanics*, Part 2, pp. 224-5 (1913 edition).

[5] See Cayley, *Elliptic Functions*, 1876, 65-66.

[6] For these formula reference may be made to Greenhill, *Applications of Elliptic Functions*, (1892), 255.

we have

$$\left|\frac{d\zeta}{d\omega}\right|^2 = \frac{k^2}{k'^2}\cdot\frac{\mathrm{Cn}\, i\,\beta - \mathrm{Cn}\,\alpha}{\mathrm{Cn}\, i\,\beta + \mathrm{Cn}\,\alpha}\cdot\frac{(\mathrm{Cn}\, i\,\beta\, dn\,\alpha - \mathrm{Cn}\,\alpha\, dn\, i\,\beta)^2}{(dn\,\alpha - dn\, i\,\beta)^2}, \quad .. \quad (10)$$

and hence (7) and (8) give

$$\chi(x, y) = \frac{m}{8\pi}\log\frac{4(\mathrm{Cn}^2\alpha - \mathrm{Cn}^2 i\,\beta)}{\mathrm{Sn}^2\alpha\,\mathrm{Sn}^2\beta} f'(z) f'(z')$$

$$= \frac{m}{8\pi}\log 4\,(n\mathrm{S}^2\alpha - n\mathrm{S}^2 i\,\beta) f'(z) f'(z')$$

$$= \frac{m}{8\pi}\log 4\,[n\mathrm{S}^2(\alpha, k) + n\mathrm{S}^2(\beta, k') - 1] f'(z) f'(z') \quad .. \quad (11)$$

If we put

$$\omega = \alpha + i\,\beta = f(z) = f_1(x, y) + i\, f_2(x, y),$$

we get

$$\chi(x, y) = \frac{m}{8\pi}\log 4\,[n\mathrm{S}^2\{f_1(x, y), k\} + n\mathrm{S}^2\{f_2(x, y), k'\} - 1]$$

$$\times f'(z) f'(z'); \quad .. \quad .. \quad .. \quad .. \quad .. \quad (12)$$

so that the curve described by the vortex is given by

$$[n\mathrm{S}^2\{f_1(x, y), k\} + n\mathrm{S}^2\{f_2(x, y), k'\} - 1] f'(z) f'(z') = \text{a constant}. \quad (13)$$

When the rectangle in the z-plane is rectilinear we have $\omega = z$, and the stream lines are given from (6) as

$$\frac{(cn^2\tfrac{1}{2}z - cn^2\tfrac{1}{2}z_0)(cn^2\tfrac{1}{2}z' - cn^2\tfrac{1}{2}z_0')}{(cn^2\tfrac{1}{2}z - cn^2\tfrac{1}{2}z_0')(cn^2\tfrac{1}{2}z' - cn^2\tfrac{1}{2}z_0)} = \text{constant}.$$

The curve described by the vortex is

$$ns^2(x, k) + ns^2(y, k') = \text{a constant}^7 \quad .. \quad .. \quad .. \quad (14)$$

Referred to the centre of the rectangle as origin it becomes

$$k'^2\, nc^2(x, k) + k^2\, nc^2(y, k') = \text{a constant} \quad .. \quad .. \quad .. \quad (15)$$

For a square cross-section we have

$$\mathrm{K} = \mathrm{K}' = 1{\cdot}854; \quad k = k' = 1/\sqrt{2},$$

and (15) reduces to

$$nc^2(x, k) + nc^2(y, k) = c \text{ (say)}.$$

In Fig. 3 I have traced some of these curves in the first quadrant by taking $c = 3, 4, 7, 31$. Obviously these curves are symmetrical both about the co-ordinate axes and the lines $x \pm y = 0$.

The vortex is stationary when at the centre of the rectangle. Using (3) and (4) and noticing that $\zeta = i$ corresponds to $z = 0$ we get

[7] This result is the same as obtained by Greenhill, *loc. cit.*, 25.

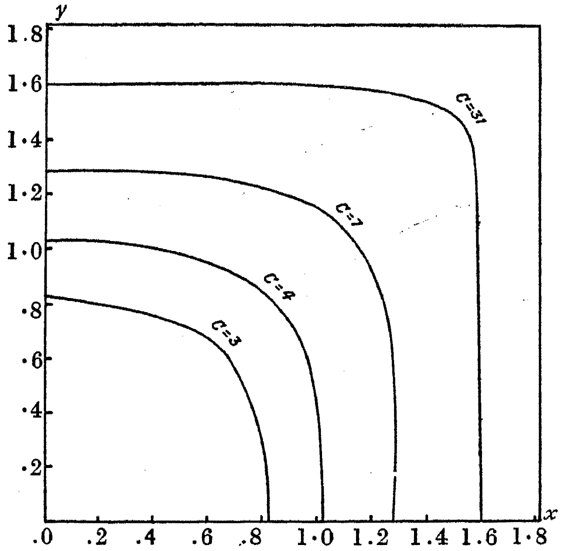

FIG. 3.

$$\phi + i\,\psi = \frac{mi}{2\pi}\log\frac{\zeta - i}{\zeta + i}$$
$$= \frac{mi}{2\pi}\log\frac{1 - \mathrm{C}n\,z + (k + i\,k')\,\mathrm{S}n\,z}{1 + \mathrm{C}n\,z + (k - i\,k')\,\mathrm{S}n\,z}$$
$$= \frac{mi}{2\pi}\log\frac{(k + i\,k')\,\mathrm{S}n\,z}{1 + \mathrm{C}n\,z} = \frac{mi}{4\pi}\log\frac{1 - \mathrm{C}n\,z}{1 + \mathrm{C}n\,z}, \quad \ldots \quad (16)$$

neglecting a constant.

The stream function[8] of the motion is given by

$$\psi = \frac{m}{8\pi}\log\frac{(1 - \mathrm{C}n\,z)\,(1 - \mathrm{C}n\,z')}{(1 + \mathrm{C}n\,z)\,(1 + \mathrm{C}n\,z')}$$
$$= \frac{m}{4\pi}\log\frac{\mathrm{C}n\,x - \mathrm{C}n\,i\,y}{\mathrm{C}n\,x + \mathrm{C}n\,i\,y} = \frac{m}{2\pi}\tanh^{-1}\frac{\mathrm{C}n\,x}{\mathrm{C}n\,i\,y}$$
$$= \frac{m}{2\pi}\tanh^{-1}\{\mathrm{C}n\,(x, k)\,\mathrm{C}n\,(y, k')\} \quad \ldots \quad \ldots \quad \ldots \quad (17\cdot1)$$

In like manner

$$\phi = -\frac{m}{2\pi}\tan^{-1}\frac{\mathrm{S}n\,(x, k)\,dn\,(y, k')}{\mathrm{S}n\,(y, k')\,dn\,(x, k)} \quad \ldots \quad \ldots \quad \ldots \quad (17\cdot2)$$

The equation of the stream lines is, therefore, given by

$$\mathrm{C}n\,(x, k)\,\mathrm{C}n\,(y, k') = \text{a constant} = c \text{ (say)} \quad \ldots \quad \ldots \quad (18)$$

By taking $c = \cdot1, \cdot25, \cdot5, \cdot75$, I have traced the corresponding stream lines in Fig. 4 for the first quadrant of a square cross-section.

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8 Greenhill, *loc. cit.*, 26.

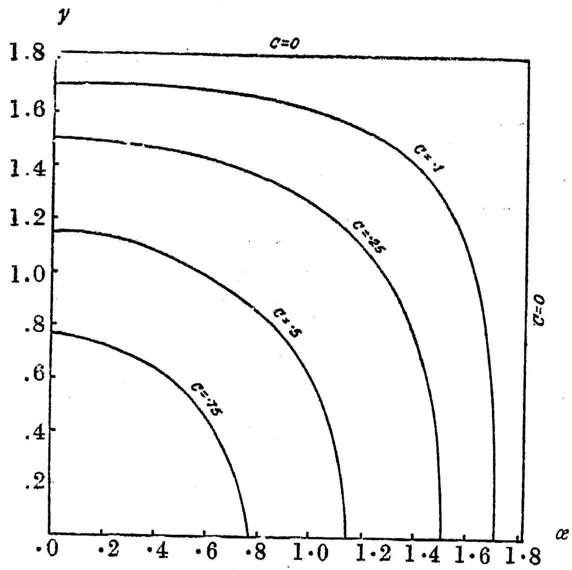

FIG. 4.

When the cross-section is bounded by two concentric arcs of radii a and b, and two radii inclined at an angle ϵ we have the relation

$$\omega = \alpha + i\,\beta = \frac{2K'}{\epsilon} \log \frac{z}{a} = \frac{2K'}{\epsilon}\left(\log \frac{r}{a} + i\,\theta\right) \quad \ldots \quad \ldots \quad (19)$$

where it is assumed that $\alpha = 0$, $\alpha = 2K$, $\beta = 0$, $\beta = 2K'$ correspond with $r = a$, $r = b$, $\theta = 0$, $\theta = \epsilon$ respectively. (6) and (13) give the stream function of the motion as

$$\psi = \frac{m}{4\pi} \log \frac{[Cn^2\{(K'/\epsilon) \log (z/a)\} - Cn^2\{K'/\epsilon) \log (z_0/a)\}]}{[Cn^2\{(K'/\epsilon) \log (z/a)\} - Cn^2\{K'/\epsilon) \log (z_0'/a)\}]} \times$$
$$\frac{[Cn^2\{(K'/\epsilon) \log (z'/a)\} - Cn^2\{K'/\epsilon) \log (z_0'/a)\}]}{[Cn^2\{(K'/\epsilon) \log (z'/a)\} - Cn^2\{K'/\epsilon) \log (z_0/a)\}]} \quad \ldots \quad (20)$$

and the curve described by the vortex as

$$\frac{1}{r^2}\left[nS^2\left(\frac{2K'}{\epsilon} \log \frac{r}{a}, k\right) + nS^2\left(\frac{2K'\theta}{\epsilon}, k'\right) - 1\right] = \text{a constant} \quad \ldots \quad (21)$$

a result quite laboriously obtained by Greenhill.[9]

I do not think Greenhill is right when he says that the vortex is stationary when $r = \sqrt{ab}$, $\theta = \frac{1}{2}\,\epsilon$. This can be easily seen from (11) if we form the values of $\partial\chi/\partial x$ and $\partial\chi/\partial y$. What actually happens is that the corresponding vortex at the centre of the ω-rectangle is always stationary. The point $r = \sqrt{ab}$, $\theta = \frac{1}{2}\,\epsilon$, in the z-plane corresponds to the centre, C, in the ω-plane. Apparently Greenhill assumes that corresponding vortices in two conformally represented planes continue to move so as to occupy corresponding points. We easily see from (7) that in general this is not true.

[9] *Loc. cit.*, 28.

In the present case the point at which the vortex is stationary is readily found from the condition

$$\frac{\partial \chi}{\partial r} = 0, \quad \frac{\partial \chi}{\partial \theta} = 0$$

Using (11) we get

$$\beta = \mathrm{K}', \ i.e., \quad \theta = \tfrac{1}{2}\epsilon,$$

and
$$1 + \frac{2\mathrm{K}'}{\epsilon}\frac{\mathrm{C}n\ \alpha\ dn\ \alpha}{\mathrm{S}n\ \alpha} = 0,$$

i.e.,
$$\frac{dn\ 2\alpha + \mathrm{C}n\ 2\alpha}{dn\ 2\alpha - \mathrm{C}n\ 2\alpha} = \frac{\epsilon^2}{4k'^2\,\mathrm{K}'^2},$$

or
$$\mathrm{S}n\ (2\alpha - \mathrm{K}) = \mathrm{S}n\left(\frac{4\mathrm{K}'}{\epsilon}\log\frac{r}{a} - \mathrm{K}\right) = \frac{4k'^2\mathrm{K}'^2 - \epsilon^2}{4k'^2\mathrm{K}'^2 + \epsilon^2}, \quad \ldots \quad (22)$$

which determines r. We also have the relation

$$\frac{\mathrm{K}}{\mathrm{K}'} = \frac{\log\ (b/a)}{\epsilon}$$

To take a simple example we put $\mathrm{K} = \mathrm{K}' = 1{\cdot}854$, $k = k' = 1/\sqrt{2}$, $a = 1$, $\epsilon = \frac{1}{2}\pi$. The value of b is $e^{\frac{1}{2}\pi} = 4{\cdot}810$. From (22) we get

$$\mathrm{S}n\left(\frac{8\mathrm{K}}{\pi}\log r - \mathrm{K}\right) = 0{\cdot}472,$$

$$i.e., \log r = \tfrac{1}{8}\pi\left(1 + \frac{0{\cdot}503}{\mathrm{K}}\right) = 0{\cdot}499$$

$$\text{or} \quad r = 1{\cdot}647.$$

Thus the point at which the vortex is stationary is given by $r = 1{\cdot}647$, $\theta = \frac{1}{4}\pi$. According to Greenhill the value of r should be $\sqrt{b}$, *i.e.*, $2{\cdot}193$.

When the rectangle is formed by arcs of co-axial circles we can take

$$\omega + (\alpha_1 + i\,\beta_1) = -\,\mu\log\frac{z-a}{z+a} \quad \ldots \quad \ldots \quad \ldots \quad (23)$$

so that the ω-rectangle is given by $\alpha = 0$, $\alpha = \alpha_2 - \alpha_1 = 2\mathrm{K}$, $\beta = 0$, $\beta = \beta_2 - \beta_1 = 2\mathrm{K}'$. Putting $z - a = r_1\, e^{i\theta_1}$, $z + a = r_2\, e^{i\theta_2}$, we get from (11)

$$-\frac{1}{r_1^2\, r_2^2}\left[n\mathrm{S}^2\left\{\left(\mu\log\frac{r_1}{r_2} + \alpha_1, k\right)\right\} + n\mathrm{S}^2\{\mu\,(\theta_1-\theta_2) + \beta_1, k'\} - 1\right] = \text{a constant} \quad \ldots \quad (24)$$

as the curve described by the vortex.

In like manner when the sides are arcs of confocal conics we can get the corresponding results by using the relation

$$\omega + (\alpha_1 + i\,\beta_1) = \sin^{-1}\frac{z}{c},$$

the ω-rectangle being the same as in the last example.