TRANSVERSE VIBRATIONS OF RECTILINEAR PLATES

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The transverse vibration of a thin plate was first investigated by Lord Kelvin and Lord Rayleigh. It was suggested by Lord Kelvin that the plate be considered as a fluid mass, and the vibrations of the plate are then described by the wave equation. The plate is then treated as an elastic solid. This method of treating the plate was further developed by Lord Rayleigh and Lord Kelvin.

The wave equation is given by:

\[ \frac{\partial^2 u}{\partial t^2} - \nabla^2 u = 0 \]

where \( u \) is the displacement, \( t \) is time, and \( \nabla^2 \) is the Laplacian operator.

If the plate is clamped along one edge, we have the boundary condition that the displacement is zero on that edge. If the plate is fixed along one edge, the displacement and its first derivative are zero on that edge. These conditions are:

\[ u = 0 \quad \text{at } y = 0 \]

\[ \frac{\partial u}{\partial y} = 0 \quad \text{at } y = 0 \]

These conditions are illustrated in Figure 1.

At a corner of the plate, the boundary conditions are:

\[ u = 0 \quad \text{at } y = 0 \]

\[ \frac{\partial u}{\partial y} = 0 \quad \text{at } y = 0 \]
At a supported edge these conditions become

\[ W = 0, \sigma \nabla^2 W = \sigma \left( \frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} \right) W = (\sigma - 1) \frac{\partial^2 W}{\partial n^2} \]  \tag{2.4}

For rectilinear plates the distinction between \( s \) and \( \mu \) disappears.

**Supported Edges**

To begin with we take the case of a rectilinear plate whose edges are supported. Rayleigh\(^2\) has given the solution for a rectangular plate. It can be shown that the problem of the rectilinear plate can be reduced to that of a rectilinear membrane.

For the transverse vibrations of a membrane the differential equation to be satisfied by \( W \) is

\[ \nabla^2 W + k^2 W = 0 \]  \tag{3}

with the condition that \( W = 0 \) over the boundary. Operating (3) by \( \nabla \) we get

\[ \nabla^4 W = k^4 W \]  \tag{4}

which is the same as (1). Any solution of (3), therefore, also satisfies (1). Over the boundary \( W = 0 \), and hence from (3) \( \nabla^2 W = 0 \). We can now show that (2.4) also reduces to this condition.

As \( W = 0 \) over the boundary, we have, differentiating along the boundary \( \partial W/\partial s = 0, \partial^2 W/\partial s^2 = 0 \). Also

\[ \nabla^2 W = \frac{\partial^2 W}{\partial s^2} + \frac{1}{\rho} \frac{\partial W}{\partial n} + \frac{\partial^2 W}{\partial n^2}, \]

which for a rectilinear boundary \( (\rho = \infty) \) reduces to

\[ \nabla^2 W = \frac{\partial^2 W}{\partial s^2} + \frac{\partial^2 W}{\partial n^2} = \frac{\partial^2 W}{\partial n^2} \]

Thus (2.4) also becomes \( \nabla^2 W = 0 \). For a simply supported rectilinear boundary the boundary conditions can therefore be put in the form \( W = \nabla^2 W = 0 \).\(^9,10\)

Hence, if we know the solution for a membrane, the corresponding solution for a plate can be easily written down. In fact the form of \( W \) given by Rayleigh for a rectangular plate is the same as that for a membrane.

**Triangular Plates**

We have already discussed in another paper\(^11\) the transverse vibrations of triangular membranes. For triangular plates it is therefore quite sufficient to give the main results,
Fatigue life of rectangular plates

We consider the case of a rectangular plate. The equations are given by:

\[ W = \frac{C_1 \cdot D^{\frac{3}{2}}}{m^2} \sqrt{m_0} \]

where

\[ m = \frac{1}{2} \left[ \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} \right] \sqrt{m_0} \]

(6)

The general solution is given by:

\[ \Phi = \frac{1}{d} \sqrt{m_0} \]

(7)

The corresponding results are given by:

\[ W = \sin^2 \theta \cdot \frac{m_0}{d} \cdot \sin m \cdot \sin m \cdot \sin m \]

(8)

**Equilateral plate**

The corresponding results are given by:

\[ W = 2 \cdot \sin \left( \frac{m_0}{d} \cdot \cos \left( \frac{m}{d} \cdot \cos \left( \frac{m_0}{d} \cdot \cos \right) \right) \right) \]

(9.1)

\[ \pm \frac{1}{d} \sqrt{m_0} \]

(9.2)

\[ \sqrt{m_0} \]

(9.3)

\[ W = 2 \cdot \sin \left( \frac{m_0}{d} \cdot \cos \left( \frac{m}{d} \cdot \cos \left( \frac{m_0}{d} \cdot \cos \right) \right) \right) \]

(9.4)

The sides of the plate being \( a \) and \( b \), we have \( \cos \frac{a}{2} \).

**Isosceles triangular plate containing an angle of 120°**

In this case, the results are:

\[ W = 2 \cdot \sin \frac{m_0}{d} \cdot \sin \left( \frac{m_x}{d} \cdot \sin \left( \frac{m_0}{d} \cdot \sin \right) \right) \]

\[ + \frac{2 \cdot \cos \left( \frac{2n+1}{2} \cdot \frac{m}{d} \cdot \sin \right)}{2n} \cdot \cos \left( \frac{2n+1}{2} \cdot \frac{m}{d} \cdot \sin \right) \]

(10.1)
\[ p = \frac{\pi}{a^2} \left[ (2m + 1)^2 + (2m + 1)(2n + 1) + (2n + 1)^2 \right] \sqrt{\frac{D}{m_0}}, \quad (10.2) \]

\[ p_0 = \frac{7\pi}{2} \frac{\sqrt{D}}{a^2}, \quad (10.3) \]

\[ W_0 = 2 \sin \frac{\pi y}{a} \sin \frac{\pi y \sqrt{3}}{a} - 2 \cos \frac{5\pi x}{2a} \cos \frac{\pi y \sqrt{3}}{2a} \]

\[ + 2 \cos \frac{\pi x}{2a} \cos \frac{3\pi y \sqrt{3}}{2a} \quad (10.4) \]

In this case the modes are all symmetrical. The sides are given by \( x = a, y = x \sqrt{3}, y + x \sqrt{3} = 2a/\sqrt{3}. \)

**Right-angled triangle containing an angle of 60°.**—If the sides are taken as \( x = a, y = a/\sqrt{3}, y = x \sqrt{3}, \) the results given in (10) hold good in this case as well.

**Rhombus and a Regular Hexagon**

If we take the sides of the rhombus, which contains an angle of 120°, as \( x = 0, y = x/\sqrt{3}, y = x/\sqrt{3} + 2a/\sqrt{3}, \) and those of the regular hexagon as \( x = \pm a, y = \pm x/\sqrt{3} \pm 2a/\sqrt{3}, \) the solution in (9) holds good.\(^{12}\)

In all the above cases we find that the frequency is proportional to \( \frac{1}{a^2} \sqrt{\frac{D}{m_0}}. \)

**Clamped Edges**

**Square-plate.**—If we take the sides of the plate as \( y = \pm x \pm 2a, \) we find that the solutions given in (6) and (7) also satisfy the condition \( \nabla W/\partial n = 0 \) over the boundary. Thus the conditions \( W = 0, \nabla W/\partial n = 0 \) for a clamped edge are satisfied on all the sides of the plate. It is found that these conditions are also satisfied on the lines \( y = \pm x. \) Hence (6) and (7) give the symmetrical vibrations.

If we use this solution for a right-angled isosceles plate we find that the conditions for a clamped edge are satisfied on the equal sides \( y = \pm x, \) but that on the edge \( x = a \) the conditions for a supported edge are only satisfied. Hence it may be used when the equal sides of the plate are clamped and the base is supported.

**Free vibrations of a square-plate.**—If the edges are free the boundary condition (2.1) must also be satisfied. Since \( \nabla^2 W = -k^2 W \) and
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\[
\frac{\partial^2 W}{\partial n \partial s^2} = \frac{1}{2} \frac{\partial}{\partial n} \left( \frac{\partial}{\partial x} \pm \frac{\partial}{\partial y} \right)^2 W = \frac{1}{2} \frac{\partial}{\partial n} \left[ \nabla^2 W \pm 2 \frac{\partial^2 W}{\partial x \partial y} \right]
\]

\[
= \frac{1}{2} \frac{\partial}{\partial n} \left[ -k^2 W \pm \frac{\pi^2}{2a^2} (2m+1) (2n+1) W \right],
\]

\[
= \frac{1}{2} \left[ -k^2 \pm \frac{\pi^2}{2a^2} (2m+1) (2n+1) \right] \frac{\partial W}{\partial n}.
\]

We see that (2.1) is also satisfied on all the sides of the plate. The solution in (6) and (7) can therefore also be used for the free symmetrical vibrations of a square plate.

As in the case of clamped edges, this solution can be used for a right-angled isosceles triangle whose equal sides are free but whose base is supported.

**SUMMARY**

The problem of the vibrations of a rectilinear plate with supported edges can be reduced to the corresponding problem of a vibrating membrane. Exact solutions are given for a number of triangular plates. The free and clamped vibrations of a square and a right-angled isosceles triangular plate have also been discussed.

**REFERENCES**

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