FINITE STRAIN IN A ROTATING SHAFT

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THE theory of finite strain in elastic problems has been developed on the hypothesis that we do not neglect the second order terms in the components of strain.[1,3] Some applications of it have been given in recent papers.[2,4–9] When applied to the case of a solid rotating shaft we get an exact solution of the stress equations of equilibrium, and the comparison of the results with those given by the small strain theory becomes quite interesting.

We treat the problem as one of plane strain, with an allowance for uniform longitudinal extension, α. Since the shaft is strained symmetrically we can take the components of displacement as

$$u = x(1-\beta),\ v = y(1-\beta),\ w = \alpha z, \tag{1}$$

where β is a function of $r = (x^2+y^2)^{\frac{1}{2}}$ only.

The stress components are given by

$$\widehat{xx} = \lambda\delta + \mu\left[(1-\beta^2) - x^2\left(\beta'^2 + \frac{2\beta\beta'}{r}\right)\right], \tag{2·1}$$

$$\widehat{yy} = \lambda\delta + \mu\left[(1-\beta^2) - y^2\left(\beta'^2 + \frac{2\beta\beta'}{r}\right)\right], \tag{2·2}$$

$$\widehat{zz} = \lambda\delta + \mu(2\alpha - \alpha^2), \tag{2·3}$$

$$\widehat{yz} = \widehat{zx} = 0, \tag{2·4}$$

$$\widehat{xy} = -\mu xy\left(\beta'^2 + \frac{2\beta\beta'}{r}\right), \tag{2·5}$$

where

$$\beta' = d\beta/dr, \text{ and}$$

$$\delta = 1 - \beta^2 - \tfrac{1}{2}(r^2\beta'^2 + 2r\beta\beta') + \alpha - \tfrac{1}{2}\alpha^2. \tag{3}$$

In polar co-ordinates these stresses are given by

$$\widehat{rr} = \lambda\delta + \mu[1 - (r\beta' + \beta)^2], \tag{4·1}$$

$$\widehat{\theta\theta} = \lambda\delta + \mu(1-\beta^2), \tag{4·2}$$

$$\widehat{r\theta} = \widehat{rz} = \widehat{\theta z} = 0, \tag{4·3}$$

with $\widehat{zz}$ given by (2·3).

The only stress-equation of equilibrium which is not identically satisfied is given by

$$\frac{\partial \widehat{rr}}{\partial r}+\frac{\partial \widehat{rz}}{\partial z}+\frac{\widehat{rr}-\widehat{\theta\theta}}{r}+\rho\, r\,\omega^2=0, \tag{5·1}$$

which, on substituting from (4), reduces to

$$\frac{\partial}{\partial r}\left[\beta^2+(r\beta'+\beta)^2+c\int r\beta'^2\,dr-\frac{c}{2\mu}\rho r^2\omega^2\right]=0,$$

where

$$c=\frac{2\mu}{\lambda+2\mu}=\frac{1-2\eta}{1-\eta},$$

and ω is the angular velocity of the shaft.

Since η, the Poisson's ratio, lies between 0 and $\frac{1}{2}$, c lies between 0 and 1.

The differential equation statisfied by β is therefore

$$\beta^2+(r\beta'+\beta)^2+c\int r\beta'^2\,dr-\frac{c}{2\mu}\rho r^2\,\omega^2=\mathrm{K}_1, \tag{5·2}$$

K_1 being a constant.

The general solution involving one arbitrary constant can be obtained in the form of an infinite series. In the present paper we propose to discuss only the particular solution obtained from (5·2) by putting $\mathrm{K}_1=0$. This is easily seen to be

$$\beta=\mathrm{A}r=\frac{c\,\rho\,\omega^2\,r}{\mu\,(10+c)}. \tag{6}$$

The radial displacement is therefore given by

$$\mathrm{U}=r\left(1-\frac{c\,\rho\omega^2\,r}{\mu\,(10+c)}\right). \tag{7}$$

The boundary condition over the curved surface is $\widehat{rr}=0$ over $r=a$, being the radius of the shaft in the strained condition. Using (4·1) we get

$$(3-2c)-(1-c)\,(1-\alpha)^2=a^2\mathrm{A}^2\,(5-c). \tag{8}$$

The boundary condition over the plane ends is $\widehat{zz}=0$ over $x=\pm l$, $2l$ being the length of the shaft. This cannot be exactly satisfied. But we can make the resultant longitudinal tension vanish over the plane ends. This requires

$$\int_0^a r\widehat{zz}\,dr=0, \tag{9·1}$$

which gives

$$(3-2c)-(1-\alpha)^2=\tfrac{5}{2}\,a^2\,\mathrm{A}^2\,(1-c). \tag{9·2}$$

We have now the three equations (6), (8), (9·2) to determine the two constants A and α. This implies an identical relation between ρ, a and ω. This is the natural consequence of the fact that (6) is only a particular solution, and not the general solution.

The constants α and A are given by

$$(1-\alpha)^2 = \frac{(3-2c)(5+3c)}{5+8c-5c^2}, \tag{10·1}$$

$$A^2 a^2 = \frac{2c(3-2c)}{5+8c-5c^2}, \tag{10·2}$$

and the identical relation is given by

$$\rho\, a^2 \omega^2 = \frac{2\mu(3-2c)(10+c)}{5+8c-5c^2}. \tag{10·3}$$

The non-vanishing stresses are now given by

$$\widehat{rr} = \frac{5-c}{10+c}\rho\,\omega^2 (a^2-r^2), \tag{11·1}$$

$$\widehat{\theta\theta} = \frac{\rho\,\omega^2}{10+c}\left[a^2(5-c) - r^2(5-4c)\right], \tag{11·2}$$

$$\widehat{zz} = \frac{5\rho\omega^2(1-c)}{2(10+c)}(a^2-2r^2). \tag{11·3}$$

If $\widehat{rr}_0$, $\widehat{\theta\theta}_0$, $\widehat{zz}_0$, denote the corresponding stresses of the small strain theory, we have[10]

$$\widehat{rr}_0 = \tfrac{1}{8}(4-c)\rho\omega^2(a^2-r^2), \tag{12·1}$$

$$\widehat{\theta\theta}_0 = \tfrac{1}{8}\rho\omega^2\left[a^2(4-c) - r^2(4-3c)\right] \tag{12·2}$$

$$\widehat{zz}_0 = \tfrac{1}{4}(1-c)\rho\omega^2(a^2-2r^2). \tag{12·3}$$

If α_0 is the corresponding value of α_1 we have for ω given by (10·3),

$$\alpha_0 = -\frac{1}{2}\cdot\frac{(1-c)(10+c)}{5+8c-5c^2}. \tag{13}$$

Instead of making the resultant longitudinal tension vanish we can suppose that the tension is adjusted on the plane ends so that the length is maintained constant. In such a case $\alpha = 0$, and the identical relation (10·3) takes the form

$$\rho a^2\omega^2 = \frac{\mu(2-c)(10+c)}{c(5-c)}. \tag{14}$$

$\widehat{rr}$, $\widehat{\theta\theta}$ retain the form given in (11), but $\widehat{zz}$ and $\widehat{zz}_0$ become

$$\widehat{zz} = \frac{\rho\omega^2(1-c)}{(2-c)(10+c)}\left[2(5-c)a^2 - 5(2-c)r^2\right], \tag{15·1}$$

$$\widehat{zz}_0 = \frac{\rho\omega^2(1-c)}{4(2-c)}\left[(4-c)a^2 - 2(2-c)r^2\right]. \tag{15·2}$$

If U_0 is the radial displacement of the small strain theory, we have[10]

$$U_0 = \frac{\rho\omega^2 r}{16\mu(3-2c)}\left[(2-c)(1+2c)\,a^2 - c(3-2c)\,r^2\right]. \tag{16}$$

If r_1 and r_{10} are the values of r for which U and U_0 vanish, we have

$$\frac{r_1}{r_{10}} = \left[\frac{(5+8c-5c^2}{2(2-c)(1+2c)}\right]^{\frac{1}{2}}. \tag{17}$$

Since $0 < c < 1$, this ratio lies between $1\cdot118$ and $1\cdot155$. For $\eta = \frac{1}{4}$ $(c = \frac{2}{3})$, the difference can be as much as 12 per cent.

In like manner we find that (α/α_0) lies betewen 1 and $0\cdot732$. For $\eta = \frac{1}{4}$, the difference between the two values is about 10 per cent.

For comparing the stresses we find

$$\frac{\widehat{rr}}{\widehat{rr}_0} = \frac{8(5-c)}{(10+c)(4-c)}, \tag{18·1}$$

$$\frac{\widehat{zz}}{\widehat{zz}_0} = \frac{10}{10+c}. \tag{18·2}$$

The ratio in the first case varies between 1 and $0\cdot97$, and in the second case between 1 and $0\cdot91$. For the cross-radial stress we find that $(\widehat{\theta\theta}/\widehat{\theta\theta}_0)$ lies between $1\cdot2$ and $0\cdot97$.

It appears that, though the stresses given by the two theories are similar in form, the radial displacements have not the same character. Also, the mistake involved in using the small strain theory can be as much as 10 to 12 per cent.

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