VISCOUS SOLUTIONS OBTAINED BY SUPERPOSITION OF EFFECTS

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WHEN a solid moves slowly through a viscous liquid, so that the inertia terms can be neglected in the equations of motion, the solution of the corresponding problem may be obtained by the method of superposition of two solutions. One of these is the irrotational solution obtained by assuming the solid to be moving through a non-viscous liquid with a particular velocity, and the other is that due to a concentrated force applied in the direction of motion of the solid in an infinite viscous liquid. As is to be expected, the concentrated force in each case is the drag suffered by the solid. We propose to illustrate this method by discussing the cases of a sphere and a circular cylinder.

Motion of a Sphere

For a sphere of radius a moving with a velocity U in the direction of the x-axis the component velocities for the irrotational solution are given by

$$u_1 = -\frac{1}{2}\,\mathrm{U}\left(\frac{a}{r}\right)^3 + \frac{3}{2}\,\mathrm{U}\,\frac{x^2}{a^2}\left(\frac{a}{r}\right)^5, \qquad (1\cdot1)$$

$$v_1 = \frac{3}{2}\,\mathrm{U}\,\frac{xy}{a^2}\left(\frac{a}{r}\right)^5, \qquad (1\cdot2)$$

$$w_1 = \frac{3}{2}\,\mathrm{U}\,\frac{xz}{a^2}\left(\frac{a}{r}\right)^5. \qquad (1\cdot3)$$

We also know that for a force P acting at the origin along the x-axis in an infinite viscous liquid the component velocities are given by[1]

$$u_2 = \frac{\mathrm{P}}{8\pi\mu}\left(\frac{x^2}{r^3} + \frac{1}{r}\right), \qquad (2\cdot1)$$

$$v_2 = \frac{\mathrm{P}}{8\pi\mu}\cdot\frac{xy}{r^3}, \qquad (2\cdot2)$$

$$w_2 = \frac{\mathrm{P}}{8\pi\mu}\cdot\frac{xz}{r^3}, \qquad (2\cdot3)$$

μ being the coefficient of viscosity.

Let

$$u = Au_1 + u_2, \qquad (3\cdot1)$$

$$v = Av_1 + v_2, \qquad (3\cdot2)$$

$$w = Aw_1 + w_2, \qquad (3\cdot3)$$

where A is a constant to be determined from the boundary conditions. If u, v, w are to give the required solution, we should have $u = U$, $v = 0$, $w = 0$ over $r = a$. These conditions give $A = -\frac{1}{2}$, $P = 6\mu\pi a U$. P is obviously the value of the drag given by Stokes's solution. We also see that the velocity of the solid in the irrotational motion is $-\frac{1}{2}U$.

Motion of a Circular Cylinder

The corresponding results for a circular cylinder are given below.

$$u_1 = -\frac{Ua^2}{r^2} + \frac{2\,Ua^2x^2}{r^4}, \qquad (4\cdot1)$$

$$v_1 = \frac{2\,Ua^2xy}{r^4}, \qquad (4\cdot2)$$

$$u_2 = -\frac{P}{4\pi\mu}\left(\log r - \frac{x^2}{r^2}\right), \qquad (6\cdot1)$$

$$v_2 = \frac{Pxy}{4\pi\mu r^2}. \qquad (6\cdot2)$$

The solution given by (6) is defective as it makes the velocity logarithmically infinite at infinity.[1] Putting

$$u = Au_1 + u_2, \qquad (7\cdot1)$$

$$v = Av_1 + v_2, \qquad (7\cdot2)$$

we find

$$A = \frac{1}{2\log a - 1}, \qquad (8\cdot1)$$

$$P = \frac{4\pi\mu U}{\frac{1}{2} - \log a}. \qquad (8\cdot2)$$

This solution has been obtained by Berry and Swain.[2] The velocity in the irrotational motion is $-U/(1-2\log a)$, while the drag suffered

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[1] Lamb, *Hydrodynamics*, 5th ed. (1930), pp. 579–81.

[2] Berry and Swain, *Proc. Roy. Soc.* (A), 1923, **102**, 770 ; also Seth, *Phil. Mag.*, 1939, ser. 7, **27**, 218.

by the cylinder is given by (8·2). As has been already pointed out, the solution is defective at infinity.

The values of the component velocities given by (2) and (6) are the displacements produced in an infinite elastic body by applying a force at the origin in the direction of the x-axis.[3] This analogy suggests the following known results.

(i) The solutions given by (3) and (7) hold good only for small values of the deformation velocities.

(ii) (3) and (7) give sufficiently correct results near the boundary of the body. Near the surface of the sphere Stokes's solution is known to give a better approximation than the one given by Oseen.[4]

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[3] Love, *Theory of Elasticity*, 4th. ed. (1927), **185**, 209.

[4] Lamb, *Hydrodynamics*, 5th. ed. 1930, p. 579.