GENERALIZED SINGULAR POINTS WITH APPLICATIONS TO FLOW PROBLEMS*

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SUMMARY

Generalized singular points are used to discuss irrotational and viscous flows produced in an infinite liquid by a moving solid. It is found that the irrotational motion of translation is the same as that due to a generalized doublet and that of rotation the same as that due to a "rotation singular point". The corresponding viscous problems are solved by superposing on the irrotational motion a solution due to a concentrated force or a couple.

1. INTRODUCTION

SINGULAR points play an important part in all branches of mathematics. They appear as sources, sinks, doublets, rectilinear vortices, electric charges, magnetic particles, electric currents, etc. It is customary to think of them as the limiting cases of spherical or circular elements. It is well known that all potential flow can be produced by a suitable distribution of them. But, excepting for simple boundaries like planes, spheres or circles, this distribution is infinite in number, and, hence for practical applications it is not very suitable. It is not generally appreciated that a generalization of these points as ultimate forms of a family of closed surface bodies can simplify a large number of boundary problems by reducing the infinite number of spherical or circular points to a finite number of generalized singular points. For example, an ellipsoidal source or doublet can play the same part for ellipsoidal boundaries as an ordinary source or doublet does for spherical boundaries. In fact the uniform non-viscous flow due to the motion of an ellipsoid can be produced by an ellipsoidal doublet.

It is proposed to discuss how such generalized singular points may be obtained and used for flow problems. It is found that:—

(i) the irrotational motion of a solid through a non-viscous liquid is the same as that due to a generalized doublet;

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(ii) the irrotational motion of rotation is the same as that due to "a rotation singular point";

(iii) the slow viscous motion of the solid can be obtained by superposing on an irrotational motion due to a generalized doublet a solution due to a concentrated force in the direction of motion;

(iv) the slow viscous motion due to rotation of the solid, if stable, can be obtained by superposing on the motion due to "a rotation singular point," the solution due to "a centre of rotation".

The corresponding analogies in electricity and magnetism and heat conduction can be easily written down.

2. IRROTATIONAL FLOW

Let us take a family of closed surfaces given by

$$f(x, y, z, \xi)=0, \tag{1}$$

and let us assume that the surface of the moving solid A is given by $\xi=\xi_0$. The family given by ξ=a constant, is not necessarily a family of equipotential surfaces. In order that it be so we should have

$$\nabla^2 [F(\xi)]=0, \tag{2}$$

which gives that[1]

$$\nabla^2 \xi \Big/ \left[\left(\frac{\partial \xi}{\partial X}\right)^2 + \left(\frac{\partial \xi}{\partial y}\right)^2 + \left(\frac{\partial \xi}{\partial z}\right)^2\right]$$

should be a function of ξ only, say $\psi'(\xi)$. When this condition is satisfied the corresponding potential is found to be

$$c \int \exp\left[-\psi(\xi)\right] d\xi + d \tag{3}$$

c and d being constants.

Let V be the potential at an external point due to the solid A whose mass is M. Then, by Gauss' theorem the normal flux across any surface enclosing S is

$$\iint_s -\frac{\partial V}{\partial n}\, ds = -4\pi M,$$

the element of normal δn being drawn outwards. The limiting form of A when it reduces to a point or a line will be called the generalized source of strength M. Its potential will be taken as V. V satisfies the following conditions:—

(i) $\nabla^2 V=0$ throughout all space unoccupied by matter;

(ii) the flow across any closed surface containing it is constant;

(iii) Grad. V vanishes at infinity. The generalized source thus satisfies all the requisite conditions.

A doublet in the x-direction is obtained by displacing the source or solid through a small distance in the x-direction. The corresponding potential is $\mu\, \partial V/\partial x$, μ being the strength of the doublet. Again, the composite body whose density at any point Q is the difference of the densities at Q of the given body in two neighbouring positions is equivalent to a boundary layer of density $\rho \cos\theta$ placed on the surface of the solid A, θ being the angle which the normal makes with the x-axis. Thus the doublet can be taken to be equivalent to a boundary layer on the surface of A of surface density $\rho \cos\theta$.

If, instead of giving a translatory displacement, we turn the body round the z-axis through a small angle, we get a " rotation singular point ", whose potential

$$\phi=\mu\left(x\frac{\partial V}{\partial y}-y\frac{\partial V}{\partial x}\right), \tag{4}$$

is the same as that of a boundry layer on the surface of A of surface density $\mu\,(mx-ly)$, l, m, n being the direction cosines of the normal to the surface S of the body A.

We shall now show how these singular points may be used to discuss the motion due to the translation and rotation of the solid A in an infinite non-viscous liquid.

At first we show that a generalized doublet or boundary layer gives the flow due to the motion of the solid A through an infinite non-viscous liquid.

By Green's theorem, if N, N′ are the normal intensities due to the boundary layer of density $\rho\cos\theta$ at points just inside and just outside the layer, we have

$$N'-N=4\pi\rho\cos\theta. \tag{5}$$

Now, since the surface S is an equipotential, we can take $N=0$. Thus we see that the normal component of the potential due to the doublet assumes a value proportional to $\cos\theta$, which is the condition to be satisfied by the potential in the corresponding irrotational motion. The constants can be easily adjusted in terms of the velocity of the solid A.

We have thus shown that the motion of the solid A through an infinite non-viscous liquid can be represented by a generalised doublet, whose potential can be obtained from the static potential due to the solid A.

In the case of a sphere moving through an infinite liquid we have the following results:—

The potential V at any external point is

$$\frac{M}{r}.$$

The potential due to the corresponding doublet is

$$\phi = -\mu \frac{\partial}{\partial x}\left(\frac{1}{r}\right) = \frac{\mu x}{r^3},$$

which is the known result for the potential of liquid flow.

In like manner for a circular cylinder we have

$$V = c - 2M \log r,$$

so that

$$\phi = \frac{\mu x}{r^2},$$

which is again a known result.

Let us take the case of an ellipsoid. The confocal family of ellipsoids is given by

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} = 1.$$

The potential due to an ellipsoidal source is[2]

$$V = \int_{\lambda}^{\infty}\left(1 - \frac{x^2}{a^2+u} - \frac{y^2}{b^2+u} - \frac{z^2}{c^2+u}\right)\frac{du}{\Delta}. \qquad (6.1)$$

The potential of the ellipsoidal doublet is therefore given by

$$\phi = -\mu \frac{\partial V}{\partial x} = \mu_1 x \alpha, \qquad (6.2)$$

μ_1, being a constant, and Δ, α are given by

$$\Delta = [(a^2+u)(b^2+u)(c^2+u)]^{\frac{1}{2}}, \quad \alpha = \int_{\lambda}^{\infty}\frac{du}{(a^2+u)\Delta}, \qquad (6.3)$$

which result is again known.

For an elliptic cylinder we have[3]

$$V = -2\pi\rho \frac{ab}{a'+b'}\left(\frac{x^2}{a'}+\frac{y^2}{b'}\right) - 2\pi\rho ab \log(a'+b') + c \qquad (7.1)$$

a', b' being the axes of the confocal ellipse through the attracted point. This value of V gives

$$\phi = \mu \frac{ab}{a'+b'} \frac{x}{a'}, \; a' = c\cosh\xi, \; b' = c\sinh\xi,$$

$$x = c\cosh\xi\cos\eta, \quad y = c\sinh\xi\sin\eta. \qquad (7.2)$$

Substituting these values we get the known result

$$\phi = \mu_1 \exp(-\xi)\cos\eta.$$

For the elliptic source V is of the form[4]

$$V = -\tfrac{1}{2}\pi\rho ab\,(e^{-2\xi}\cos 2\eta + 2\xi). \qquad (7.3)$$

For the motion of rotation, we easily get the following known results for the value of ϕ given in (4) :—

(i) for a sphere or a circular cylinder $\phi = 0$;

(ii) for an ellipsoid* $\phi = \mu_1(\beta - \alpha)\,xy$,

(iii) for an elliptic cylinder $\phi = \mu_1 e^{-2\xi}\sin 2\eta$. (8)

Thus we see that the irrotational motion of an infinite liquid due to the translation or rotation of the solid A can be easily deduced from the static potential of the body at any external point, by using the idea of generalised singular points.

For any internal point V satisfies Poisson's equation

$$\nabla^2 V = -4\pi\rho \text{ or } -2\pi\rho \qquad (9)$$

according as it is a three-dimensional or two-dimensional problem. ρ is assumed to be a constant, and hence V does not satisfy Laplace's equation. But its first partial derivatives with respect to x, y, z satisfy it. The potential for the transational and rotational flows of the liquid contained in the surface S of the body A remain of the form

$$\mu \frac{\partial V}{\partial x} \text{ and } \mu\left(x\frac{\partial V}{\partial y} - y\frac{\partial V}{\partial x}\right). \qquad (10)$$

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* β is obtained from α in (6.3) by interchanging a and b.

The following results now follow immediately:—

(1) for the motion of translation along the x-axis of any solid $\phi = -Ux$

(2) for the rotation of sphere or circular cylinder $\phi = 0$;

(3) for rotation of an ellipsoid, we have to use $V = D - Ax^2 - By^2 - Cz^2$, D, A, B, C being constants. This gives

$$\phi = \mu_1 xy$$

(4) for an elliptic cylinder

$$V = D - 2\pi\rho \frac{ab}{a+b}\left(\frac{x^2}{a} + \frac{y^2}{b}\right), \ \phi = \mu_1 xy.$$

Thus liquid motion contained in a closed surface given by (1) can be deduced from the static potential at any point inside the body A.

3. Viscous Flow

For the slow motion of a solid through a viscous liquid we have to consider generalised singular points of various types. In the first type we determine the displacements produced in an infinite elastic solid wherein a concentrated force X_0 acts at the origin in the direction of the X-axis. In this case we know that the displacements components can be put in the form[5]

$$u = \frac{\partial \phi}{\partial x} + \frac{\partial H}{\partial y} - \frac{\partial G}{\partial z}, \text{ etc.} \tag{11.1}$$

and the body force X, Y, Z in the form

$$X = \frac{\partial \Phi}{\partial x} + \frac{\partial N}{\partial y} - \frac{\partial M}{\partial z}, \text{ etc.} \tag{11.2}$$

where

$$(\lambda + 2\mu)\nabla^2\phi + \rho\Phi = 0,\ \mu\nabla^2 F + \rho L = 0,\ \mu\nabla^2 G + \rho M = 0,$$
$$\mu\nabla^2 H + \rho N = 0. \tag{12}$$

The Laplace's equation in orthogonal curvilinear co-ordinates ξ, η, ξ is

$$\frac{\partial}{\partial \xi}\left(\frac{h_1}{h_2 h_3}\frac{\partial V}{\partial \xi}\right) + \text{two similar terms} = 0,$$

where

$$\frac{1}{h_1^2} = \left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2 + \left(\frac{\partial z}{\partial \xi}\right)^2, \text{ etc.} \tag{13}$$

Let the surface of the moving solid A be given by the equipotential $\xi = \xi_0$. There is a solution of (13) which is a function of ξ only. Let it be denoted by $\psi(\xi)$. It should be such, that its flux across any equi-potential surface is equal to 4π in three dimensions and 2π in two dimensions.

Putting

$$\rho \int\int\int X' \, dx' dy' dz' = X_0,$$

we get[5]

$$\Phi = -\frac{1}{4\pi\rho} X_0 \frac{\partial\psi}{\partial x}, \; L = 0, \; M = \frac{1}{4\pi\rho} X_0 \frac{\partial\psi}{\partial z}, \; N = -\frac{1}{4\pi\rho} X_0 \frac{\partial\psi}{\partial y}. \tag{14}$$

Since

$$\nabla^2 (x\psi) = 2 \frac{\partial\psi}{\partial x},$$

we can put

$$\phi = \frac{X_0}{8\pi(\lambda + 2\mu)} x\psi, \; F = 0, \; G = -\frac{X_0}{8\pi\mu} z\psi,$$

$$H = \frac{X_0}{8\pi\mu} y\psi. \tag{15}$$

u, v, w now take the form

$$u = -\frac{(\lambda + \mu) X_0}{8\pi\mu(\lambda + 2\mu)} \frac{\partial}{\partial x}(x\psi) + \frac{X_0}{4\pi\mu} \psi,$$

$$v, w = -\frac{(\lambda + \mu) X_0}{8\pi\mu(\lambda + 2\mu)} \left(\frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) (x\psi). \tag{16}$$

These give

$$\Delta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{X_0}{4\pi(\lambda + 2\mu)} \frac{\partial\psi}{\partial x}, \tag{17.1}$$

$$\tau_{xx} = \frac{(3\lambda + 4\mu) X_0}{4\pi(\lambda + 2\mu)} \frac{\partial\psi}{\partial x} - \frac{(\lambda + \mu) X_0}{4\pi(\lambda + 2\mu)} \frac{\partial^2}{\partial x^2}(x\psi)$$

$$\tau_{yz} = -\frac{(\lambda + \mu) X_0}{4\pi(\lambda + 2\mu)} \frac{\partial^2}{\partial y \partial z}(x\psi), \text{ etc.} \tag{17.2}$$

The equations of equilibrium are satisfied with these values of τ_{xx}, τ_{yz}, etc. They have to satisfy the condition that across any surface $\xi =$ constant, they give a force X_0 in the x direction, independent of ξ.

X_ν, the traction component along the x-axis is

$$X_\nu = l\tau_{xx} + m\tau_{xy} + n\tau_{xz}$$

$$= -\frac{(\lambda + \mu) X_0}{4\pi(\lambda + 2\mu)} \left[l \frac{\partial^2}{\partial x^2} + m \frac{\partial^2}{\partial x \partial y} + n \frac{\partial^2}{\partial x \partial z} \right] (x\psi)$$

$$+ \frac{X_0}{4\pi} \left(l \frac{\partial}{\partial x} + m \frac{\partial}{\partial y} + n \frac{\partial}{\partial z} \right) \psi + \frac{\lambda + \mu}{2\pi(\lambda + 2\mu)} X_0 \frac{\partial\psi}{\partial x}, \tag{18.1}$$

l, m, n being the direction cosines of the normal drawn to $\xi =$ constant. If $\delta\gamma$ is an element of this normal we can write (18.1) as

$$X_\gamma = -\frac{(\lambda+\mu)X_0}{4\pi(\lambda+2\mu)}\, x\, \frac{\partial}{\partial\gamma}\left(\frac{\partial\psi}{\partial x}\right) + \frac{\mu X_0}{4\pi(\lambda+2\mu)}\frac{\partial\psi}{\partial\gamma} + \frac{(\lambda+\mu)X_0}{4\pi(\lambda+2\mu)}\, l\, \frac{\partial\psi}{\partial x}. \qquad (18.2)$$

We shall now show that

$$\iint X_\gamma ds = X_0,$$

the integration being taken over any closed surface $\xi =$ constant.

We shall assume that the liquid is bounded by the surface S ($\xi = \xi_0$) and an equipotential surface at infinity given by $\xi = \infty$. In the usual notation

$$\iint X_\gamma ds = \iint_s X_\gamma ds - \iint_\infty X_\gamma ds. \qquad (19.1)$$

But X_γ is given by (18.2). Also

$$\iint_s x \frac{\partial}{\partial\gamma}\left(\frac{\partial\psi}{\partial x}\right) ds - \iint_\infty x \frac{\partial}{\partial\gamma}\left(\frac{\partial\psi}{\partial x}\right) ds$$

$$= \iint_s \frac{\partial\psi}{\partial x}\frac{\partial x}{\partial\gamma} ds - \iint \frac{\partial\psi}{\partial x}\frac{\partial x}{\partial\gamma} ds$$

$$= \iint_s l \frac{\partial\psi}{\partial x} ds - \iint_\infty l \frac{\partial\psi}{\partial x} ds \qquad (19.2)$$

But, since $\psi \to 1/r$, at infinity, we have

$$\iint_\infty x \frac{\partial}{\partial\gamma}\left(\frac{\partial\psi}{\partial x}\right) ds = \frac{8}{3}\pi, \quad \iint_\infty l \frac{\partial\psi}{\partial x} ds = -\frac{4}{3}\pi.$$

Hence

$$\iint_s x \frac{\partial}{\partial\gamma}\left(\frac{\partial\psi}{\partial x}\right) ds = \iint_s l \frac{\partial\psi}{\partial x} ds + 4\pi. \qquad (19.3)$$

Similarly

$$\iint_s \frac{\partial\psi}{\partial\gamma} ds - \iint_\infty \frac{\partial\psi}{\partial\gamma} ds = 0,$$

so that

$$\iint_s \frac{\partial\psi}{\partial\gamma} ds = -4\pi. \qquad (19.4)$$

Thus

$$\iint_s X_y \, ds = -\frac{X_0 \mu}{\lambda + 2\mu} - \frac{X_0 (\lambda + \mu)}{\lambda + 2\mu} = -X_0, \qquad (19.5)$$

which shows that a force X_0 should be applied in the x-direction. Similarly

$$Y_y = -\frac{(\lambda+\mu) X_0}{4\pi(\lambda+2\mu)} \, x \, \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right) + \frac{(\lambda+\mu) X_0}{4\pi(\lambda+2\mu)} \, m \, \frac{\partial \psi}{\partial x},$$

$$Z_y = -\frac{(\lambda+\mu) X_0}{4\pi(\lambda+2\mu)} \, x \, \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial z}\right) + \frac{(\lambda+\mu) X_0}{4\pi(\lambda+2\mu)} \, n \, \frac{\partial \psi}{\partial x} \qquad (19.6)$$

from which we easily see that

$$\iint_s Y_y \, ds = 0, \quad \iint_s Z_y \, ds = 0.$$

For the vanishing of the couple we should have

$$\iint_s (z Y_y - y z_y) \, ds = 0, \qquad (20)$$

or

$$\iint_s \left[\frac{\partial \psi}{\partial y}(lz + nx) - \frac{\partial \psi}{\partial z}(ly + mx) - ny \frac{\partial \psi}{\partial y} - mz \frac{\partial \psi}{\partial x}\right] ds = 0.$$

Transforming into volume integrals, we see that this condition is also satisfied.

For a liquid $\Delta \to 0$, $\lambda \to \infty$, such that $\lambda \Delta \to -p$. Thus we get from (16)

$$\tau_{xx} = -p + 2\mu B \left[\frac{\partial^2}{\partial x^2}(x\psi) - 2\frac{\partial \psi}{\partial x}\right],$$

$$\tau_{yy} = -p + 2\mu B \frac{\partial^2}{\partial y^2}(x\psi),$$

$$\tau_{zz} = -p + 2\mu B \frac{\partial^2}{\partial z^2}(x\psi),$$

$$\tau_{yz} = 2\mu B \frac{\partial^2}{\partial y \partial z}(x\psi), \quad \tau_{zx} = 2\mu B \frac{\partial^2}{\partial x \partial z}(x\psi) - 2\mu B \frac{\partial \psi}{\partial z} \qquad (21.1)$$

$$\tau_{xy} = 2\mu B \frac{\partial^2}{\partial x \partial y}(x\psi) - 2\mu B \frac{\partial \psi}{\partial y}, \qquad (21.2)$$

where $\nabla^2 \psi = 0$ and $B = -X_0/8\pi\mu$.

The tractions become

$$X_y = 2\mu B \left[l \frac{\partial^2}{\partial x^2} + m \frac{\partial^2}{\partial x \partial y} + n \frac{\partial^2}{\partial x \partial z} \right] (x\psi)$$

$$- 2\mu B \left(l \frac{\partial}{\partial x} + m \frac{\partial}{\partial y} + n \frac{\partial}{\partial z} \right) \psi - lp \qquad (21.3)$$

$$= 2\mu Bx \frac{\partial}{\partial y} \left(\frac{\partial \psi}{\partial x} \right) - lp,$$

$$Y_y, Z_y = 2\mu Bx \frac{\partial}{\partial y} \left(\frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial z} \right).$$

The displacement u, v, w take the form

$$u = B \left[\frac{\partial}{\partial x} (x\psi) - 2\psi \right],$$

$$v, w = B \left(\frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) (x\psi). \qquad (21.4)$$

Let V be the potential of the corresponding irrotational flow. On the boundary of the moving solid ($\xi =$ constant) we have

$$- h_1 \frac{\partial V}{\partial \xi} = h_1 U \frac{\partial x}{\partial \xi}, \text{ or } \frac{\partial V}{\partial \xi} = - U \frac{\partial x}{\partial \xi}. \qquad (22)$$

If V is of the form $x\alpha(\xi)$, the velocities for the slow viscous motion can be at once obtained by superposing on the generalized doublet of the corresponding irrotational motion a generalized singular point of the first type. Thus we get

$$u = A \frac{\partial V}{\partial x} - \frac{X_0}{8\pi\mu} \left[\frac{\partial}{\partial x} (\partial\psi) - 2\psi \right],$$

$$v = A \frac{\partial V}{\partial y} - \frac{X_0}{8\pi\mu} \frac{\partial}{\partial y} (\psi), \qquad (23)$$

$$w = A \frac{\partial V}{\partial z} - \frac{X_0}{8\pi\mu} \frac{\partial}{\partial z} (x\psi),$$

where the constant A can be determined from the boundary conditions.

If the irrotational motion is represented by harmonics of the type V_n, V in (23) will have V_n and all harmonies of an order less than V_n.

When $V = x\alpha(\xi)$ we get from (22)

$$A \left[\alpha_0 + x\alpha_0' \frac{\partial \xi}{\partial x} \right] - \frac{X_0}{8\pi\mu} \left[x\psi_0' \frac{\partial \xi}{\partial x} - \psi_0 \right] = U \qquad (24)$$

where α_0, ϕ_0 are the values of α, ϕ on the surface of the solid, dashes denote differentiation with respect to ξ, and U is the uniform velocity of the solid A. Thus

$$A\alpha_0 - \frac{X_0\phi_0}{8\pi\mu} = U, \quad A\alpha_0' = \frac{X_0\phi_0'}{8\pi\mu}. \tag{25.1}$$

These values of X_0 and A also ensure that $v = 0$, $w = 0$ on the boundary. The drag suffered by the solid and the constant A are given by

$$X_0 = \frac{8\pi\mu\,\alpha_0' U}{\phi_0\alpha_0' - \phi_0'\alpha_0}, \quad A = \frac{\phi_0'}{\phi_0\alpha_0' - \phi_0'\alpha_0}. \tag{25.2}$$

The second case when V is to be represented by a series of harmonics can be similarly treated.

In the particular case of the sphere we get

$$V = x\alpha = \frac{2}{3}\frac{x}{r^3}, \quad \phi = \frac{1}{r}, \quad A = \frac{3}{8}a^3, \quad X_0 = 6\pi a\mu U. \tag{26}$$

These give the known values of u, v, w

For a circular cylinder we get[6]

$$V = x\alpha = \frac{x}{r^2}, \quad \phi = -\log r,$$

$$X_0 = \frac{4\pi\mu\,\alpha_0' U}{\phi_0\alpha_0' - \phi_0'\alpha_0} = \frac{8\pi\mu U}{1 - 2\log a}. \tag{27}$$

An infinite constant is neglected in the value of ϕ, which is a well-known drawback.

For an elliptic cylinder we have

$$\alpha = c\,\xi\,\mathrm{sech}\,\xi, \quad \phi = -\xi \tag{7}$$

$$X_0 = \frac{4\pi\mu U(a+b)}{a - \xi_0(a+b)},$$

$$a = c\cosh\xi_0, \quad b = c\sinh\xi_0, \quad c^2 = a^2 - b^2. \tag{28}$$

For an ellipsoid we have the following results:

$$\phi_0' = -\frac{1}{2abc}, \quad \alpha_0' = -\frac{1}{a^3bc}, \quad A = \frac{Ua^2}{2\phi_0 + a^2\alpha_0}, \tag{29.1}$$

$$X_0 = \frac{16\pi\mu U}{2\phi_0 + a^2\alpha_0}, \tag{29.2}$$

and the values for u, v, w which are known.[7]

For the slow rotatory motion of a sphere or a cylinder in a viscous liquid we have to use the generalised singular point, called " the centre of rotation". In both cases there is no irrotational motion, and hence we can take $V = 0$. For a sphere, if G is the couple of rotation, the corresponding displacements in an incompressible infinite solid are

$$u, v = \frac{G}{8\pi\mu}\left(\frac{\partial}{\partial y}, -\frac{\partial}{\partial x}\right)\frac{1}{r}, \quad w = 0 \tag{30.1}$$

If ω_0 is the angular velocity and a the radius of the sphere

$$G = 8\pi\mu\, a^3\, \omega_0 \tag{30.2}$$

For a circular cylinder of radius a we get the following results:—

$$u, v = \frac{G}{4\pi\mu}\left(-\frac{\partial}{\partial y}, \frac{\partial}{\partial x}\right)\log r, \quad w = 0 \tag{31.1}$$

$$G = 4\pi\mu a^2\, \omega_0 \tag{31.2}$$

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