SOME SOLUTIONS OF THE WAVE EQUATION

By B. R. Seth, F.A.Sc.
(From the Department of Mathematics, Iowa State College, U.S.A., and Hindu College, Delhi)

Received October 5, 1950

1. For rectilinear boundaries a number of solutions of the two-dimensional wave equation

$$\nabla^2\phi = c^2 \frac{\partial^2\phi}{\partial t^2} \tag{1}$$

are known, in which the boundary condition satisfied by ϕ is

$$\phi = 0, \tag{2.1}$$

or

$$\frac{\partial\phi}{\partial n} = 0, \tag{2.2}$$

∂n denoting a normal element drawn to the boundary. In the transverse vibrations of membranes the condition (2.1) amounts to a fixed edge, and in Hydro-dynamics of inviscid liquids (2.2) is to be satisfied over a fixed boundary. Some such solutions satisfying (2.1) have been given by B. R. Seth.[1] For hexagonal and equilateral boundaries* incomplete solutions have been obtained by D. G. Christopherson[2] and B. B. Sen.[3] Christopherson's solution has been completed by P. N. Sharma.[4]

2. When the boundary condition is nonhomogeneous solutions of the wave equation for rectilinear boundaries have not received much attention, though formal solutions can be obtained through the Green's function. In a recent investigation of some problems in the bending of elastic plates it was required to determine solutions of (1) satisfying the boundary condition

$$\phi = k, \text{ a constant.} \tag{2.3}$$

It is proposed to give some of these solutions. It is found that for equilateral, some rhombus and pentagonal boundaries the solution can be obtained in a finite number of trigonometrical terms.

3. Assuming that $\theta \propto \cos(pt/c)$ we can rewrite (1) as

$$\nabla^2\phi + p^2\phi = 0. \tag{3.1}$$

* The complete solution for an equilateral boundary was originally obtained by Lamé in *Leçons sur l'Elasticite.*

A typical solution of (3·1) is

$$\phi = {}^{\sin}_{\cos}(p\sqrt{\cos\theta})\,{}^{\sin}_{\cos}(py\sin\phi). \tag{3·2}$$

It may be noted that when applied to the vibrations of a membrane the condition (2.3) means that its edge is vibrating harmonically.

4. For a rectangular section with sides $x = \pm a$, $y = \pm b$, we see that $k\cos px \sec pa$ satisfies the boundary condition (2·3) over $x = \pm a$. Using the Fourier's expansion for $\cos px$ which vanishes for $x = \pm a$, we get

$$\phi = k\left[\cos px \sec pa - \sum_0^\infty A_n \sec b\sqrt{p^2 - \alpha_n^2}\cos\alpha_n x \cos\sqrt{p^2-\alpha_n^2}\,y\right], \tag{4}$$

where $\alpha_n = (2n+1)\pi/2a$,

$$A_n = \frac{2(-1)^{n+1}kp^2}{a\alpha_n(p^2-\alpha_n^2)}.$$

For a right-angled isosceles triangular section with sides $x = a$, $y = a$, $y + x = 0$, we see that the functions to be used for the individual sides can be taken to be of the form $\sin px$, $\sin py$ and $\cos\frac{p}{\sqrt{2}}(x+y)$, respectively.

On account of the symmetry which exists about the line $y = x$ we now take two Fourier series which are such that their sum vanishes over $y = -x$ and in which terms can be obtained from one another by interchanging x and y. Thus we get

$$\begin{aligned}\phi = {} & k\cos\frac{p}{\sqrt{2}}(x+y) + \tfrac{1}{2}k(1-\cos pa\sqrt{2})\left[\frac{\sin px}{\sin pa} + \frac{\sin py}{\sin pa}\right] \\ & - \sum_0^\infty A_n \operatorname{cosec} a\sqrt{p^2-\alpha_n^2}\left[\cos\alpha_n x \sin\sqrt{p^2-\alpha_n^2}\,y + \cos\alpha_n y \sin\sqrt{p^2-\alpha_n^2}\,x\right] \\ & - \sum_0^\infty B_n \sec a\sqrt{p^2-\beta_n^2}\left[\sin\beta_n x \cos\sqrt{p^2-\beta_n^2}\,y + \sin\alpha_n y \cos\sqrt{p^2-\beta_n^2}\,x\right]\end{aligned} \tag{5}$$

where $\alpha_n = (2n+1)\pi/2a$, $\beta_n = n\pi/a$, and

$$A_n = \frac{(-1)^{n+1}kp^2\cos^2\frac{1}{2}pa\sqrt{2}}{a\alpha_n(\frac{1}{2}p^2-\alpha_n^2)},\quad B_n = \frac{(-1)^{n+1}k\beta_n p^2\sin^2\frac{1}{2}pa\sqrt{2}}{a(\frac{1}{2}p^2-\beta_n^2)(p^2-\beta_n^2)}.$$

For an equilateral section with sides $y = a$, $y = \pm x\sqrt{3}$, we should take functions of the type $\cos py$, $\sin py$, $\cos\frac{1}{2}px\sqrt{3}\cos\frac{1}{2}py$ and $\cos\frac{1}{2}px\sqrt{3}\sin\frac{1}{2}py$. It is found that no Fourier expansions are required, and we get

$$\begin{aligned}\phi = {} & k\left[2\cos\tfrac{1}{2}py\cos\tfrac{1}{2}px\sqrt{3} - \cos py\right] \\ & - k\cot\tfrac{1}{2}pa\left[2\sin\tfrac{1}{2}py\cos\tfrac{1}{2}px\sqrt{3} - \sin py\right].\end{aligned} \tag{6}$$