ON THE GRAVEST MODE OF SOME VIBRATING SYSTEMS

By B. R. Seth

(From the Department of Mathematics, Hindu College, Delhi)

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The gravest mode of any vibrating system should be devoid of internal nodal lines. In some well-known cases only the symmetrical solutions have been obtained, and hence the gravest mode remains undetermined. The following are some examples :

(*a*) Transverse oscillations of water contained in a canal (*i*) with sides inclined at an angle of 60° to the vertical, (*ii*) whose section is a hyperbola of eccentricity 2.

If we take the axes of x and y respectively horizontal and vertical, the solution due to Greenhill for (*i*) is given by the stream function[1]

$$\psi = Ax(x^2 - 3y^2)\cos(\sigma t + \epsilon),\ \sigma^2 = g/h, \tag{1}$$

h being the depth of water in the canal. The slowest mode, as Lamb remarks, is asymmetrical and has not yet been determined.

For (*ii*) we find[2]

$$\psi = Ax(y^2 - \tfrac{1}{3}x^2 - a^2)\cos(\sigma t + \epsilon),\ \sigma^2 = g/h, \tag{2}$$

which again gives a symmetrical type.

(*b*) Standing waves between two transverse partitions in a canal with sides inclined at 60° to the vertical.

If the x-axis be parallel to the length of the canal, the z-axis drawn vertically upwards, the y-axis horizontal and transverse to the canal, and (π/k) be the distance between the transverse partitions, Macdonald[3] finds that the velocity potential of the motion is given by

$$\phi = A\Big[\cosh k(z-h) + \frac{\sigma^2}{gk}\sinh k(z-h)$$
$$+ 2\cosh\frac{ky\sqrt{3}}{2}\left\{\cosh k\left(\frac{z}{2}+h\right) - \frac{\sigma^2}{gk}\sinh k\left(\frac{z}{2}+h\right)\right\}\Big]\cos kx, \tag{3}$$

[1] Lamb, *Hydrodynamics*, 5th Ed., 1930, p. 419.

[2] Seth, *Phil. Mag.*, Ser. 7, 1937, **23**, 113–14.

[3] Macdonald, *Proc. Lond. Math. Soc.*, 1894, **25**, 101.

where

$$2\left(\frac{\sigma^2}{gk}\right)^2 - 3\frac{\sigma^2}{gk}\coth\frac{3\,kh}{2} + 1 = 0,$$

omitting the time-factor $\cos(\sigma t + \epsilon)$.

The motion is symmetrical about the z-axis, and hence (3) cannot give the gravest mode.

(c) Transverse vibrations of an isosceles triangular membrane containing an angle of 120°.

If the sides are given by $x = a$, $y = x\sqrt{3}$, $y + x\sqrt{3} = 2a/\sqrt{3}$, it is given in a recent paper[4] that the displacement z should be

$$\begin{aligned} z &= 2\sin\frac{(m-n)\pi x}{a}\sin\frac{(m+n+1)\pi y\sqrt{3}}{a} \\ &\quad - 2\cos\left(2m+1+\frac{2n+1}{2}\right)\frac{\pi x}{a}\cos\frac{(2n+1)\pi y\sqrt{3}}{2a} \\ &\quad + 2\cos\left(2n+1+\frac{2m+1}{2}\right)\frac{\pi x}{a}\cos\frac{(2m+1)\pi y\sqrt{3}}{2a}, \end{aligned} \tag{4}$$

$$\frac{\sigma}{2\pi} = \frac{1}{2}\frac{c}{a}\left\{(2m+1)^2 + (2n+1)^2 + (2m+1)(2n+1)\right\}^{\frac{1}{2}},$$

m and n being integers.

This is again symmetrical about the median bisecting 120°.

(d) Transverse vibrations of a rhombus containing an angle of 120°[4] and those of a regular hexagon.[5]

If we take the sides as $x = 0$, $x = a$, $y = x/\sqrt{3}$, $y = x/\sqrt{3} + 2a/\sqrt{3}$, in the first case, and as $x = \pm a$, $y = \pm x/\sqrt{3} \pm 2a/\sqrt{3}$ in the second case, it is found that the displacement z is given by

$$\begin{aligned} z &= 2\sin\frac{(m-n)\pi x}{a}\cos\frac{(m+n)\pi y\sqrt{3}}{a} \\ &\quad - 2\sin\frac{(2m+n)\pi x}{a}\cos\frac{n\pi y\sqrt{3}}{a} \\ &\quad + 2\sin\frac{(2n+m)\pi x}{a}\cos\frac{m\pi y\sqrt{3}}{a}, \end{aligned} \tag{5}$$

$$\frac{\sigma}{2\pi} = \frac{c}{a}(m^2 + n^2 + mn)^{\frac{1}{2}},$$

m and n being integers as before.

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[4] Seth, *Proc. Ind. Acad. Sci.* (A), 1940, **5**.

[5] D. G. Christopherson, *Quart. J. of Math.*, 1940, **11**, 65.

The solution is symmetrical about the diagonal bisecting 120° in the first case, and about all the three diagonals in the second.

In all the above cases we see that whenever there is symmetry about the bisector of the angle 120° the modes are also symmetrical about the same line.

The determination of the asymmetrical types in each of the above cases, and hence that of the gravest mode, yields an infinite determinant for the frequency. Rayleigh's method can be used to get an approximate value of the gravest mode. We shall illustrate this by discussing in detail the case of transverse oscillations of water contained in a canal with its sides inclined at 60° to the vertical.

We take the x-axis vertically downwards, and the sides given by $\theta = \frac{2}{3}\pi$ and $\theta = \frac{4}{3}\pi$. Let the free surface be given by $x = -h$. Omitting the time factor we see that the asymmetrical modes are given by

$$\phi = \sum_{1}^{\infty} A_n \, r^{\frac{3}{2}(2n-1)} \cos \tfrac{3}{2}(2n-1)\theta, \qquad (6 \cdot 1)$$

$$\psi = \sum_{1}^{\infty} A_n \, r^{\frac{3}{2}(2n-1)} \sin \tfrac{3}{2}(2n-1)\theta. \qquad (6 \cdot 2)$$

This value of ψ vanishes over $\theta = \frac{2}{3}\pi$ and $\theta = \frac{4}{3}\pi$, but does not do so over $\theta = \pi$, which is the line of symmetry.

The condition at the free surface, $r \cos \theta = -h$, is

$$\sigma^2 \phi = -g \frac{\partial \phi}{\partial x}, \qquad (7)$$

and since

$$\frac{\partial \phi}{\partial x} = \cos \theta \frac{\partial \phi}{\partial r} - \sin \theta \frac{\partial \phi}{r \partial \theta},$$

we get

$$g \, [\tfrac{3}{2} A_1 \, r^{\frac{1}{2}} \cos \tfrac{1}{2}\theta + \tfrac{9}{2} A_2 \, r^{\frac{7}{2}} \cos \tfrac{7}{2}\theta + \tfrac{15}{2} A_3 r^{\frac{13}{2}} \cos \tfrac{13}{2}\theta + \ldots .]$$
$$= -\sigma^2 \, [A_1 \, r^{\frac{3}{2}} \cos \tfrac{3}{2}\theta + A_2 \, r^{\frac{9}{2}} \cos \tfrac{9}{2}\theta + A_3 \, r^{\frac{15}{2}} \cos \tfrac{15}{2}\theta + \ldots \ldots]. \qquad (8 \cdot 1)$$

Multiplying both sides by $r^{\frac{1}{2}} \cos \frac{1}{2}\theta$ we get

$$g \, [\tfrac{3}{2} A_1 r \, (1 + \cos \theta) + \tfrac{9}{2} A_2 r^4 \, (\cos 4\theta + \cos 3\theta)$$
$$+ \tfrac{15}{2} A_3 r^7 \, (\cos 7\theta + \cos 6\theta) + \ldots]$$
$$= -\sigma^2 \, [A_1 r^2 \, (\cos 2\theta + \cos \theta) + A_2 \, r^5 \, (\cos 5\theta + \cos 4\theta)$$
$$+ A_3 r^8 \, (\cos 8\theta + \cos 7\theta) + \ldots \ldots].$$

Expanding the cosines in powers of $\cos \theta$ and putting $r \cos \theta = -h$, we get

$$g \, [\tfrac{3}{2} A_1 \, (r - h) + \tfrac{9}{2} A_2 \, (r^4 - 8 \, r^2 h^2 + 8 \, h^4 + 3 \, r^3 \, h - 4 \, r \, h^3) + \ldots \ldots]$$
$$= -\sigma^2 \, [A_1 \, (2 \, h^2 - r^2 - rh) + A_2 \, (-5 \, r^4 \, h + 20 \, r^2 \, h^3 - 16 \, h^5 + r^5$$
$$- 8 \, r^3 h^2 + 8 \, r \, h^4) + \ldots \ldots]. \qquad (8 \cdot 2)$$

Putting $t = \sigma^2 h/g$, and equating the coefficients of like powers of r, we get the following set of infinite equations:—

$$A_1 (2t - \tfrac{3}{2}) - A_2 h^3 (2^4 t - \tfrac{9}{2} \cdot 2^3) + A_3 h^6 (2^7 t - \tfrac{15}{2} \cdot 2^6)$$
$$- A_4 h^9 (2^{10} t - \tfrac{21}{2} \cdot 2^9) + \ldots\ldots = 0,$$

$$A_1 (t - \tfrac{3}{2}) - A_2 h^3 (2^3 t - \tfrac{9}{2} \cdot 2^2) + A_3 h^6 (2^6 t - \tfrac{15}{2} \cdot 2^5)$$
$$- A_4 h^9 (2^9 t - \tfrac{21}{2} \cdot 2^8) + \ldots\ldots\ldots = 0,$$

$$A_1 t \qquad - A_2 h^3 (\tfrac{5}{11} 2^2 t - \tfrac{9}{2} \cdot \tfrac{4}{11} \cdot 2) + A_3 h^6 (\tfrac{8}{11} 2^5 t - \tfrac{15}{2} \cdot 2^4 \cdot \tfrac{7}{11})$$
$$- A_4 h^9 (\tfrac{11}{11} 2^8 t - \tfrac{21}{2} \cdot \tfrac{10}{11} \cdot 2^7) + \ldots = 0,$$

$$A_2 h^3 (\tfrac{4}{11} 2t - \tfrac{9}{2} \cdot \tfrac{3}{11}) - A_3 h^6 (\tfrac{7}{11} 2^4 t - \tfrac{15}{2} \cdot \tfrac{6}{11} \cdot 2^3)$$
$$+ A_4 h^9 (\tfrac{10}{11} 2^7 t - \tfrac{21}{2} \cdot \tfrac{9}{11} \cdot 2^6) + \ldots = 0,$$

$$A_2 h^3 (\tfrac{5 \cdot 2}{21} t - \tfrac{9}{2}) - A_3 h^6 (\tfrac{8 \cdot 5}{21} 2^3 t - \tfrac{15}{2} \cdot \tfrac{7 \cdot 4}{21} 2^2)$$
$$+ A_4 h^9 (\tfrac{11 \cdot 8}{21} 2^6 t - \tfrac{15}{2} \cdot \tfrac{10 \cdot 7}{21} \cdot 2^5) + = 0, \qquad (9)$$

.. ..

Eliminating the A's we get an infinite determinant for t.

To get an approximate value of the frequency of the gravest mode we take

$$\phi = A_1 r^{\frac{3}{2}} \cos \tfrac{3}{2} \theta \cos (\sigma t + \epsilon) \qquad (10)$$

as a constrained type. If T and V represent the kinetic and potential energies respectively, we find

$$T = \tfrac{1}{2} \rho \int_{\frac{2}{3}\pi}^{\frac{4}{3}\pi} \int_{0}^{-h \sec \theta} \tfrac{9}{4} A_1^2 \cos^2 (\sigma t + \epsilon) r^2 \, dr \, d\theta$$

$$= \tfrac{3}{8} A_1^2 \rho h^3 [2\sqrt{3} + \log (2 + \sqrt{3})] \cos^2 (\sigma t + \epsilon),$$

$$V = \tfrac{1}{2} \rho g \int_{-h\sqrt{3}}^{h\sqrt{3}} \frac{\sigma^2 A_1^2}{g^2} \sin^2 (\sigma t + \epsilon) r^3 \cos^2 \tfrac{3}{2} \theta \, dy, \qquad (y = - h \tan \theta)$$

$$= \frac{1}{16} \frac{\rho A_1^2 \sigma^2}{g} h^4 [38 \sqrt{3} + 3 \log (2 + \sqrt{3})] \sin^2 (\sigma t + \epsilon).$$

Hence

$$\sigma^2 = \frac{g}{h} \cdot \frac{6 [2\sqrt{3} + \log (2 + \sqrt{3})]}{38 \sqrt{3} + 3 \log (2 + \sqrt{3})} = \frac{g}{h} \cdot (0 \cdot 4112)$$

$$\therefore \sigma = (0 \cdot 641) \sqrt{\frac{g}{h}}. \qquad (11)$$

The lowest symmetrical mode is, of course, $\sigma = \sqrt{g/h}$.

For a hyperbola with eccentsicity 2 we should take

$$\phi = \sum_1^\infty A_n \cosh \tfrac{3}{2}(2n-1)\xi \cos \tfrac{3}{2}(2n-1)\eta, \tag{12·1}$$

$$\psi = \sum_1^\infty A_n \sinh \tfrac{3}{2}(2n-1)\xi \sin \tfrac{3}{2}(2n-1)\eta, \tag{12·2}$$

where (ξ, η) are the elliptic co-ordinates, and the hyperbola is given by $\eta = \frac{2}{3}\pi$.

The first term may be taken as a constrained type to get an approximate value of the frequency of the gravest mode.

In the case of (b) we should take

$$\phi = \sum_1^\infty A_n I_{\frac{3}{2}(2n-1)}(kr) \cos \tfrac{3}{2}(2n-1)\theta, \tag{13}$$

$I_{\frac{3}{2}(2n-1)}$ which are Bessel's function of fractional order with an imaginary argument, can be expressed in finite terms. If we take the first term for the constrained type, we get

$$\phi = A_1 I_{\frac{3}{2}}(kr) \cos \tfrac{3}{2}(2-1)\theta$$

$$= \frac{A}{\sqrt{kr}}\left[\frac{\sinh kr}{kr} - \cosh kr\right] \cos \tfrac{3}{2}(2n-1)\theta. \tag{13·1}$$

To give an example from the vibrations of membranes we can take (c), and, in this case we easily see that the displacement should be taken as

$$z = \sum_1^\infty A_n J_{\frac{3}{2}(2n-1)}(kr) \sin \tfrac{3}{2}(2n-1)\theta, \tag{14}$$

where $k = \sigma/c$. $J_{\frac{3}{2}(2n-1)}(kr)$ can again be expressed in finite terms.

For the constrained mode we have

$$z = \frac{A_1}{\sqrt{kr}}\left[\frac{\sin kr}{kr} - \cos kr\right] \sin \tfrac{3}{2}\theta. \tag{14·1}$$