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THE DEFORMED FIGURES OF THE DEDEKIND ELLIPSOIDS IN THE POST-NEWTONIAN APPROXIMATION TO GENERAL RELATIVITY

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ABSTRACT

The effects of general relativity, in the post-Newtonian approximation, on the Dedekind figures of equilibrium of homogeneous masses are determined. It is shown how the post-Newtonian figures can be obtained by first altering the velocity field in the Dedekind ellipsoid, appropriately, and then subjecting it to a suitable Lagrangian displacement cubic in the coordinates. The solution exhibits a singularity at a point where the axes of the Dedekind ellipsoid are in the ratios 1:0.6158:0.4412. However, in contrast to what happens along the Jacobian sequence, the occurrence of the singularity along the Dedekind sequence is not associated with the onset of any instability at that point by a strict Newtonian-like dynamic perturbation.

Subject headings: hydrodynamics — relativity — rotation

I. INTRODUCTION

In earlier papers, the post-Newtonian effects of general relativity on the figures of equilibrium of the classical Maclaurin spheroids (Chandrasekhar 1965, 1967*a*, 1971*a*; also Bardeen 1971) and Jacobi ellipsoids (Chandrasekhar 1967*b*, 1971*b*) were considered. The principal result of these studies is the disclosure that in an exact relativistic theory, equilibrium sequences of rotating masses may exhibit features which are qualitatively different from those in the Newtonian theory: already, in the post-Newtonian approximation the solution for the equilibrium figures diverges when, on the Newtonian theory, the configurations are unstable to certain specific fourth-harmonic deformations. Moreover, it has since appeared that the occurrence of a Dedekind-like point of bifurcation along an axisymmetric sequence may, in fact, play a central role in the evolution of collapsing rotating masses in the framework of general relativity, since it is by a Dedekind mode of deformation that gravitational radiation-reaction induces secular instability (Chandrasekhar 1970). Since a triaxial stationary Jacobi-like object is strictly impossible in general relativity, a Dedekind-like object is the only remaining alternative for the stationary existence of a nonaxisymmetric state: besides axisymmetric objects, Dedekind-like objects are the only kinds that can exist in a stationary nonradiating state. For these reasons, it has seemed worthwhile to include a consideration of the Dedekind ellipsoids in the framework of the post-Newtonian equations of hydrodynamics (Chandrasekhar 1965).

It will appear that while the treatment of the deformed figures of the Dedekind ellipsoids in the post-Newtonian approximation is closely related to that of the Jacobian ellipsoids (Chandrasekhar 1967*b*, 1971*b*; these papers will be referred to hereafter as Papers III and VI, respectively) in some respects, there are yet essential differences in other respects; and these differences illuminate certain characteristic aspects of general relativity.

II. THE NEWTONIAN FIGURES

It was Dedekind's discovery in 1858 that a sequence of ellipsoids, congruent to the Jacobi ellipsoids, exists; and that this sequence bifurcates from the Maclaurin sequence at the same point that the Jacobian sequence does, but by a different mode of deformation: the Jacobi mode is neutral at the point of bifurcation in a frame of reference rotating with the angular velocity of the Maclaurin spheroid while the Dedekind mode is neutral at the *same* point in the inertial frame (for an account of all these matters, see Chandrasekhar 1969, §§ 5, 36, 44, and 45; this book will be referred to hereafter as *E.F.E.*).

The Dedekind ellipsoids, unlike the Jacobi ellipsoids, are stationary in the inertial frame; they maintain their ellipsoidal figures by virtue of internal motions with a uniform vorticity ζ (about the x_3-axis, say) derived from the motions

$$v_1 = Q_1 x_2\,, \qquad v_2 = Q_2 x_1\,, \qquad \text{and} \qquad v_3 = 0\,, \tag{1}$$

where

$$Q_1 = -\frac{a_1^2}{a_1^2 + a_2^2}\,\zeta \qquad \text{and} \qquad Q_2 = +\frac{a_2^2}{a_1^2 + a_2^2}\,\zeta\,, \tag{2}$$

and a_1, a_2, and a_3 denote the semiaxes of the ellipsoid. The ellipsoid with a vorticity ζ has semiaxes determined by the equations (E.F.E., § 44)

$$-Q_1 Q_2 = \frac{a_1^2 a_2^2}{(a_1^2 + a_2^2)^2}\,\zeta^2 = 2\pi G\rho B_{12} \qquad \text{and} \qquad a_1^2 a_2^2 A_{12} = a_3^2 A_3\,, \tag{3}$$

where A_1, A_{12}, B_{12}, etc., are the "index symbols" defined in E.F.E., § 21.

It may be noted here that Q_1 and Q_2, as defined in equations (2), satisfy the relations

$$\frac{Q_1{}^2 a_2{}^2}{a_1{}^2} = \frac{Q_2{}^2 a_1{}^2}{a_2{}^2} = -Q_1 Q_2 = 2\pi G \rho B_{12}\,,$$

and

$$Q_1 a_2{}^2 + Q_2 a_1{}^2 = 0\,. \tag{4}$$

III. THE POST-NEWTONIAN EQUATIONS GOVERNING EQUILIBRIUM

The post-Newtonian equations governing a stationary fluid mass with internal motions are (cf. Paper III, eqs. [1]–[5])

$$\frac{\partial}{\partial x_\alpha}(\sigma v_\alpha) = 0\,, \tag{5}$$

$$\sigma v_\beta \frac{\partial v_\alpha}{\partial x_\beta} + \frac{\partial}{\partial x_\alpha}\left[\left(1 + \frac{2U}{c^2}\right)p\right] - \rho \frac{\partial U}{\partial x_\alpha} + \frac{4}{c^2}\rho v_\beta \frac{\partial U_\beta}{\partial x_\alpha} + \frac{4}{c^2}\rho v_\beta \frac{\partial}{\partial x_\beta}(v_\alpha U - U_\alpha) - \frac{2}{c^2}\rho\left(\phi \frac{\partial U}{\partial x_\alpha} + \frac{\partial \Phi}{\partial x_\alpha}\right) = 0\,, \tag{6}$$

where

$$\sigma = \rho + \frac{1}{c^2}\rho\left(v^2 + 2U + \Pi + \frac{p}{\rho}\right),$$

and

$$\phi = v^2 + U + \frac{1}{2}\Pi + \frac{3}{2}\frac{p}{\rho}\,, \tag{7}$$

and U_α and Φ are defined as solutions of the equations

$$\nabla^2 U_\alpha = -4\pi G \rho v_\alpha \qquad \text{and} \qquad \nabla^2 \Phi = -4\pi G \rho \phi\,. \tag{8}$$

An alternative form of equation (6) which we shall find useful is

$$\rho v_\beta \frac{\partial v_\alpha}{\partial x_\beta} + \rho \frac{\partial}{\partial x_\alpha}\left(\Pi + \frac{p}{\rho}\right) - \rho \frac{\partial U}{\partial x_\alpha} - \frac{1}{c^2}\rho \frac{\partial}{\partial x_\alpha}\left[2\Phi + 2v^2 U + \frac{1}{2}\left(\Pi + \frac{p}{\rho}\right)^2\right]$$
$$+ \frac{1}{c^2}\rho\left[2U \frac{\partial v^2}{\partial x_\alpha} + v_\beta \frac{\partial v_\alpha}{\partial x_\beta}(v^2 + 4U) + 4 v_\alpha v_\beta \frac{\partial U}{\partial x_\beta} + 4 v_\beta\left(\frac{\partial U_\beta}{\partial x_\alpha} - \frac{\partial U_\alpha}{\partial x_\beta}\right)\right] = 0\,. \tag{9}$$

In the rest of this paper, we shall be concerned with a configuration in which the energy-density $\epsilon = \rho c^2 + \rho \Pi$ is a constant. This assumption that ϵ is a constant is formally equivalent to the assumption

$$\rho = \text{constant} \qquad \text{and} \qquad \Pi = 0\,, \tag{10}$$

and the assignment to ρ the meaning ϵ/c^2. On this understanding, equation (9) can be rewritten in the form

$$\frac{\partial}{\partial x_\alpha}\frac{p}{\rho} = \frac{\partial U}{\partial x_\alpha} + \frac{1}{c^2}\frac{\partial}{\partial x_\alpha}\left[2\Phi + 2v^2 U + \frac{1}{2}\left(\frac{p}{\rho}\right)^2\right] - v_\beta \frac{\partial v_\alpha}{\partial x_\beta}$$
$$- \frac{1}{c^2}\left[2U \frac{\partial v^2}{\partial x_\alpha} + (v^2 + 4U) v_\beta \frac{\partial v_\alpha}{\partial x_\beta} + 4 v_\alpha v_\beta \frac{\partial U}{\partial x_\beta} + 4 v_\beta\left(\frac{\partial U_\beta}{\partial x_\alpha} - \frac{\partial U_\alpha}{\partial x_\beta}\right)\right]. \tag{11}$$

The equations describing the Dedekind figures, given in § II, represent a solution of equations (5) and (11) in the Newtonian approximation, i.e., in an approximation in which the terms in equations (5) and (11) which occur with the factor $1/c^2$ are ignored. We now seek a solution of these same equations when the post-Newtonian terms are retained.

First we make the following observations concerning equations (5) and (11).

i) In the terms which occur explicitly with the factor $1/c^2$, we may insert expressions which are valid in the Newtonian approximation.

ii) The velocity-field specified in equation (1) is not consistent with the equation of continuity (5) to the required order: it must, therefore, be modified appropriately.

iii) The terms on the right-hand side of equation (11) must be expressible as the gradient of a scalar function. We shall now turn to a consideration of these matters.

a) The Terms in Equation (11) Which are Explicitly of Post-Newtonian Order

The expressions for the gravitational potential U and the pressure distribution p/ρ in the Newtonian approximation are given by

$$\frac{U}{\pi G\rho} = I - (A_1x_1^2 + A_2x_2^2 + A_3x_3^2) \tag{12}$$

and

$$\frac{1}{\pi G\rho}\frac{p}{\rho} = a_3^2A_3\left(1 - \frac{x_1^2}{a_1^2} - \frac{x_2^2}{a_2^2} - \frac{x_3^2}{a_3^2}\right); \tag{13}$$

and using these expressions we can write

$$\frac{\phi}{\pi G\rho} = Q_1^2x_2^2 + Q_2^2x_1^2 + I - \sum_{\mu=1}^{3} A_\mu x_\mu^2 + \tfrac{3}{2}a_3^2A_3\left(1 - \sum_{\mu=1}^{3}\frac{x_\mu^2}{a_\mu^2}\right), \tag{14}$$

where here, and in the sequel, Q_1 and Q_2 are measured in the unit $(\pi G\rho)^{1/2}$.

The solutions for U_1, U_2, and Φ can now be written in the forms

$$\frac{U_1}{(\pi G\rho)^{1/2}} = Q_1\mathfrak{D}_2, \qquad \frac{U_2}{(\pi G\rho)^{1/2}} = Q_2\mathfrak{D}_1, \tag{15}$$

and

$$\frac{\Phi}{\pi G\rho} = (I + \tfrac{3}{2}a_3^2A_3)U + \left(Q_2^2 - A_1 - \frac{3}{2}\frac{a_3^2}{a_1^2}A_3\right)\mathfrak{D}_{11} - \tfrac{5}{2}A_3\mathfrak{D}_{33} + \left(Q_1^2 - A_2 - \frac{3}{2}\frac{a_3^2}{a_2^2}A_3\right)\mathfrak{D}_{22}, \tag{16}$$

where $\mathfrak{D}_i$ and $\mathfrak{D}_{ij}$ are the Newtonian potentials appropriate to the distributions ρx_i and $\rho x_i x_j$. Inserting the known expressions for $\mathfrak{D}_i$ and $\mathfrak{D}_{ij}$ (given in E.F.E., § 22) we can readily write down the explicit expressions for U_1, U_2, and Φ.

Turning next to the post-Newtonian terms on the right-hand side of equation (11) which are not expressed as gradients and inserting for the various quantities, we find, apart from a factor $(\pi G\rho)^2/c^2$,

$$\begin{aligned}&-x_1[4Q_2(Q_1 + Q_2)I + 8a_2^2B_{12}(A_1 + A_2)]\\ &+ x_1^3[4A_1Q_2(Q_1 + Q_2) - Q_1Q_2^3 + 8a_2^2B_{12}(3A_{11} + A_{12})]\\ &+ x_1x_2^2[4A_2Q_2(Q_1 + Q_2) - Q_1^3Q_2 + 8a_2^2B_{12}(3A_{22} + A_{12}) + 8Q_1(A_1Q_1 + A_2Q_2)]\\ &+ x_1x_3^2[4A_3Q_2(Q_1 + Q_2) + 8a_2^2B_{12}(A_{13} + A_{23})] \qquad (\alpha = 1),\end{aligned} \tag{17}$$

and

$$\begin{aligned}&- x_2[4Q_1(Q_1 + Q_2)I + 8a_1^2B_{12}(A_1 + A_2)]\\ &+ x_2^3[4A_2Q_1(Q_1 + Q_2) - Q_2Q_1^3 + 8a_1^2B_{12}(3A_{22} + A_{12})]\\ &+ x_2x_1^2[4A_1Q_1(Q_1 + Q_2) - Q_2^3Q_1 + 8a_1^2B_{12}(3A_{11} + A_{12}) + 8Q_2(A_1Q_1 + A_2Q_2)]\\ &+ x_2x_3^2[4A_3Q_1(Q_1 + Q_2) + 8a_1^2B_{12}(A_{23} + A_{13})] \qquad (\alpha = 2).\end{aligned} \tag{18}$$

We observe that the coefficient of $x_1x_2^2$ in the expression (17) is not equal to the coefficient of $x_2x_1^2$ in the expression (18); these terms cannot, therefore, be expressed as the gradient of a scalar function. But equation (11) requires that when these terms are combined with those derived from $v_\beta\partial v_\alpha/\partial x_\beta$, they must be so expressible.

b) The Post-Newtonian Velocity-Field

As we have already remarked, the Newtonian velocity-field, specified in equations (1) and (2), is not consistent with the post-Newtonian equation of continuity (5) to the requisite order. To rectify this situation, we let

$$\frac{v_1}{(\pi G\rho)^{1/2}} = Q_1x_2 + \frac{\pi G\rho}{c^2}\,\delta v_1 \quad \text{and} \quad \frac{v_2}{(\pi G\rho)^{1/2}} = Q_2x_1 + \frac{\pi G\rho}{c^2}\,\delta v_2, \tag{19}$$

where δv_1 and δv_2 are quantities that are to be determined consistently with equation (5). With Q_1 and Q_2 defined as in equation (2), equation (5) is satisfied in zero order (as we should indeed expect); in the next higher order we obtain

$$\frac{\partial}{\partial x_1}\,\delta v_1 + \frac{\partial}{\partial x_2}\,\delta v_2 = -\frac{1}{\pi G\rho}\left(Q_1 x_2 \frac{\partial}{\partial x_1} + Q_2 x_1 \frac{\partial}{\partial x_2}\right)\left(v^2 + 2U + \frac{p}{\rho}\right); \tag{20}$$

or, inserting for v^2, U, and p/ρ their Newtonian values (as we may for determining the post-Newtonian terms δv_1 and δv_2), we obtain

$$\frac{\partial}{\partial x_1}\,\delta v_1 + \frac{\partial}{\partial x_2}\,\delta v_2 = -2\left[Q_1\left(Q_2^2 - 2A_1 - \frac{a_3^2}{a_1^2}A_3\right) + Q_2\left(Q_1^2 - 2A_2 - \frac{a_3^2}{a_2^2}A_3\right)\right]x_1 x_2\,. \tag{21}$$

A *particular solution* of equation (21) is given by

$$\delta v_1 = q_1 x_1^2 x_2 \quad \text{and} \quad \delta v_2 = q_2 x_2^2 x_1\,, \tag{22}$$

provided

$$q_1 + q_2 = -\left[Q_1\left(Q_2^2 - 2A_1 - \frac{a_3^2}{a_1^2}A_3\right) + Q_2\left(Q_1^2 - 2A_2 - \frac{a_3^2}{a_2^2}A_3\right)\right]. \tag{23}$$

We make this particular solution *determinate* as follows.

For the velocity field (19) with δv_1 and δv_2 given by equations (22), we find that

$$\frac{v_\beta}{\pi G\rho}\frac{\partial v_1}{\partial x_\beta} = Q_1 Q_2 x_1 + \frac{\pi G\rho}{c^2}\left[(Q_1 q_2 + 2q_1 Q_1)x_1 x_2^2 + Q_2 q_1 x_1^3\right] \tag{24}$$

and

$$\frac{v_\beta}{\pi G\rho}\frac{\partial v_2}{\partial x_\beta} = Q_1 Q_2 x_2 + \frac{\pi G\rho}{c^2}\left[(Q_2 q_1 + 2q_2 Q_2)x_2 x_1^2 + Q_1 q_2 x_2^3\right]. \tag{25}$$

The Newtonian terms on the right-hand sides of equations (24) and (25) are clearly expressible as the gradient of $\frac{1}{2}Q_1Q_2(x_1^2 + x_2^2)$. We now require that when the terms (17) and (18) are combined with the terms in (24) and (25) (in accordance with eq. [11]), they are also expressible as the gradient of a scalar function. This latter requirement can be met if

$$\begin{aligned}&4A_2Q_2(Q_1 + Q_2) - Q_1^3Q_2 + 8a_2^2B_{12}(3A_{22} + A_{12}) + 8Q_1(A_1Q_1 + A_2Q_2)\\ &- \left[4A_1Q_1(Q_1 + Q_2) - Q_2^3Q_1 + 8a_1^2B_{12}(3A_{11} + A_{12}) + 8Q_2(A_1Q_1 + A_2Q_2)\right]\\ &\qquad = Q_1(q_2 + 2q_1) - Q_2(q_1 + 2q_2)\,.\end{aligned} \tag{26}$$

Equation (26), together with equation (23), will determine q_1 and q_2 and make determinate the particular solution (22).

We now proceed to write down the general solution of equation (21). It will appear that in the post-Newtonian Dedekind configuration, the velocity field can, at most, be a cubic polynomial in the coordinates. Consistent with this requirement we shall write the general solution in the form

$$\delta v_1 = (q_1 + q)x_2 x_1^2 + r_1 x_2^3 + t_1 x_2 x_3^2$$

and

$$\delta v_2 = (q_2 - q)x_1 x_2^2 + r_2 x_1^3 + t_2 x_1 x_3^2\,, \tag{27}$$

where q, r_1, r_2, t_1, and t_2 are constants, unspecified for the present. By virtue of equation (23), equation (21) is identically satisfied by the solution (27); the constants in the solution (27) are, therefore, not restricted in any manner by equation (21).

The additional terms in the solution (27) contribute to the right-hand sides of equations (24) and (25) the further post-Newtonian terms

$$(Q_1 r_2 + Q_2 q)x_1^3 + (Q_1 t_2 + Q_2 t_1)x_1 x_3^2 + (3r_1 Q_2 + qQ_1)x_1 x_2^2 \tag{28}$$

and

$$(Q_2 r_1 - Q_1 q)x_2^3 + (Q_1 t_2 + Q_2 t_1)x_2 x_3^2 + (3r_2 Q_1 - qQ_2)x_2 x_1^2\,, \tag{29}$$

respectively. To satisfy the requirement that the right-hand side of equation (11) continues to be the gradient of a scalar function, we impose the condition

$$3r_1 Q_2 + qQ_1 = 3r_2 Q_1 - qQ_2 ,$$

or

$$q(Q_1 + Q_2) = 3(r_2 Q_1 - r_1 Q_2) , \tag{30}$$

which maintains the equality of the coefficients of $x_1 x_2^2$ and $x_2 x_1^2$.

With the reductions that have until now been effected, we can write the integral of equation (11) in the form

$$\begin{aligned}
\frac{1}{\pi G\rho}\frac{p}{\rho} = {} & a_3^2 A_3\left(1 - \sum_{\mu=1}^{3}\frac{x_\mu^2}{a_\mu^2}\right) + \delta U \\
& + \frac{\pi G\rho}{c^2}\left\{2\Phi + 2v^2 U + \frac{1}{2}\left(\frac{p}{\rho}\right)^2\right. \\
& + \tfrac{1}{4}x_1^4[4A_1Q_2(Q_1+Q_2) - Q_1Q_2^3 + 8a_2^2B_{12}(3A_{11}+A_{12}) - Q_2q_1 - (Q_1r_2 + Q_2q)] \\
& + \tfrac{1}{4}x_2^4[4A_2Q_1(Q_1+Q_2) - Q_2Q_1^3 + 8a_1^2B_{12}(3A_{22}+A_{12}) - Q_1q_2 - (Q_2r_1 - Q_1q)] \\
& + \tfrac{1}{2}x_1^2x_2^2\begin{bmatrix} 4A_2Q_2(Q_1+Q_2) - Q_1^3Q_2 + 8a_2^2B_{12}(3A_{22}+A_{12}) + 8Q_1(A_1Q_1 + A_2Q_2) \\ - Q_1q_2 - 2q_1Q_1 - (Q_1q + 3r_1Q_2) \\ \textit{or} \\ 4A_1Q_2(Q_1+Q_2) - Q_2^3Q_1 + 8a_1^2B_{12}(3A_{11}+A_{12}) + 8Q_2(A_1Q_1 + A_2Q_2) \\ - Q_2q_1 - 2q_2Q_2 + (Q_2q - 3r_2Q_1) \end{bmatrix} \\
& + \tfrac{1}{2}x_1^2x_3^2[4A_3Q_2(Q_1+Q_2) + 8a_2^2B_{12}(A_{13}+A_{23}) - (Q_1t_2 + Q_2t_1)] \\
& + \tfrac{1}{2}x_2^2x_3^2[4A_3Q_1(Q_1+Q_2) + 8a_1^2B_{12}(A_{13}+A_{23}) - (Q_1t_2 + Q_2t_1)] \\
& - \tfrac{1}{2}x_1^2[4Q_2(Q_1+Q_2)I + 8a_2^2B_{12}(A_1+A_2)] \\
& \left. - \tfrac{1}{2}x_2^2[4Q_1(Q_1+Q_2)I + 8a_1^2B_{12}(A_1+A_2)]\right\} ,
\end{aligned} \tag{31}$$

where δU is the change in the Newtonian gravitational potential of the deformed post-Newtonian configuration. For the sake of brevity, we shall rewrite equation (31) in the form

$$\begin{aligned}
\frac{1}{\pi G\rho}\frac{p}{\rho} = {} & a_3^2 A_3\left(1 - \sum_{\mu=1}^{3}\frac{x_\mu^2}{a_\mu^2}\right) + \delta U \\
& + \frac{\pi G\rho}{c^2}\{\alpha_1 x_1^2 + \alpha_2 x_2^2 + \alpha_3 x_3^2 + \alpha_{33}x_3^4 \\
& + [\alpha_{11} - \tfrac{1}{4}(Q_1r_2 + Q_2q)]x_1^4 + [\alpha_{22} - \tfrac{1}{4}(Q_2r_1 - Q_1q)]x_2^4 \\
& + [\alpha_{13} - \tfrac{1}{2}(Q_1t_2 + Q_2t_1)]x_1^2x_3^2 + [\alpha_{23} - \tfrac{1}{2}(Q_1t_2 + Q_2t_1)]x_2^2x_3^2 \\
& + [\alpha_{12} - \tfrac{1}{2}(Q_1q + 3r_1Q_2)\ \textit{or}\ + \tfrac{1}{2}(Q_2q - 3r_2Q_1)]x_1^2x_2^2\} ,
\end{aligned} \tag{32}$$

where α_1, α_{11}, α_{12}, etc., are quantities which can be read off by comparison with equation (31).

IV. THE NATURE OF THE POST-NEWTONIAN DEFORMATION AND THE CHANGE IN THE GRAVITATIONAL POTENTIAL CAUSED BY IT

We shall suppose that the post-Newtonian figure is obtained by a deformation of the Newtonian figure by the application of a suitable Lagrangian displacement at each point of its interior and the boundary. It is clear that the nature of the deformation considered in Paper III, § IV, in the context of the Jacobian figures will suffice equally in the present context of the Dedekind figures. We shall suppose, then, that (cf. Paper III, eqs. [47], [48], and [56])

$$\boldsymbol{\xi} = \frac{\pi G \rho a_1^2}{c^2} \sum_{i=1}^{5} S_i \boldsymbol{\xi}^{(i)} , \tag{33}$$

where

$$\boldsymbol{\xi}^{(1)} = (x_1, 0, -x_3) , \qquad \boldsymbol{\xi}^{(2)} = (0, x_2, -x_3) , \qquad \boldsymbol{\xi}^{(3)} = \frac{1}{a_1^2} (\tfrac{1}{3}x_1^3, -x_1^2 x_2, 0) ,$$

$$\boldsymbol{\xi}^{(4)} = \frac{1}{a_1^2} (0, \tfrac{1}{3}x_2^3, -x_2^2 x_3) , \qquad \text{and} \qquad \boldsymbol{\xi}^{(5)} = \frac{1}{a_1^2} (-x_3^2 x_1, 0, \tfrac{1}{3}x_3^3) . \tag{34}$$

The deformation of the Dedekind ellipsoid by the displacement (33) will change the gravitational potential U by the amount

$$\delta U = \frac{\pi G \rho a_1^2}{c^2} \sum_{i=1}^{5} S_i \delta U^{(i)} , \tag{35}$$

where expressions for $\delta U^{(i)}$ are given in Paper III, equations (70)–(73); and as in Paper III, we can write

$$\delta U = \frac{(\pi G \rho)^2}{c^2} \Bigg\{ a_1^2 \sum_{i=1}^{2} S_i \Big[u_0^{(i)} + \sum_{\mu=1}^{3} u_\mu^{(i)} x_\mu^2 \Big] + \sum_{i=3}^{5} S_i \Big[u_0^{(i)} + \sum_{\mu=1}^{3} u_\mu^{(i)} x_\mu^2 + \sum_{\mu=1}^{3} u_{\mu\mu}^{(i)} x_\mu^4 + \sum_{\mu\nu}^{12,23,31} u_{\mu\nu}^{(i)} x_\mu^2 x_\nu^2 \Big] \Bigg\} . \tag{36}$$

V. THE DETERMINATION OF THE POST-NEWTONIAN FIGURE

Returning to equation (32), we shall now rewrite it in the form

$$\frac{1}{\pi G \rho} \frac{p}{\rho} = a_3^2 A_3 \Big(1 - \sum_{\mu=1}^{3} \frac{x_\mu^2}{a_\mu^2} \Big) + \frac{\pi G \rho}{c^2} \Big[\sum_{\mu=1}^{3} P_\mu x_\mu^2 + \sum_{\mu=1}^{3} P_{\mu\mu} x_\mu^4 + \sum_{\mu,\nu}^{12,23,31} P_{\mu\nu} x_\mu^2 x_\nu^2 \Big] , \tag{37}$$

where

$$\begin{aligned}
P_\mu &= \alpha_\mu + \sum_{i=1}^{3} S_i a_i^2 u_\mu^{(i)} + \sum_{i=3}^{5} S_i u_\mu^{(i)} , \\
P_{11} &= \alpha_{11} - \tfrac{1}{4}(Q_1 r_2 + Q_2 q) + \sum_{i=3}^{5} S_i u_{11}^{(i)} , \\
P_{22} &= \alpha_{22} - \tfrac{1}{4}(Q_2 r_1 - Q_1 q) + \sum_{i=3}^{5} S_i u_{22}^{(i)} , \\
P_{33} &= \alpha_{33} + \sum_{i=3}^{5} S_i u_{33}^{(i)} , \\
P_{12} &= \alpha_{12} - \tfrac{1}{2}(Q_1 q + 3 r_1 Q_2) + \sum_{i=3}^{5} S_i u_{12}^{(i)} , \\
P_{13} &= \alpha_{13} - \tfrac{1}{2}(Q_1 t_2 + Q_2 t_1) + \sum_{i=3}^{5} S_i u_{13}^{(i)} , \\
P_{23} &= \alpha_{23} - \tfrac{1}{2}(Q_1 t_2 + Q_2 t_1) + \sum_{i=3}^{5} S_i u_{23}^{(i)} .
\end{aligned} \tag{38}$$

It remains to apply the proper boundary conditions to the solutions which we have found for the velocity field (eqs. [19] and [27]) and the pressure distribution (eq. [38]) and determine the ten constants q, r_1, r_2, t_1, t_2, S_1, S_2, S_3, S_4, and S_5.

The boundary conditions that have to be applied on the bounding surface,

$$S(x) = \sum_{\mu=1}^{3} \frac{x_\mu^2}{a_\mu^2} - 1 - \frac{2\pi G\rho}{c^2}\left[S_1 a_1^2\left(\frac{x_1^2}{a_1^2} - \frac{x_3^2}{a_3^2}\right) + S_2 a_1^2\left(\frac{x_2^2}{a_2^2} - \frac{x_3^2}{a_3^2}\right) + S_3\left(\frac{x_1^4}{3a_1^2} - \frac{x_1^2 x_2^2}{a_2^2}\right) + S_4\left(\frac{x_2^4}{3a_2^2} - \frac{x_2^2 x_3^2}{a_3^2}\right) + S_5\left(\frac{x_3^4}{3a_3^2} - \frac{x_3^2 x_1^2}{a_1^2}\right)\right] = 0\,, \tag{39}$$

of the deformed ellipsoid, are that *the normal component of the velocity and the pressure vanish on it identically.*

The requirement that the normal component of the velocity vanishes on the surface defined by equation (39) is

$$v_\mu \frac{\partial S}{\partial x_\mu} = 0\,. \tag{40}$$

For the velocity field specified by equation (19), equation (40) gives

$$\left(Q_1 x_2 + \frac{\pi G\rho}{c^2}\,\delta v_1\right)\frac{\partial S}{\partial x_1} + \left(Q_2 x_1 + \frac{\pi G\rho}{c^2}\,\delta v_2\right)\frac{\partial S}{\partial x_2} = 0\,. \tag{41}$$

For Q_1 and Q_2 related as in equation (2), equation (41) is automatically satisfied in the Newtonian approximation; in the post-Newtonian approximation, the equation gives

$$-x_1 x_2\left(2S_1 Q_1 + 2S_2 Q_2 \frac{a_1^2}{a_2^2}\right) + x_1^3 x_2\left[\frac{1}{a_1^2}(q_1 + q) + \frac{r_2}{a_2^2} - \frac{4S_3}{3a_1^2} Q_1 + 2\frac{S_3}{a_2^2} Q_2\right] + x_2^3 x_1\left[\frac{1}{a_2^2}(q_2 - q) + \frac{r_1}{a_1^2} - \frac{4S_4}{3a_2^2} Q_2 + 2\frac{S_3}{a_2^2} Q_1\right] + x_1 x_2 x_3^2\left[\left(\frac{t_1}{a_1^2} + \frac{t_2}{a_2^2}\right) + 2\frac{S_5}{a_1^2} Q_1 + 2\frac{S_4}{a_3^2} Q_2\right] = 0\,. \tag{42}$$

Accordingly, we must require that the coefficient of each of the terms in equation (42) vanishes separately. By virtue of the relation (cf. eq. [4]),

$$Q_1 = -Q_2 \frac{a_1^2}{a_2^2}\,, \tag{43}$$

the vanishing of the coefficient of $x_1 x_2$ in equation (42) requires

$$S_1 = S_2\,. \tag{44}$$

And from the vanishing of the remaining coefficients, we obtain

$$\left[a_1^2 \frac{Q_1 + Q_2}{3Q_1} - (a_1^2 - a_2^2)\right] q = [\tfrac{4}{3}a_2^2 Q_1 - 2a_1^2(Q_1 + Q_2)]S_3 + \tfrac{4}{3}a_1^2 Q_2 S_4 - (a_1^2 q_2 + a_2^2 q_1)\,, \tag{45}$$

$$r_1 = \frac{a_1^2}{a_2^2}[\tfrac{4}{3}Q_2 S_4 - 2Q_1 S_3 + (q - q_2)]\,, \tag{46}$$

$$r_2 = \frac{1}{a_1^2}[(\tfrac{4}{3}a_2^2 Q_1 - 2a_1^2 Q_2)S_3 - (q + q_1)a_2^2]\,, \tag{47}$$

and

$$\frac{t_1}{a_1^2} + \frac{t_2}{a_2^2} = -2\left(\frac{S_5}{a_1^2} Q_1 + \frac{S_4}{a_3^2} Q_2\right)\cdot \tag{48}$$

In obtaining the solutions (45), (46), and (47) for q, r_1, and r_2 we have made use of the relations (30).

We observe that the constants q, r_1, and r_2 are expressed in terms of S_3, S_4, and S_5; but the constants t_1 and t_2 are not so separately expressed. Accordingly, by virtue of equations (45), (46), and (47), only P_{13} and P_{23} among the coefficients listed in equations (38) are not solely expressed in terms of S_1 $(=S_2)$, S_3, S_4, and S_5.

Turning next to the boundary condition which requires the vanishing of the pressure p on the bounding surface defined by equation (39), we observe that in view of the formal identity of equation (37) and equation (76) of

Paper III, the discussion of the boundary condition in Paper III, § VII (eqs. [76]–[87]) applies unchanged. Therefore, with the definitions (cf. Paper III, eq. [81])

$$\bar{Q}_1 = P_1 - 2a_3^2 A_3 S_1 \,, \qquad (S_1 = S_2)$$

$$\bar{Q}_2 = P_2 - 2a_3^2 \frac{a_1^2}{a_2^2} S_2 \,,$$

$$\bar{Q}_3 = P_3 + 2a_1^2 A_3 (S_1 + S_2)$$

$$Q_{11} = P_{11} - \frac{2a_3^2 A_3}{3a_1^2} S_3 \,; \qquad Q_{12} = P_{12} + \frac{2a_3^2 A_3}{a_2^2} S_3 \,,$$

$$Q_{22} = P_{22} - \frac{2a_3^2 A_3}{3a_2^2} S_4 \,; \qquad Q_{23} = P_{23} + 2A_3 S_4 \,,$$

and

$$Q_{33} = P_{33} - \tfrac{2}{3} A_3 S_5 \,; \qquad Q_{31} = P_{31} + \frac{2a_3^2 A_3}{a_1^2} S_5 \tag{49}$$

the boundary condition yields the equations

$$a_1^4 Q_{11} + a_2^4 Q_{22} - a_1^2 a_2^2 Q_{12} = 0 \,, \tag{50}$$

$$a_2^4 Q_{22} + a_3^4 Q_{33} - a_2^2 a_3^2 Q_{23} = 0 \,, \tag{51}$$

$$a_3^4 Q_{33} + a_1^4 Q_{11} - a_3^2 a_1^2 Q_{31} = 0 \,, \tag{52}$$

$$a_1^4 Q_{11} - a_2^4 Q_{22} + a_1^2 \bar{Q}_1 - a_2^2 \bar{Q}_2 = 0 \,, \tag{53}$$

and

$$a_3^4 Q_{33} - a_1^4 Q_{11} + a_3^2 \bar{Q}_3 - a_1^2 \bar{Q}_1 = 0 \,. \tag{54}$$

Equations (50), (53), and (54), by virtue of equations (45)–(47), provide three linear equations for S_1 ($=S_2$), S_3, S_4, and S_5. Equations (51) and (52) involve in addition the quantity $(Q_1 t_2 + Q_2 t_1)$ which occurs in the definitions of P_{13} and P_{23}; but we can eliminate its occurrence by combining equations (51) and (52) in the manner

$$a_1^2 a_2^4 Q_{22} - a_1^4 a_2^2 Q_{11} + a_3^4 (a_1^2 - a_2^2) Q_{33} - a_1^2 a_2^2 a_3^2 (Q_{23} - Q_{13}) = 0 \,. \tag{55}$$

Equations (50) and (53)–(55) now provide four linear equations for the four constants S_1, S_3, S_4, and S_5; they can accordingly be determined. With S_1 ($=S_2$), S_3, S_4, and S_5 determined in this fashion, the constants q, r_1, and r_2 follow from equations (45)–(47). Finally, equation (48), together with equation (51) or (52) will determine t_1 and t_2, thus completing the solution of the problem.

VI. THE BINDING ENERGY

The binding energy E of the post-Newtonian Dedekind configuration can be obtained by integrating the "conserved energy" (cf. Paper VI, eq. [23])

$$\mathfrak{E} = \tfrac{1}{2}\rho(v^2 - U) + \frac{1}{c^2}\rho\left(\tfrac{5}{8}v^4 + \tfrac{5}{2}v^2 U - \tfrac{5}{2}U^2 + v^2\frac{p}{\rho} - 2v_1 U_1 - 2v_2 U_2\right), \tag{56}$$

over the volume of the fluid. Inserting for the various quantities their known values, we obtain

$$\mathfrak{E} = \tfrac{1}{2}\rho(Q_1^2 x_2^2 + Q_2^2 x_1^2 - U)$$
$$+ \rho\frac{(\pi G\rho)^2}{c^2}\Big\{Q_1 x_2^2[(q_1 + q)x_1^2 + r_1 x_2^2 + t_1 x_3^2] + Q_2 x_1^2[(q_2 - q)x_2^2 + r_2 x_1^2 + t_2 x_3^2]$$
$$+ \tfrac{5}{8}(Q_1^2 x_2^2 + Q_2^2 x_1^2)^2 - \tfrac{5}{2}\left(I - \sum_{\mu=1}^{3} A_\mu x_\mu^2\right)^2 + \tfrac{5}{2}(Q_1^2 x_2^2 + Q_2^2 x_1^2)\left(I - \sum_{\mu=1}^{3} A_\mu x_\mu^2\right)$$

$$+ a_3^2 A_3(Q_1^2 x_2^2 + Q_2^2 x_1^2)\left(1 - \sum_{\mu=1}^{3} \frac{x_\mu^2}{a_\mu^2}\right)$$

$$- 2Q_1^2 a_2^2 x_2^2\left(A_2 - \sum_{\mu=1}^{3} A_{2\mu} x_\mu^2\right) - 2Q_2^2 a_1^2 x_1^2\left(A_1 - \sum_{\mu=1}^{3} A_{1\mu} x_\mu^2\right)\Bigg\}$$

$$= \tfrac{1}{2}\rho(Q_1^2 x_2^2 + Q_2^2 x_1^2 - U) + \frac{1}{c^2}\mathfrak{E}_2 \quad \text{(say)}. \tag{57}$$

The required binding energy can therefore be expressed in the form

$$E = \frac{1}{2}\int_{\text{post N.}} \rho(Q_1^2 x_2^2 + Q_2^2 x_1^2 - U) dx + \frac{1}{c^2}\int_{\text{Dedekind}} \mathfrak{E}_2 dx\,, \tag{58}$$

where the first integral on the right-hand side must be evaluated, correctly to $O(c^{-2})$, over the deformed figure of the post-Newtonian configuration, while it will suffice to evaluate the second integral over the undeformed Dedekind ellipsoid.

The contribution to E by the first integral on the right-hand side of equation (58) can be reduced in the manner

$$\frac{1}{2}\int_{\text{post N.}} \rho(Q_1^2 x_2^2 + Q_2^2 x_1^2 - U) dx = (\tfrac{1}{2}Q_1^2 I_{22} + \tfrac{1}{2}Q_2^2 I_{11} + \mathfrak{W})_{\text{Dedekind}} + \tfrac{1}{2}Q_1^2 V_{22} + \tfrac{1}{2}Q_2^2 V_{11} + \delta\mathfrak{W}\,. \tag{59}$$

The first term on the right-hand side of equation (59) represents the binding energy of the Newtonian Dedekind ellipsoid having the same density and coordinate volume as the post-Newtonian configuration; and the remaining terms represent the contribution arising from the fact that the moments of inertia and the potential energy of the post-Newtonian configuration differ from the Newtonian configuration by the amounts δI_{11} $(=V_{11})$, δI_{22} $(=V_{22})$, and $\delta\mathfrak{W}$. We may accordingly write

$$E = E_0 + (\Delta E)_{\text{coord. vol.}}\,, \tag{60}$$

where

$$E_0 = (\tfrac{1}{2}Q_1^2 I_{22} + \tfrac{1}{2}Q_2^2 I_{11} + \mathfrak{W})_{\text{Dedekind}}$$
$$= (\tfrac{4}{15}\pi a_1 a_2 a_3 \rho)\pi G\rho[B_{12}(a_1^2 + a_2^2) - 2I] \tag{61}$$

[1]

and

$$(\Delta E)_{\text{coord. vol.}} = \tfrac{1}{2}Q_1^2 V_{22} + \tfrac{1}{2}Q_2^2 V_{11} + \delta\mathfrak{W} + \frac{1}{c^2}\int_{\text{Dedekind}} \mathfrak{E}_2 dx\,. \tag{62}$$

It can be verified by using the appropriate formulae in E.F.E. (eq. [148] on page 60, Lemma 7 on page 54, and eq. [132] on page 125) that

$$\delta\mathfrak{W} = \pi G\rho B_{12}(V_{11} + V_{22})\,. \tag{63}$$

We can, accordingly, write

$$(\Delta E)_{\text{coord. vol.}} = \pi G\rho[(\tfrac{1}{2}Q_2^2 + B_{12})V_{11} + (\tfrac{1}{2}Q_1^2 + B_{12})V_{22}] + \frac{1}{c^2}\int_{\text{Dedekind}} \mathfrak{E}_2 dx\,. \tag{64}$$

Expressions for V_{11} and V_{22} are given in Paper III, equation (64); we have

$$V_{11} = (\tfrac{4}{15}\pi a_1 a_2 a_3 \rho)\frac{\pi G\rho}{c^2}\tfrac{2}{7}(7a_1^4 S_1 + a_1^4 S_3 - a_1^2 a_3^2 S_5)$$

and

$$V_{22} = (\tfrac{4}{15}\pi a_1 a_2 a_3 \rho)\frac{\pi G\rho}{c^2}\tfrac{2}{7}(7a_1^2 a_2^2 S_2 + a_2^4 S_4 - a_1^2 a_2^2 S_3)\,. \tag{65}$$

[1] We observe that this binding energy is the same as that of the congruent Jacobi ellipsoid.

The integration of $\mathfrak{E}_2$ over the volume of the ellipsoid is readily effected. In this manner we finally obtain

$$
\begin{aligned}
(\Delta E)_{\text{coord. vol.}} = \frac{1}{c^2}(\pi G\rho)^2(\tfrac{4}{15}\pi a_1 a_2 a_3 \rho) \\
\times \Big\{ \tfrac{3}{7}a_1^4\Big[\tfrac{5}{8}Q_2^4 + Q_2^2\Big(2a_1^2A_{11} - \frac{a_3^2}{a_1^2}A_3 - \tfrac{5}{2}A_1\Big) - \tfrac{5}{2}A_1^2 + Q_2 r_2\Big] \\
+ \tfrac{3}{7}a_2^4\Big[\tfrac{5}{8}Q_1^4 + Q_1^2\Big(2a_2^2A_{22} - \frac{a_3^2}{a_2^2}A_3 - \tfrac{5}{2}A_2\Big) - \tfrac{5}{2}A_2^2 + Q_1 r_1\Big] - \tfrac{15}{14}a_3^4A_3^2 \\
+ \tfrac{1}{7}a_1^2a_2^2\Big[\tfrac{5}{4}Q_1^2Q_2^2 + Q_1^2\Big(2a_2^2A_{12} - \frac{a_3^2}{a_1^2}A_3 - \tfrac{5}{2}A_1\Big) - 5A_1A_2 \\
+ Q_2^2\Big(2a_1^2A_{12} - \frac{a_3^2}{a_2^2}A_3 - \tfrac{5}{2}A_2\Big) + Q_1(q_1 + q) + Q_2(q_2 - q)\Big] \\
+ \tfrac{1}{7}a_1^2a_3^2[Q_2^2(2a_1^2A_{13} - 3.5A_3) - 5A_1A_3 + Q_2 t_2] \\
+ \tfrac{1}{7}a_2^2a_3^2[Q_1^2(2a_2^2A_{23} - 3.5A_3) - 5A_2A_3 + Q_1 t_1] \\
+ a_2^2[Q_1^2(\tfrac{5}{2}I + a_3^2A_3 - 2a_2^2A_2) + 5IA_2] \\
+ a_1^2[Q_2^2(\tfrac{5}{2}I + a_3^2A_3 - 2a_1^2A_1) + 5IA_1] + 5a_3^2IA_3 - 12.5I^2 \\
+ \tfrac{2}{7}(\tfrac{1}{2}Q_2^2 + B_{12})(7a_1^4S_1 + a_1^4S_3 - a_1^2a_3^2S_5) \\
+ \tfrac{2}{7}(\tfrac{1}{2}Q_1^2 + B_{12})(7a_1^2a_2^2S_2 + a_2^4S_4 - a_1^2a_2^2S_3)\Big\} . \qquad (66)
\end{aligned}
$$

a) *Adjustment to Equal Baryon Number*

It is clear that by the choice of the divergence-free Lagrangian displacement (33), we have arranged that the *coordinate volumes* of the Newtonian and the post-Newtonian configurations are the same. But their *proper*

TABLE 1

THE CONSTANTS OF THE POST-NEWTONIAN DEDEKIND CONFIGURATIONS

a_2/a_1	q_1	q_2	$S_1\ (=S_2)$	S_3	S_4	S_5
0.85	+1.1623	−1.3095	−0.426	+0.0559	−0.2690	−1.245
0.80	+1.1214	−1.3214	−0.444	+0.0797	−0.3872	−2.264
0.75	+1.0742	−1.3283	−0.489	+0.1150	−0.5256	−4.146
0.70	+1.0202	−1.3293	−0.602	+0.1887	−0.6970	−8.40
0.66	+0.9717	−1.3248	−0.898	+0.3722	−0.903	−18.86
0.65	+0.9589	−1.3230	−1.084	+0.4874	−0.988	−25.30
0.64	+0.9458	−1.3207	−1.424	+0.7004	−1.120	−37.08
0.63	+0.9323	−1.3181	−2.246	+1.217	−1.394	−65.4
0.62	+0.9185	−1.3151	−7.034	+4.234	−2.845	−230.2
0.618	+0.9157	−1.3145	−13.30	+8.187	−4.710	−446.0
0.616	+0.9129	−1.3138	−173.8	+109.38	−52.30	−5966.4
0.614	+0.9101	−1.3131	+15.11	−9.732	+3.703	+531.5
0.612	+0.9073	−1.3125	+7.104	−4.684	+1.3226	+256.1
0.61	+0.9044	−1.3118	+4.584	−3.0951	+0.5692	+169.4
0.60	+0.8900	−1.3080	+1.533	−1.1742	−0.3664	+64.34
0.59	+0.8753	−1.3038	+0.8430	−0.7428	−0.6018	+40.54
0.58	+0.8603	−1.2992	+0.5391	−0.5548	−0.7226	+30.01
0.57	+0.8449	−1.2942	+0.3681	−0.4509	−0.8038	+24.04
0.56	+0.8292	−1.2886	+0.2586	−0.3861	−0.8663	+20.18
0.55	+0.8132	−1.2827	+0.1824	−0.3427	−0.9183	+17.47
0.50	+0.7285	−1.2452	−0.0004	−0.2543	−1.105	+10.60
0.45	+0.6361	−1.1934	−0.0730	−0.2400	−1.217	+7.270
0.40	+0.5370	−1.1252	−0.1107	−0.2464	−1.240	+4.723
0.35	+0.4328	−1.0382	−0.1301	−0.2541	−1.145	+2.259
0.32	+0.3688	−0.9761	−0.1344	−0.2538	−1.021	+0.7264
0.30	+0.3260	−0.9303	−0.1344	−0.2502	−0.910	−0.3194
0.28	+0.2835	−0.8809	−0.132	−0.243	−0.777	−1.38
0.25	+0.2210	−0.7998	−0.124	−0.226	−0.541	−2.97

volumes (in their respective approximations) are different. The adjustment to equal proper volumes or, equivalently, equal baryon numbers, can be made by subjecting the post-Newtonian configuration to the uniform expansion

$$\xi^{(0)} = \frac{\pi G\rho a_1^2}{c^2} S_0 \boldsymbol{x} , \tag{67}$$

where (cf. Paper VI, eqs. [17] and [18])

$$\frac{\pi G\rho a_1^2}{c^2}(3S_0) = -\frac{1}{(\frac{4}{3}\pi a_1 a_2 a_3 \rho)c^2}\int_{\text{Dedekind}} (\tfrac{1}{2}Q_1^2 x_2^2 + \tfrac{1}{2}Q_2^2 x_1^2 + 3U)d\boldsymbol{x} . \tag{68}$$

Equation (68) gives

$$S_0 = -\frac{1}{3a_1^2}[\tfrac{1}{10}(a_2^2 Q_1^2 + a_1^2 Q_2^2) + \tfrac{12}{5}I]. \tag{69}$$

By this adjustment to equal baryon numbers, the binding energy is altered in the manner (cf. Paper VI, eq. [30])

$$(\Delta E)_{\text{proper vol.}} = (\Delta E)_{\text{coord. vol.}} - \tfrac{4}{15}\pi a_1 a_2 a_3 \rho \frac{(\pi G\rho)^2}{c^2} \tfrac{1}{3}[B_{12}(a_1^2 + a_2^2) - 2I][\tfrac{1}{12}(a_2^2 Q_1^2 + a_1^2 Q_2^2) + 12I] . \tag{70}$$

VII. NUMERICAL RESULTS AND CONCLUDING REMARKS

In table 1, the various constants which determine the deformed figures of the Dedekind ellipsoids are listed. The table also includes the binding energy, $(\Delta E)_{\text{proper vol.}}$, and the expansion factor, S_0, required to equalize the baryon numbers of the Newtonian and the post-Newtonian configurations (for equal values of Q_1 and Q_2).

It will be observed that the solution of the post-Newtonian equations diverges at

$$a_2/a_1 = 0.61583 \quad \text{and} \quad a_3/a_1 = 0.44119 . \tag{71}$$ [2]

[2] It is important to observe that the location of this point does not depend on the choice of the comparison Newtonian-configuration. In the text, the comparison is made with a Newtonian configuration having the same values of Q_1 and Q_2. If, instead, we make the comparison with a Newtonian configuration appropriate for $Q_1 + (\pi G\rho/c^2)\delta Q_1$ and $Q_2 + (\pi G\rho/c^2)\delta Q_2$, then equation (44) will be replaced by

$$S_1 - S_2 = \frac{1}{2Q_1}\left(\frac{\delta Q_1}{a_1^2} + \frac{\delta Q_2}{a_2^2}\right) ,$$

while none of the other equations are affected. Therefore, by replacing S_1 and S_2 by $S_1 + \epsilon$ and $S_2 - \epsilon$, we shall obtain the same equations for the various constants of integration as those considered in the paper, except that the inhomogeneous terms will now contain terms in ϵ as well. The occurrence of the singularity at $a_2/a_1 = 0.6158$ is related to the vanishing of the determinant of the linear system; and this will not be affected by the terms in ϵ among the inhomogeneous terms.

With the freedom thus available, in the choice of the comparison Newtonian-configuration, one can eliminate, if one so desires, the singularity in the binding energy $(\Delta E)_{\text{proper vol.}}$ at $a_2/a_1 = 0.6158$.

TABLE 1—*Continued*

r_1	r_2	q	t_1	t_2	S_0	$(\Delta E)_{\text{prop. vol.}}$
−0.3372	+0.1098	−1.448	−2.39	+1.15	−0.9419	−0.558
−0.3869	+0.0749	−1.440	−3.88	+1.185	−0.8828	−0.4992
−0.4289	+0.0316	−1.438	−6.73	+1.118	−0.8227	−0.4429
−0.4263	−0.0414	−1.472	−13.40	+0.799	−0.7618	−0.3932
−0.2565	−0.1956	−1.634	−30.11	−0.151	−0.7125	−0.3658
−0.1180	−0.2874	−1.752	−40.48	−0.755	−0.7001	−0.3648
+0.1564	−0.4541	−1.982	−59.45	−1.864	−0.6877	−0.3718
+0.8549	−0.8534	−2.557	−105.2	−4.544	−0.6752	−0.4030
+5.0444	−3.167	−5.984	−371.2	−20.12	−0.6627	−0.6247
+10.56	−6.195	−10.49	−719.6	−40.52	−0.6602	−0.9369
+151.9	−83.68	−125.9	−9634.	−562.7	−0.6577	−8.97
−14.47	+7.524	+9.957	+858.8	+51.90	−0.6552	+0.490
−7.428	+3.658	+4.201	+414.0	+25.86	−0.6527	+0.0908
−5.214	+2.441	+2.391	+274.0	+17.66	−0.6502	−0.0336
−2.555	+0.9659	+0.2156	+104.5	+7.722	−0.6377	−0.1874
−1.977	+0.6315	−0.2604	+66.08	+5.466	−0.6251	−0.2100
−1.738	+0.4834	−0.4584	+49.10	+4.461	−0.6125	−0.2159
−1.617	+0.3995	−0.5597	+39.49	+3.886	−0.5999	−0.2154
−1.551	+0.3455	−0.6158	+33.29	+3.509	−0.5873	−0.2119
−1.514	+0.3077	−0.6468	+28.93	+3.236	−0.5747	−0.2069
−1.506	+0.2151	−0.6457	+17.89	+2.473	−0.5113	−0.1743
−1.582	+0.1757	−0.5440	+12.34	+1.980	−0.4478	−0.1401
−1.642	+0.1494	−0.4048	+7.69	+1.509	−0.3845	−0.1080
−1.632	+0.1249	−0.2563	+2.64	+1.021	−0.3218	−0.0792
−1.571	+0.1090	−0.1734	−0.816	+0.732	−0.2848	−0.0637
−1.503	+0.0978	−0.1237	−3.31	+0.548	−0.2604	−0.0541
−1.413	+0.0861	−0.080	−5.98	+0.377	−0.2363	−0.0452
−1.238	+0.0684	−0.029	−10.23	+0.152	−0.2010	−0.0335

This divergence occurs much earlier along the Dedekind sequence than the corresponding point ($a_2/a_1 = 0.2971$) where a similar divergence occurs along the Jacobian sequence (cf. Paper VI, table 3). But there is an even more important difference. While along the Jacobian sequence, the point, where the solution of the post-Newtonian equations diverges, is associated with the occurrence, at the same point, of a fourth-harmonic neutral mode of deformation of the Newtonian ellipsoid, there is no such association along the Dedekind sequence. Indeed, as we show in the Appendix, the Dedekind ellipsoid does not admit, along its entire sequence, a nontrivial fourth-harmonic neutral mode of deformation.[3] The occurrence, nevertheless, of a divergence in the solution of the post-Newtonian equation (which, in effect, subjects the Newtonian ellipsoid to a fourth-harmonic deformation) must be traced to the circumstance that, unlike the post-Newtonian Jacobian configuration, the Dedekind configuration cannot be considered as the result of a strict Newtonian-like *dynamic* perturbation of the Newtonian ellipsoid. In other words (as Bardeen has emphasized in some private discussions), the relativistic singularity at $a_2/a_1 = 0.6158$ must result from the existence of a Newtonian sequence bifurcating at this same point, but one to which a transition from the Dedekind sequence cannot be accomplished by any allowed hydrodynamical motion consistent with the Newtonian equations of motion. The impossibility of such a transition could arise from the fact that the internal motions along the bifurcating sequence are not of uniform vorticity and include terms which are cubic in the coordinates.

The divergence at $a_2/a_1 = 0.6158$ exhibited by the solution of the post-Newtonian equations is earlier than the point ($a_2/a_1 = 0.4413$) where the Dedekind ellipsoid becomes unstable (for the first time) by a third-harmonic deformation (cf. E.F.E., page 217). On this account, the occurrence of the divergence along the Dedekind sequence is much more meaningful than the occurrence of similar divergences along the Maclaurin and the Jacobian sequences. Also, the occurrence of the divergence itself, in the post-Newtonian approximation, must mean that in the exact framework of general relativity, the sequence of Dedekind-like configurations, starting from the point of bifurcation along an axisymmetric sequence, and the sequence, starting from the opposite prolate end, do not join—and this fact must be significant for the last stages of evolution of rotating masses in general relativity.

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APPENDIX

THE CONDITIONS FOR THE OCCURRENCE OF A NEUTRAL POINT, BELONGING TO THE FOURTH HARMONICS, ALONG THE DEDEKIND SEQUENCE

The question, whether the Dedekind ellipsoid, along its sequence, allows a nontrivial neutral mode of deformation belonging to the fourth harmonics, can be answered by availing ourselves of the first variations of the relevant virial equations of the fourth order derived in an earlier paper (Chandrasekhar 1968; this paper will be referred to as V.E.). As in the analogous considerations relating to the Jacobi ellipsoids, it will suffice, in our present context, to restrict ourselves to the six "even" equations[4] (V.E., eq. [33])

$$2\delta\mathfrak{W}_{ii;ii} + \delta\mathfrak{W}_{ii;i;i} - (4\delta\mathfrak{W}_{ij;ij} + 2\delta\mathfrak{W}_{jj;ii} + 2\delta\mathfrak{W}_{ij;i;j} + \delta\mathfrak{W}_{jj;i;i}) + 6\delta\mathfrak{T}_{ii;ii} - 6(2\delta\mathfrak{T}_{ij;ij} + \delta\mathfrak{T}_{jj;ii}) = 0\,.$$

$$[\text{no summation over repeated indices; } i \neq j \text{ and } (i, j) \text{ an ordered pair}]\,. \quad \text{(A1)}$$

In V.E. (Appendix, eq. [A1]) an explicit expression, for the particular combinations of $\delta\mathfrak{W}_{ij;kl}$ and $\delta\mathfrak{W}_{ij;k;l}$ which occur in equation (A1), is given in terms of the symmetrized virials

$$V_{ijkl} = \int_V \rho\xi_m \frac{\partial}{\partial x_m}(x_i x_j x_k x_l)d\boldsymbol{x} = V_{i;jkl} + V_{j;ikl} + V_{k;ijl} + V_{l;ijk}\,,$$

and

$$V_{ij} = \int_V \rho\xi_m \frac{\partial}{\partial x_m}(x_i x_j)d\boldsymbol{x} = V_{i;j} + V_{j;i}\,. \quad \text{(A2)}$$

[3] But as we remark in the Appendix, the absence of a neutral point, as strictly defined, does not foreclose the possibility of a zero-frequency mode occurring along the Dedekind sequence.

[4] The displacements (34) leading to the post-Newtonian deformation belong to the class comprising these equations.

It remains to express

$$2\delta\mathfrak{T}_{ij;kl} = \delta\int_V \rho u_i u_j x_k x_l d\boldsymbol{x}$$
$$= \int_V \rho(u_j\Delta u_i + u_i\Delta u_j)x_k x_l d\boldsymbol{x} + \int_V \rho u_i u_j(\xi_k x_l + \xi_l x_k)d\boldsymbol{x}\,. \tag{A3}$$

We are presently interested only in the case in which the velocity field is a linear function of the coordinates and is of the form

$$u_i = Q_{ij}x_j\,, \tag{A4}$$

where Q_{ij} represents a constant matrix. (In eq. [A4] and in the equations following, the summation convention is restored.) In this special case, equation (A2) becomes

$$2\delta\mathfrak{T}_{ij;kl} = Q_{jm}\int_V \rho\Delta u_i x_k x_l x_m d\boldsymbol{x} + Q_{im}\int_V \rho\Delta u_j x_k x_l x_m d\boldsymbol{x} + Q_{im}Q_{jn}(V_{k;lmn} + V_{l;kmn})\,. \tag{A5}$$

The integrals over Δu_i and Δu_j in equation (A5) can also be expressed in terms of the virials with the aid of the formula

$$\frac{dV_{i;jkl}}{dt} = \int_V \rho\Delta u_i x_j x_k x_l d\boldsymbol{x} + \int_V \rho\xi_i(u_j x_k x_l + u_k x_l x_j + u_l x_j x_k)d\boldsymbol{x}\,, \tag{A6}$$

which *under stationary conditions* and for the assumed form of the velocity field gives

$$\int_V \rho\Delta u_i x_j x_k x_l d\boldsymbol{x} = -(Q_{jm}V_{i;klm} + Q_{km}V_{i;ljm} + Q_{lm}V_{i;jkm})\,. \tag{A7}$$

Inserting this relation in equation (A5), we find after some rearrangements

$$2[\delta\mathfrak{T}_{ij;kl}]_0 = -Q_{jn}{}^2V_{i;kln} - Q_{in}{}^2V_{j;kln} + Q_{im}Q_{jn}(V_{k;lmn} + V_{l;kmn})$$
$$- Q_{jm}(Q_{kn}V_{i;lmn} + Q_{ln}V_{i;kmn}) - Q_{im}(Q_{kn}V_{j;lmn} + Q_{ln}V_{j;kmn})\,, \tag{A8}$$

where $\delta\mathfrak{T}_{i;jkl}$ has been enclosed in brackets with a distinguishing subscript "0" to emphasize that the expression is valid only for quasi-stationary deformations.

For the particular case of the Dedekind ellipsoid

$$\boldsymbol{Q} = \begin{vmatrix} 0 & Q_1 & 0 \\ Q_2 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \quad \text{and} \quad \boldsymbol{Q}^2 = \begin{vmatrix} Q_1Q_2 & 0 & 0 \\ 0 & Q_1Q_2 & 0 \\ 0 & 0 & 0 \end{vmatrix}\,, \tag{A9}$$

and we find from equation (A8)

$$[6\delta\mathfrak{T}_{11;11}]_0 = -6(Q_1Q_2V_{1;111} + Q_1{}^2V_{1;122})\,,$$
$$[6\delta\mathfrak{T}_{22;22}]_0 = -6(Q_1Q_2V_{2;222} + Q_2{}^2V_{2;211})\,,$$
$$[6\delta\mathfrak{T}_{33;33}]_0 = 0\,,$$
$$[2\delta\mathfrak{T}_{12;12} + \delta\mathfrak{T}_{22;11}]_0 = -Q_1Q_2(4V_{2;211} + V_{1;122}) - Q_1{}^2V_{2;222}\,,$$
$$[2\delta\mathfrak{T}_{12;12} + \delta\mathfrak{T}_{11;22}]_0 = -Q_1Q_2(4V_{1;122} + V_{2;211}) - Q_2{}^2V_{1;111}\,,$$
$$[2\delta\mathfrak{T}_{23;23} + \delta\mathfrak{T}_{33;22}]_0 = -Q_1Q_2V_{3;322} - Q_2{}^2V_{3;311}\,,$$
$$[2\delta\mathfrak{T}_{13;13} + \delta\mathfrak{T}_{33;11}]_0 = -Q_1Q_2V_{3;311} - Q_1{}^2V_{3;322}\,,$$
$$[2\delta\mathfrak{T}_{23;23} + \delta\mathfrak{T}_{22;33}]_0 = -Q_1Q_2(V_{3;322} + V_{2;233})$$

and

$$[2\delta\mathfrak{T}_{13;13} + \delta\mathfrak{T}_{11;33}]_0 = -Q_1Q_2(V_{3;311} + V_{1;133})\,. \tag{A10}$$

On inserting in equation (A1) for the various potential-energy and kinetic-energy tensors, in accordance with the equations V.E. (A1) and (A10), we shall obtain a set of six linear homogeneous equations for the *nine* fourth-order virials

$$V_{1;111}\,,\quad V_{2;222}\,,\quad V_{3;333}\,,\quad V_{1;122}\,,\quad V_{2;211}\,,$$
$$V_{2;233}\,,\quad V_{3;322}\,,\quad V_{3;311}\,,\quad \text{and}\quad V_{1;133}\,, \tag{A11}$$

and the *three* second-order virials

$$V_{11}\,,\quad V_{22}\,,\quad \text{and}\quad V_{33}\,. \tag{A12}$$

The *six* equations provided by equation (A1) must be supplemented by the *two* second-order equations (cf. E.F.E., page 149)

$$\delta\mathfrak{W}_{11} - \delta\mathfrak{W}_{33} + 2[\delta\mathfrak{T}_{11} - \delta\mathfrak{T}_{33}]_0 = \delta\mathfrak{W}_{11} - \delta\mathfrak{W}_{33} - Q_1Q_2V_{11} = 0$$

and

$$\delta\mathfrak{W}_{22} - \delta\mathfrak{W}_{33} + 2[\delta\mathfrak{T}_{22} - \delta\mathfrak{T}_{33}]_0 = \delta\mathfrak{W}_{22} - \delta\mathfrak{W}_{33} - Q_1Q_2V_{22} = 0\,, \tag{A13}$$

and the *four* equations (V.E., eqs. [24] and [25])

$$V_{ii} = \sum_{k=1}^{3} \frac{V_{iikk}}{a_k^2} \qquad (\text{no summation over } i;\ i = 1, 2, 3) \tag{A14}$$

and

$$\sum_{k=1}^{3} \frac{V_{kk}}{a_k^2} = 0\,, \tag{A15}$$

which ensure that the displacement considered is divergence-free.

Equations (A1), (A13), (A14), and (A15) provide a set of 12 linear homogeneous equations for the 12 virials listed in (A11) and (A12). Of these 12 equations, the three equations (A13) and (A15) involve only the second-order virials (A12). It is known that the determinant of this 3×3 system does not vanish anywhere along the Dedekind sequence (E.F.E., page 125). Hence

$$V_{11} = V_{22} = V_{33} = 0\,; \tag{A16}$$

and equation (A14) becomes

$$\sum_{k=1}^{3} \frac{V_{iikk}}{a_k^2} = 0 \qquad (\text{no summation over } i;\ i = 1, 2, 3)\,. \tag{A17}$$

In view of equation (A16), equations (A1) and (A17) now provide nine linear homogeneous equations for the nine fourth-order virials (A11). By eliminating V_{1111}, V_{2222}, and V_{3333} in favor of V_{1122}, V_{2233}, and V_{3311}, equation (A1) becomes linear and homogeneous in the six virials $V_{1;122}$, $V_{2;211}$, $V_{2;233}$, $V_{3;322}$, $V_{1;133}$, and $V_{3;311}$. It is found that the resulting six equations break up into a set of four homogeneous equations for V_{1122}, $V_{2;211}$, V_{2233}, and V_{1133} and a set of two equations which involves all six virials. It is found that the determinant of the 4×4 system vanishes nowhere along the Dedekind sequence. Accordingly, we must require

$$V_{1;122} = V_{2;211} = V_{2233} = V_{1133} = 0\,. \tag{A18}$$

The remaining two equations then show that, along the entire sequence, a trivial[5] neutral deformation is possible for which

$$V_{2;233} : V_{1;133} = a_2^2 : a_1^2\,. \tag{A19}$$

We conclude that *the Dedekind ellipsoid allows, nowhere along its sequence, a nontrivial neutral deformation belonging to the fourth harmonics.* But this fact does not foreclose the possibility that by seeking solutions with a time dependence of the form $e^{\lambda t}$ we may obtain for the parameter λ a characteristic value zero somewhere along the Dedekind sequence. To ascertain whether or not such a possibility occurs, we must discuss the fully time-dependent virial equations along the following lines.

Using the general time-dependent relation (A6) and considering the variation of V.E., equation (13) (with the terms in Ω^2 suppressed), we readily obtain the equation

$$\frac{d^2V_{i;jkl}}{dt^2} - 2Q_{jm}\frac{dV_{i;klm}}{dt} - 2Q_{km}\frac{dV_{i;ljm}}{dt} - 2Q_{lm}\frac{dV_{i;jkm}}{dt}$$
$$= 2[\delta\mathfrak{T}_{ij;kl} + \delta\mathfrak{T}_{ik;lj} + \delta\mathfrak{T}_{il;jk}]_0 + \delta_{ij}\delta\Pi_{kl} + \delta_{ik}\delta\Pi_{lj} + \delta_{il}\delta\Pi_{jk}$$
$$+ \tfrac{1}{3}(2\delta\mathfrak{W}_{ij;kl} + 2\delta\mathfrak{W}_{ik;lj} + 2\delta\mathfrak{W}_{il;jk} + \delta\mathfrak{W}_{ij;k;l} + \delta\mathfrak{W}_{ik;l;j} + \delta\mathfrak{W}_{il;j;k})\,, \tag{A20}$$

where Q and $[\delta\mathfrak{T}_{i;jkl}]_0$ have the same meanings as in equations (A4) and (A8).

[5] Trivial, because it corresponds to a simple rotation of the ellipsoid as a rigid body.

For the particular case of the Dedekind ellipsoids, when $\boldsymbol{Q}$ has the value given in equation (A9), we obtain from equation (A20) the following sixteen equations for determining the modes that are even in the index 3:

$$\frac{d^2V_{1;111}}{dt^2} - 6Q_1\frac{dV_{1;112}}{dt} = 6[\delta\mathfrak{T}_{11;11}]_0 + (2\delta\mathfrak{W}_{11;11} + \delta\mathfrak{W}_{11;1;1}) + 3\delta\Pi_{11}\,,$$

$$\frac{d^2V_{2;222}}{dt^2} - 6Q_2\frac{dV_{2;221}}{dt} = 6[\delta\mathfrak{T}_{22;22}]_0 + (2\delta\mathfrak{W}_{22;22} + \delta\mathfrak{W}_{22;2;2}) + 3\delta\Pi_{22}\,,$$

$$\frac{d^2V_{3;333}}{dt^2} = 6[\delta\mathfrak{T}_{33;33}]_0 + (2\delta\mathfrak{W}_{33;33} + \delta\mathfrak{W}_{33;3;3}) + 3\delta\Pi_{33}\,,$$

$$\frac{d^2V_{1;122}}{dt^2} - 2Q_1\frac{dV_{1;222}}{dt} - 4Q_2\frac{dV_{1;121}}{dt} = 2[\delta\mathfrak{T}_{11;22} + 2\delta\mathfrak{T}_{12;12}]_0 + \delta\Pi_{22} + \tfrac{1}{3}(2\delta\mathfrak{W}_{11;22} + 4\delta\mathfrak{W}_{12;12} + \delta\mathfrak{W}_{11;2;2} + 2\delta\mathfrak{W}_{12;1;2})\,,$$

$$\frac{d^2V_{2;211}}{dt^2} - 2Q_2\frac{dV_{2;111}}{dt} - 4Q_1\frac{dV_{2;212}}{dt} = 2[\delta\mathfrak{T}_{22;11} + 2\delta\mathfrak{T}_{21;21}]_0 + \delta\Pi_{11} + \tfrac{1}{3}(2\delta\mathfrak{W}_{22;11} + 4\delta\mathfrak{W}_{21;21} + \delta\mathfrak{W}_{22;1;1} + 2\delta\mathfrak{W}_{21;2;1})\,,$$

$$\frac{d^2V_{1;133}}{dt^2} - 2Q_1\frac{dV_{1;332}}{dt} = 2[\delta\mathfrak{T}_{11;33} + 2\delta\mathfrak{T}_{13;13}]_0 + \delta\Pi_{33} + \tfrac{1}{3}(2\delta\mathfrak{W}_{11;33} + 4\delta\mathfrak{W}_{13;13} + \delta\mathfrak{W}_{11;3;3} + 2\delta\mathfrak{W}_{13;1;3})\,,$$

$$\frac{d^2V_{2;233}}{dt^2} - 2Q_2\frac{dV_{2;331}}{dt} = 2[\delta\mathfrak{T}_{22;33} + 2\delta\mathfrak{T}_{23;23}]_0 + \delta\Pi_{33} + \tfrac{1}{3}(2\delta\mathfrak{W}_{22;33} + 4\delta\mathfrak{W}_{23;23} + \delta\mathfrak{W}_{22;3;3} + 2\delta\mathfrak{W}_{23;2;3})\,,$$

$$\frac{d^2V_{3;311}}{dt^2} - 4Q_1\frac{dV_{3;312}}{dt} = 2[\delta\mathfrak{T}_{33;11} + 2\delta\mathfrak{T}_{31;31}]_0 + \delta\Pi_{11} + \tfrac{1}{3}(2\delta\mathfrak{W}_{33;11} + 4\delta\mathfrak{W}_{31;31} + \delta\mathfrak{W}_{33;1;1} + 2\delta\mathfrak{W}_{31;3;1})\,,$$

$$\frac{d^2V_{3;322}}{dt^2} - 4Q_2\frac{dV_{3;321}}{dt} = 2[\delta\mathfrak{T}_{33;22} + 2\delta\mathfrak{T}_{32;32}]_0 + \delta\Pi_{22} + \tfrac{1}{3}(2\delta\mathfrak{W}_{33;22} + 4\delta\mathfrak{W}_{32;32} + \delta\mathfrak{W}_{33;2;2} + 2\delta\mathfrak{W}_{32;3;2})\,,$$

$$\frac{d^2V_{1;112}}{dt^2} - 4Q_1\frac{dV_{1;122}}{dt} - 2Q_2\frac{dV_{1;111}}{dt} = 2[2\delta\mathfrak{T}_{11;12} + \delta\mathfrak{T}_{12;11}]_0 + 2\delta\Pi_{12} + \tfrac{1}{3}(4\delta\mathfrak{W}_{11;12} + 2\delta\mathfrak{W}_{12;11} + 2\delta\mathfrak{W}_{11;1;2} + \delta\mathfrak{W}_{12;1;1})\,,$$

$$\frac{d^2V_{2;221}}{dt^2} - 4Q_2\frac{dV_{2;211}}{dt} - 2Q_1\frac{dV_{2;222}}{dt} = 2[2\delta\mathfrak{T}_{22;21} + \delta\mathfrak{T}_{21;22}]_0 + 2\delta\Pi_{12} + \tfrac{1}{3}(4\delta\mathfrak{W}_{22;21} + 2\delta\mathfrak{W}_{21;22} + 2\delta\mathfrak{W}_{22;2;1} + \delta\mathfrak{W}_{21;2;2})\,,$$

$$\frac{d^2V_{3;312}}{dt^2} - 2Q_1\frac{dV_{3;322}}{dt} - 2Q_2\frac{dV_{3;311}}{dt} = 2[\delta\mathfrak{T}_{33;12} + \delta\mathfrak{T}_{31;32} + \delta\mathfrak{T}_{32;31}]_0 + \delta\Pi_{12} + \tfrac{1}{3}(2\delta\mathfrak{W}_{33;12} + 2\delta\mathfrak{W}_{31;32} + 2\delta\mathfrak{W}_{32;31} + \delta\mathfrak{W}_{33;1;2} + \delta\mathfrak{W}_{31;3;2} + \delta\mathfrak{W}_{32;3;1})\,,$$

$$\frac{d^2V_{1;222}}{dt^2} - 6Q_2\frac{dV_{1;221}}{dt} = 6[\delta\mathfrak{T}_{12;22}]_0 + (2\delta\mathfrak{W}_{12;22} + \delta\mathfrak{W}_{12;2;2})\,,$$

$$\frac{d^2V_{2;111}}{dt^2} - 6Q_1\frac{dV_{2;112}}{dt} = 6[\delta\mathfrak{T}_{21;11}]_0 + (2\delta\mathfrak{W}_{21;11} + \delta\mathfrak{W}_{21;1;1})\,,$$

$$\frac{d^2V_{1;233}}{dt^2} - 2Q_2\frac{dV_{1;331}}{dt} = 2[\delta\mathfrak{T}_{12;33} + 2\delta\mathfrak{T}_{13;23}]_0 + \tfrac{1}{3}(2\delta\mathfrak{W}_{12;33} + 4\delta\mathfrak{W}_{13;23} + \delta\mathfrak{W}_{12;3;3} + 2\delta\mathfrak{W}_{13;2;3})\,,$$

$$\frac{d^2V_{2;133}}{dt^2} - 2Q_1\frac{dV_{2;332}}{dt} = 2[\delta\mathfrak{T}_{21;33} + 2\delta\mathfrak{T}_{23;13}]_0 + \tfrac{1}{3}(2\delta\mathfrak{W}_{21;33} + 4\delta\mathfrak{W}_{23;13} + \delta\mathfrak{W}_{21;3;3} + 2\delta\mathfrak{W}_{23;1;3})\,. \qquad \text{(A21)}$$

The matrix elements $[\delta \mathfrak{T}_{i;jkl}]_0$ which occur in the foregoing equations, not included in the earlier listing (A10), have the values

$$[2\delta \mathfrak{T}_{11;12}]_0 = -4Q_1Q_2V_{1;112} + Q_1^2(V_{2;221} - V_{1;222}),$$

$$[2\delta \mathfrak{T}_{22;21}]_0 = -4Q_1Q_2V_{2;221} + Q_2^2(V_{1;112} - V_{2;111}),$$

$$[2\delta \mathfrak{T}_{12;11}]_0 = -Q_1Q_2(V_{1;112} + V_{2;111}) - 2Q_1^2V_{2;221},$$

$$[2\delta \mathfrak{T}_{21;22}]_0 = -Q_1Q_2(V_{2;221} + V_{1;222}) - 2Q_2^2V_{1;112},$$

$$[2\delta \mathfrak{T}_{12;33}]_0 = -Q_1Q_2(V_{1;332} + V_{2;331} - 2V_{3;312}),$$

$$[2\delta \mathfrak{T}_{13;23}]_0 = [2\delta \mathfrak{T}_{23;13}]_0 = -2Q_1Q_2V_{3;312},$$

and

$$[2\delta \mathfrak{T}_{33;12}]_0 = 0, \tag{A22}$$

and the required expressions $\delta \mathfrak{W}_{ij;kl}$ and $\delta \mathfrak{W}_{ij;k;l}$ are given in V.E., equations (21) and (22).

Equations (A20) supplemented by the divergence condition (A15) and the further condition,

$$\sum_{k=1}^{3} \frac{V_{12kk}}{a_k^2} = V_{12}, \tag{A23}$$

provide a complete set of equations for determining the characteristic values of λ. (Because of the complete separability of the second-order virial equations from those of the fourth order, we can set $V_{12} = 0$; cf. the arguments preceding eq. [A16].)

It is now clear that the full equations might very well provide a zero-frequency mode for some particular member of the Dedekind sequence. The problem here is analogous to the one which we encounter in the context in the Roche ellipsoids (cf. E.F.E., pp. 205–207, and Lebovitz 1963). (Note that eq. [A1] follows from eqs. [A20] when the state is stationary and the only virials that do not vanish are those even in all three indices 1, 2, and 3.)

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