BENDING OF AN EQUILATERAL PLATE

By B. R. Seth

(From the Department of Mathematics, Hindu College, Delhi)

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1. One of the most important of the special problems of Elasticity is the bending of plates. It has received incessant attention from many quarters.[1] The form of boundaries which have been considered are a circle, an ellipse, a rectangle, half an ellipse bounded by the transverse axis and a sector of a circle. The solutions for a circle and an ellipse are simple in character. The literature available for a rectangle can form the subject-matter of a good treatise. It will therefore be not devoid of interest to give an exact solution for an equilateral plate.

2. We assume that the plate has its four edges supported and is bent by uniform pressure applied to one face. Let (u, v, w) be the displacement of any point of the middle plane. We take the face subjected to pressure to be $z = -h$. Let p be the load per unit area and $D = \frac{2}{3} Eh^3/(1-\sigma^2)$, the flexural rigidity of the plate, E, h, σ having their usual meanings. Then, we know that w at any point $(x, y, 0)$ satisfies the differential equation[2]

$$D\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2 w = D\nabla_1^4 w = p. \tag{1}$$

The boundary conditions can be put in the form

$$w = o,\ \nabla_1^2 w = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) w = 0, \tag{2}$$

over all the three edges.

We can put (1) in the form

$$\nabla_1^2\left[\nabla_1^2 w - \frac{1}{4}\frac{p}{D}(x^2+y^2)\right] = 0. \tag{3}$$

If therefore we put

$$\psi = \nabla_1^2 w - \frac{1}{4}\frac{p}{D}(x^2+y^2), \tag{4}$$

we see that ψ satisfies $\nabla_1^2\psi = 0$ and assumes the value $-\frac{1}{4}\frac{p}{D}(x^2+y^2)$ over the boundary. The problem is now the same as that of the torsion of an equilateral prism, whose solution is known.

3. Let the edges of the plate be given by $y=a$, $y=\pm\sqrt{3}x$, then we know that

$$\psi=\tfrac{1}{2}\frac{p}{\mathrm{D}}(x^2-y^2)+\frac{1}{4a}\frac{p}{\mathrm{D}}(y^3-3x^2y), \tag{4·1}$$

so that

$$\nabla_1^2 w=\tfrac{1}{4}\frac{p}{\mathrm{D}}(x^2+y^2)+\tfrac{1}{2}\frac{p}{\mathrm{D}}(x^2-y^2)+\frac{1}{4a}\frac{p}{\mathrm{D}}(y^3-3x^2y)$$

$$=\tfrac{1}{4}\frac{p}{\mathrm{D}a}(y-a)(y^2-3x^2). \tag{5}$$

If ϕ is harmonic, then

$$\nabla_1^2(x\phi)=2\frac{\partial\phi}{\partial x},\quad \nabla_1^2(y\phi)=2\frac{\partial\phi}{\partial y}.$$

We can now re-write (5) as

$$\nabla_1^2\left[w-\frac{1}{8}\frac{p}{\mathrm{D}}x^2y^2-\frac{1}{12}\frac{p}{\mathrm{D}}y(3x^2y-y^3)+\frac{1}{8a}\frac{p}{\mathrm{D}}x(xy^3-x^3y)\right]=0, \tag{6}$$

which shows that

$$w=\frac{1}{8}\frac{p}{\mathrm{D}}x^2y^2+\frac{1}{12}\frac{p}{\mathrm{D}}y(3x^2y-y^3)+\frac{1}{8a}\frac{p}{\mathrm{D}}x(xy^3-x^3y)+w_0, \tag{6·1}$$

when w_0 is harmonic. Since w is to vanish over $y=a$ and $y=\pm\sqrt{3}x$, we put

$$w_0=\mathrm{A}_1(3x^2y-y^3)+\mathrm{A}_2(x^4-6x^2y^2+y^4)$$
$$+\mathrm{A}_3(5x^4y-10x^2y^3+y^5). \tag{6·2}$$

The boundary condition $w=0$ over the sides is found to be satisfied if we take

$$\mathrm{A}_1=-\frac{1}{48}\frac{pa}{\mathrm{D}},\quad \mathrm{A}_2=\frac{3}{64}\frac{p}{\mathrm{D}},\quad \mathrm{A}_3=\frac{1}{64}\frac{p}{\mathrm{D}a}.$$

The value of w is therefore

$$w=\frac{1}{8}\frac{p}{\mathrm{D}}x^2y^2+\frac{1}{12}\frac{p}{\mathrm{D}}y(3x^2y^2-y^3)+\frac{1}{8a}\frac{p}{\mathrm{D}}x(xy^3-x^3y)$$
$$+\frac{3}{64}\frac{p}{\mathrm{D}}(x^4-6x^2y^2+y^4)+\frac{1}{64}\frac{p}{\mathrm{D}a}(5x^4y-10x^2y^3+y^5)$$
$$-\frac{1}{48}\frac{pa}{\mathrm{D}}(3x^2y-y^3)$$
$$=\frac{1}{192}\frac{p}{\mathrm{D}a}(a-y)(3x^2-y^2)(3x^2+3y^2-4ay). \tag{7}$$

From (7) we see that w vanishes over all the three sides and also on the circle

$$x^2+\left(y-\frac{2a}{3}\right)^2=\frac{4a^2}{9},$$

which is the circumcircle of the triangle. w therefore does not vanish at any other point of the plate except at the edges.

If w_c denotes the deflection at the centre of the plate given by $x=0$, $y=\frac{2}{3}a$, we find

$$w_c=\frac{1}{4\cdot 3^5}\frac{pa^4}{D} \tag{8}$$

To get the extreme value of w we put $\frac{\partial w}{\partial x}=0$, $\frac{\partial w}{\partial y}=0$. The first condition gives

$$x=0, \text{ or } y=a, \text{ or } 3x^2+y^2-2ay=0, \tag{9·1}$$

which combined with the second gives the following points:

$$\left(\pm\frac{4a}{5\sqrt{3}}, \frac{2a}{5}\right), (0, 0), \left(\pm\frac{a}{\sqrt{3}}, a\right), \left(0, \frac{2a}{3}\right), \left(0, \frac{2a}{3}\right).$$

The only admissible value, besides (0, 0), $(\pm a/\sqrt{3}, a)$, which are the vertices of the plate, is $(0, \frac{2}{3}a)$. The maximum value of w therefore occurs at the centre of the plate, and hence if w_m denotes this value,

$$w_m=w_c=\frac{1}{4\cdot 3^5}\cdot\frac{pa^4}{D}. \tag{9}$$

For a circular plate of radius c the corresponding value of w is[3]

$$w=\frac{1}{64}\frac{p}{D}(c^2-r^2)\left(\frac{5+\sigma}{1+\sigma}c^2-r^2\right), \tag{10}$$

and hence

$$w_{c1}=w_{m1}=\frac{1}{64}\frac{p}{D}\cdot\frac{5+\sigma}{1+\sigma}c^4. \tag{10·1}$$

If, therefore, we take the in-circle of the plate for which $c=\frac{1}{3}a$, we find

$$\frac{w_c}{w_{c1}}=\frac{w_m}{w_{m1}}=\frac{16(1+\sigma)}{3(5+\sigma)}, \tag{10·2}$$

which for $0<\sigma<\frac{1}{2}$ varies between 1·07 and 1·46.

4. For a thin plate the flexural couple G is given by

$$G=-D\left\{\frac{\partial^2 w}{\partial \nu^2}+\sigma\frac{\partial^2 w}{\partial s^2}\right\} \tag{11}$$

where s is measured along any curve drawn on the middle surface and ν is the direction of the normal to this curve. Let s be measured along

$y=$ const., then (11) becomes

$$G=-D\left[\nabla_1^2 w-(1-\sigma)\frac{\partial^2 w}{\partial x^2}\right]. \tag{11·1}$$

We find

$$\frac{\partial^2 w}{\partial x^2}=\frac{1}{16}\frac{p}{Da}(a-y)(9x^2+y^2-2ay), \tag{12}$$

and $\nabla_1^2 w$ is given by (6). Hence

$$G=-\frac{1}{16}\frac{p}{a}(a-y)[4(3x^2-y^2)-(1-\sigma)(9x^2+y^2-2ay] \tag{13}$$

G vanishes over $y=a$, as it should.

For a thick plate a term of the type

$$h^2\frac{\partial^2}{\partial x^2}(\nabla_1^2 w)=\frac{3}{2}h^2\frac{p}{Da}(a-y), \tag{13·1}$$

should be added to (13). This also vanishes for $y=a$. Hence G vanishes whether the plate is thick or thin. Since there is symmetry G vanishes over the remaining two edges as well.

For extreme values of G we get

$$(a-y)[24x-18x(1-\sigma)]=0,$$

$$i.e., \; y=a, \; x=0,$$

and

$$4(3x^2-y^2)-(1-\sigma)(9x^2+y^2-2ay)$$
$$+(a-y)[8y+(1-\sigma)(2y-2a)]=0.$$

The possible points are given by

$$\left[\pm\sqrt{\frac{3+\sigma}{3(1+3\sigma)}}\,a,\; a\right],\; \left[0,\; \frac{a(7-3\sigma)\pm a\sqrt{19-6\sigma+3\sigma^2}}{3(5-\sigma)}\right].$$

The only admissible value is

$$0,\; \frac{a(7-3\sigma)+a\sqrt{19-6\sigma+3\sigma^2}}{3(5-\sigma)}, \tag{14}$$

which for $0<\sigma<\frac{1}{2}$ varies between $(0, 0\cdot72a)$ and $(0, 0\cdot76a)$. Thus the points of greatest weakness are on the medians and are very near the centre of the plate. For $\sigma=\frac{1}{3}$ the maximum value of G is found to be approximately $\frac{1}{40}pa^2$.

The other stress couple H is

$$H=D(1-\sigma)\frac{\partial}{\partial r}\left(\frac{\partial w}{\partial s}\right), \tag{15}$$

and for $y=$ constant, it is

$$H=D(1-\sigma)\frac{\partial^2 w}{\partial x\,\partial y}=-\frac{1}{16}\frac{p}{a}(1-\sigma)\,x\,[3(x^2+y^2)-4ay+2a^2],$$

which shews that it vanishes along all the medians.

The shearing force N is

$$N=-D\frac{\partial}{\partial\nu}(\nabla_1^2 w), \tag{16}$$

and for $y=$ constant, it is

$$N=-D\frac{\partial}{\partial y}(\nabla_1^2 w)=\frac{1}{4}\frac{p}{a}(3x^2-3y^2+2ay). \tag{16·1}$$

This vanishes at all the angular points. For a thick plate N remains as given in (16·1). But a term of the type in (16·1) has to be added to H given in (15·1).

w being known, the displacements (u, v) can now be easily obtained.

Other cases of supported and clamped plates will be discussed in another paper.

5. Summary

An exact solution is obtained for an equilateral plate supported at its edges and bent by uniform pressure. It is found that the maximum deflection occurs at the centre and that its value is $pa^4/4\cdot 3^5 D$. The points of greatest weakness are found to be very near the centre, the maximum value of the flexural couple G for $\sigma=\frac{1}{3}$ being approximately $\frac{1}{40}pa^2$.

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