TRANSVERSE VIBRATIONS OF RECTILINEAR PLATES

[illegible]

[illegible]

[illegible]

The [illegible] plates [illegible] very fully by [illegible] plate has been considered by [illegible] rectangular plate [illegible] of the square plate [illegible] of the square [illegible] by [illegible] and [illegible] of a square [illegible] the length of a side [illegible] D the flexural rigidity [illegible] Young's [illegible] and σ the Poisson's ratio.

If the plate [illegible] and the form W [illegible] we know that W [illegible] satisfying the equation[illegible]

$$[illegible] \qquad (1)$$

[illegible] are to be determined by adapting the solution [illegible] the boundary conditions.

[illegible] edge [illegible] along the normal and [illegible] a fixed [illegible] at the point under consideration, we know that the [illegible] edge are[illegible]

$$[illegible] \qquad (2.1)$$

$$[illegible] \qquad (2.2)$$

[illegible] has shown that these conditions [illegible] system of curvilinear orthogonal coordinates.

At a clamped edge the boundary conditions are

At a supported edge these conditions become

$$W=0, \sigma \nabla^2 W=\sigma\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) W=(\sigma-1) \frac{\partial^2 W}{\partial n^2} \cdot \qquad (2.$$

For rectilinear plates the distinction between s and μ disappears.

Supported Edges

To begin with we take the case of a rectilinear plate whose edges are a supported. Rayleigh[2] has given the solution for a rectangular plate. can be shewn that the problem of the rectilinear plate can be reduced to th of a rectilinear membrane.

For the transverse vibrations of a membrane the differential equatic to be satisfied by W is

$$\nabla^2 W+k^2 W=0 \qquad ($$

with the condition that $W=0$ over the boundary. Operating (3) by ∇ we get

$$\nabla^4 W=k^4 W \qquad ($$

which is the same as (1). Any solution of (3), therefore, also satisfies (1 Over the boundary $W=0$, and hence from (3) $\nabla^2 W=0$. We can now shc that (2.4) also reduces to this condition.

As $W=0$ over the boundary, we have, differentiating along the boundar $\partial W/\partial s=0$, $\partial^2 W/\partial s^2=0$. Also

$$\nabla^2 W=\frac{\partial^2 W}{\partial s^2}+\frac{1}{\rho} \frac{\partial W}{\partial n}+\frac{\partial^2 W}{\partial n^2},$$

which for a rectilinear boundary ($\rho=\infty$) reduces to

$$\nabla^2 W=\frac{\partial^2 W}{\partial s^2}+\frac{\partial^2 W}{\partial n^2}=\frac{\partial^2 W}{\partial n^2}$$

Thus (2.4) also becomes $\nabla^2 W=0$. For a simply supported rectiline boundary the boundary conditions can therefore be put in the form $W=$ $\nabla^2 W=0$.[9, 10]

Hence, if we know the solution for a membrane, the correspondi solution for a plate can be easily written down. In fact the form of W giv by Rayleigh for a rectangular plate is the same as that for a membrane.

Triangular Plates

We have already discussed in another paper[11] the transverse vibratio of triangular membranes. For triangular plates it is therefore quite sufficie to give the main results.

[illegible] plate. The displacement W is given by

$$W = \cos\frac{(2m+1)\pi x}{2a}\cos\frac{(2n+1)\pi y}{2a} - \cos\frac{(2n+1)\pi x}{2a}\cos\frac{(2m+1)\pi y}{2a} \quad (5)$$

[illegible] m [illegible]. The frequency [illegible]

$$p = \frac{1}{4}\frac{\pi}{a^2}[(2m+1)^2 + (2n+1)^2]\sqrt{\frac{Eh^2}{3\rho(1-\sigma^2)}}$$

$$= \frac{1}{4}\frac{\pi^2}{a^2}\left[(2m+1)^2 + (2n+1)^2\sqrt{\frac{D}{m_0}}\right] \quad (6)$$

The gravest tone is given by

$$\left.\begin{aligned} p_0 &= \frac{5\pi^2}{2a^2}\sqrt{\frac{D}{m_0}} \\ W_0 &= \cos\frac{\pi x}{2a}\cos\frac{3\pi y}{2a} - \cos\frac{3\pi x}{2a}\cos\frac{\pi y}{2a} \end{aligned}\right\} \quad (7)$$

These plates give the antisymmetrical vibrations. The symmetrical are given by

$$W = \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a} - \sin\frac{n\pi x}{a}\sin\frac{m\pi y}{a} \quad (8)$$

Equilateral plate. The corresponding results are given by

$$W = 2\sin\frac{(m-n)\pi x}{a}\cos\frac{(m+n)\pi y\sqrt{3}}{a}$$

$$+ 2\sin\frac{(2m+n)\pi x}{a}\cos\frac{n\pi y\sqrt{3}}{a} - 2\sin\frac{(2n+m)\pi x}{a}\cos\frac{m\pi y\sqrt{3}}{a}, \quad (9.1)$$

$$p = \frac{16\pi^2}{3a^2}[m^2 + mn + n^2]\sqrt{\frac{D}{m_0}} \quad (9.2)$$

$$p_0 = \frac{16\pi^2}{3a^2}\sqrt{\frac{D}{m_0}} \quad (9.3)$$

$$W_0 = 2\sin\frac{2\pi x}{a} - 4\sin\frac{\pi x}{a}\cos\frac{\pi y\sqrt{3}}{a}, \quad (9.4)$$

the sides of the plates being $x = a$, $y = \pm x\sqrt{3}$.

Isosceles triangular plate containing an angle of 120°. In this case the results are

$$W = 2\sin\frac{(m-n)\pi x}{a}\sin\frac{(m+n+1)\pi y\sqrt{3}}{a}$$

$$- 2\cos\left(2m+1+\frac{2n+1}{2}\right)\frac{\pi x}{a}\cos\frac{(2n+1)\pi y\sqrt{3}}{2a}$$

$$+ 2\cos\left(2n+1-\frac{2m+1}{2}\right)\frac{\pi x}{a}\cos\frac{(2m+1)\pi y\sqrt{3}}{2a}, \quad (10.1)$$

$$\frac{p}{\pi}=\frac{\pi}{a^2}\left[(2m+1)^2+(2m+1)(2n+1)+(2n+1)^2\right]\sqrt{\frac{D}{m_0}}, \quad (10.2)$$

$$\frac{p_0}{\pi}=\frac{7}{2}\frac{\pi}{a^2}\sqrt{\frac{D}{m_0}}, \quad (10.3)$$

$$W_0=2\sin\frac{2\pi x}{a}\sin\frac{\pi y\sqrt{3}}{a}-2\cos\frac{5\pi x}{2a}\cos\frac{\pi y\sqrt{3}}{2a}$$

$$+2\cos\frac{\pi x}{2a}\cos\frac{3\pi y\sqrt{3}}{2a} \quad (10.4)$$

In this case the modes are all symmetrical. The sides are given by $x=a$, $y=x\sqrt{3}$, $y+x\sqrt{3}=2a/\sqrt{3}$.

Right-angled triangle containing an angle of 60°.—If the sides are taken as $x=a$, $y=a/\sqrt{3}$, $y=x\sqrt{3}$, the results given in (10) hold good in this case as well.

RHOMBUS AND A REGULAR HEXAGON

If we take the sides of the rhombus, which contains an angle of 120°, as $x=0$, $x=a$, $y=x/\sqrt{3}$, $y=x/\sqrt{3}+2a/\sqrt{3}$, and those of the regular hexagon as $x=\pm a$, $y=\pm x/\sqrt{3}\pm 2a/\sqrt{3}$, the solution in (9) holds good.[12]

In all the above cases we find that the frequency is proportional to $\frac{1}{a^2}\sqrt{\frac{D}{m_0}}$.

CLAMPED EDGES

Square-plate.—If we take the sides of the plate as $y=\pm x\pm 2a$, we find that the solutions given in (6) and (7) also satisfy the condition $\partial W/\partial n=0$ over the boundary. Thus the conditions $W=0$, $\partial W/\partial n=0$ for a clamped edge are satisfied on all the sides of the plate. It is found that these conditions are also satisfied on the lines $y=\pm x$. Hence (6) and (7) give the symmetrical vibrations.

If we use this solution for a right-angled isosceles plate we find that the conditions for a clamped edge are satisfied on the equal sides $y=\pm x$, but that on the edge $x=a$ the conditions for a supported edge are only satisfied. Hence it may be used when the equal sides of the plate are clamped and the base is supported.

Free vibrations of a square-plate.—If the edges are free the boundary condition (2.1) must also be satisfied. Since $\nabla^2 W=-k^2 W$ and

$$\frac{\partial^3 W}{\partial n\, \partial s^2} = \frac{1}{2}\frac{\partial}{\partial n}\left(\frac{\partial}{\partial x} \pm \frac{\partial}{\partial y}\right)^2 W = \frac{1}{2}\frac{\partial}{\partial n}\left[\nabla^2 W \pm 2\frac{\partial^2 W}{\partial x\, \partial y}\right]$$

$$= \frac{1}{2}\frac{\partial}{\partial n}\left[-k^2 W \pm \frac{\pi^2}{2a^2}(2m+1)(2n+1) W\right],$$

$$= \frac{1}{2}\left[-k^2 \pm \frac{\pi^2}{2a^2}(2m+1)(2n+1)\right]\frac{\partial W}{\partial n}.$$

We see that (2.1) is also satisfied on all the sides of the plate. The solution in (6) and (7) can therefore also be used for the free symmetrical vibrations of a square plate.

As in the case of clamped edges, this solution can be used for a right-angled isosceles triangle whose equal sides are free but whose base is supported.

SUMMARY

The problem of the vibrations of a rectilinear plate with supported edges can be reduced to the corresponding problem of a vibrating membrane. Exact solutions are given for a number of triangular plates. The free and clamped vibrations of a square and a right-angled isosceles triangular plate have also been discussed.

REFERENCES

1. Kirchhoff .. *Journal für Math.* (Crelle), 1850, **40**.
2. Rayleigh .. *Theory of Sound*, 1894, Vol. I, Chap. X.
3. Mathieu .. *Journal de Math.* (Liouville), 1869, **14**, (Sér. 2).
4. Barthélémy .. *Toulouse Mém. de l' Acad.*, 1877, **9**.
5. Mary D. Waller .. *Proc. Phy. Soc.*, 1939, **51**, 831 ; 1941, **53**, 35.
6. Rayleigh .. *Phil. Mag.*, 1911, **22**, 225.
7. Ritz .. *Ann. Phys.*, Leipzig, 1909, **28**, 737.
8. Timoshenko .. *Vibration Problems in Engineering*, 1937, 2nd ed., § 70.
9. Stevenson .. *Phil. Mag.*, 1943, **34**, 110.
10. Timoshenko .. *Theory of Plates and Shells*, 1940, 100.
11. Seth .. *Proc. Ind. Acad. Sci.* (A), 1940, **12**, 487.
12. ——— .. *Ibid.*, 1941, 13, 390.