ON GUEST'S LAW OF ELASTIC FAILURE

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Various hypotheses have been advanced for determining the yield point of an elastic material. In one group we may combine the hypotheses of (i) principal maximum stress, (ii) maximum principal strain and (iii) maximum shear stress or maximum principal stress difference. Of these the last one seems to find favour with many writers. In fact Cook[1] maintains "that stress differences alone will be found in the expression of any theory which can be applicable" to elastic failure. Taylor and Quinney[2] also hold this view.

A second group of theories starts from the idea that there is a limit to the amount of strain energy which can be stored in the element of volume of an elastic material. The strain energy can be split into two parts, one due to the change in volume and the other due to the distortion. Haigh[4] and Beltrami[3] take into consideration both these changes, but Mises[5] and Hencky neglect the part due to the change in volume. Nadai[6] is a great supporter of the Mises-Hencky hypothesis.

A feature common to all these theories is that the criterion of failure is unaltered by a reversal of the sign of the stress. But it is well known that materials shew yield stresses in compression several times those in tension. To avoid this objection Mohr[7] has extended the maximum shear theory by combining it with the principal maximum stress theory.

Very few have found their way to agree to Guest's criterion[8] that besides the shearing stress, which is a primary factor in yield, volumetric stress of a certain type should also be taken into account. In two recent papers[9] he has elaborately discussed some recent experimental results of Mason, Becker and Cook, and has shewn how the difficulties in the explanations given by the various authors and others of their results can be overcome by taking a volumetric stress in the shearing stress hypothesis. We propose to shew how the theory of Finite Strain which has been developed in some recent papers[10, 11] gives a criterion similar to that of Guest, which contains the three commonly adopted theories of maximum shear stress, of Mohr and of Mises and Hencky.

Let p_1, p_2, p_3 be the principal stresses in descending order of magnitude. If we adopt the maximum stress difference hypothesis, the criterion can be written as

$$p_1 - p_2 = \text{Constant} = \mathrm{K} \text{ (say)}. \tag{1}$$

In the case of a simple tension in the x-direction we can put $p_1 = \widehat{xx}$, $p_2 = 0$, so that $p_1 = \widehat{xx} = \mathrm{K}$. In the case of pure shear we can put $p_1 = -p_2 = \widehat{xy}$, so that $\widehat{xy} = \frac{1}{2}\mathrm{K}$. Hence, if ξ be the ratio between the yield point in tension and the yield point in shear, we get $\xi = 0 \cdot 5$.

Mises-Hencky hypothesis can be written as

$$(p_1 - p_2)^2 + (p_2 - p_3)^2 + (p_3 - p_1)^2 = \mathrm{K} \text{ (a constant)}. \tag{2}$$

Proceeding as above we find $\xi = 0 \cdot 577$.

Guest formulates the law of failure as

$$p_1 - p_3 + k\,(p_1 + p_3) = \mathrm{K} \text{ (a constant)}, \tag{3}$$

where k is a non-dimensional constant of the material varying from zero to unity. For $k=0$ (3) reduces to (1), and for $k = 0 \cdot 155$ (3) gives the same value of ξ as (2). In general the value of ξ given by (3) is

$$\xi = \tfrac{1}{2}(1 + k). \tag{4}$$

The values of ξ determined experimentally have a wide variation, and hence (3) can be expected to give better results than either (1) or (2). It is significant that Guest does not use the average volumetric stress $\frac{1}{3}(p_1 + p_2 + p_3)$ but the stress $\frac{1}{2}(p_1 + p_3)$. In the finite strain theory it is found that $\frac{1}{3}(p_1 + p_2 + p_3)$ should be used.

For a spherical shell subjected to uniform normal tractions on the inner and outer surfaces the finite strain theory gives the limits for the principal stresses $\widehat{rr}$ and $\widehat{\theta\theta}$ $(= \widehat{\phi\phi})$ as[12]

$$-\frac{1-2\eta}{1-\eta} < \frac{2\,(\widehat{rr} - \widehat{\theta\theta})}{3k - 2\widehat{rr}} < \frac{1-2\eta}{2\eta}, \tag{5}$$

where η is Poisson's ratio and k the modulus of compression. The lower limit corresponds to infinite contraction and the upper to infinite extension.

If we take the upper limit, we get the criterion for failure due to extension as

$$\widehat{rr} - \widehat{\theta\theta} + \frac{1-2\eta}{1+2\eta}\,(\widehat{rr} + \widehat{\theta\theta}) = 3k.\frac{1-2\eta}{1+2\eta}, \tag{6}$$

which is of the form proposed by Guest. For steel we can put $\eta = 0 \cdot 3$ and we get the ratio ξ equal to 0·625, which is a very high value. But, if we write (6) as

$$\widehat{rr} - \widehat{\theta\theta} + \tfrac{1}{2}\,\frac{1-2\eta}{1+\eta}\,(\widehat{rr} + \widehat{\theta\theta} + \widehat{\phi\phi}) = \tfrac{3}{4}\,\frac{E}{1+\eta}, \tag{7}$$

where E is Young's modulus, we get a slight modification of Guest's Law as

$$p_1 - p_3 + \tfrac{1}{2}\cdot\frac{1-2\eta}{1+\eta}\,(p_1 + p_2 + p_3) = \tfrac{3}{4}\,\frac{E}{1+\eta}. \tag{8}$$

For a pure shear we can put $p_1 = -p_3 = \widehat{xy}$ (say), $p_2 = 0$, and we get

$$2\,\widehat{xy} = \tfrac{3}{4}\,\frac{E}{1+\eta}. \tag{9·1}$$

For a pure tension we can put $p_1 = \widehat{xx}$ (say), $p_2 = p_3 = 0$, and we get

$$\widehat{xx} = \tfrac{1}{2}\,E. \tag{9·2}$$

Hence

$$\xi = \tfrac{3}{4}\cdot\frac{1}{1+\eta}, \tag{9·3}$$

which lies between 0·5 and 0·75. For steel we can take $\eta = 0·3$, and the value of ξ becomes 0·577, which is exactly that given by Mises-Hencky theory.

For contraction we take the lower limit, and the criterion corresponding to (8) is found to be

$$p_1 - p_3 + \frac{1-2\eta}{1+\eta}\,(p_1 + p_2 + p_3) = \tfrac{3}{2}\,\frac{E}{1+\eta}. \tag{10}$$

For a pure shear

$$2\widehat{xy} = \tfrac{3}{2}\cdot\frac{E}{1+\eta}, \tag{11·1}$$

and for a pure tension

$$\widehat{xx} = \tfrac{3}{2}\cdot\frac{E}{2-\eta}. \tag{11·2}$$

The corresponding value of ξ is therefore

$$\xi = \tfrac{1}{2}\,\frac{2-\eta}{1+\eta},$$

which, as was to be expected, is equal to 0·577 for a higher value of η given by $\eta = 0·392$.

The ratio of the values of $\widehat{xx}$ in pure compression and in tension is found from (9·2) and (11·2) as which varies between $\frac{8}{3}$ and 2. Thus, as is proved by experiment, the yield stresses in compression are much greater than those in tension.

It should be mentioned that the laws formulated in (8) and (10)* are based on the hypothesis that the material remains within the limits of perfect elasticity. But Guest has shewn his law to hold good for brittle materials as well. Whether we use $\frac{1}{2}(p_1+p_2)$, as Guest has done, or the average volumetric stress, as has been done in (8) and (10), it appears certain that an extension of the maximum shear stress hypothesis lies in including some more linear terms in the principal stresses. If the additional terms are to reflect the effect of the change in volume, the inclusion of the average volumetric stress $\frac{1}{3}(p_1+p_2+p_3)$ seems to be more feasible than $\frac{1}{2}(p_1+p_2)$, as has been done by Guest.

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* For a thick cylindrical tube under normal tractions on the inner and outer surfaces an equation similar to (10) can be obtained.[13]