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ON SLOWLY ROTATING HOMOGENEOUS MASSES IN GENERAL RELATIVITY

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SUMMARY

The present paper is devoted to a study of slowly rotating homogeneous masses in which the energy density ϵ is a constant. The structure of such configurations is determined with the aid of equations derived by Hartle in the exact framework of general relativity. These configurations have a natural limit in that the static, non-rotating, configurations must have radii (R) exceeding 9/8 times the Schwarzschild radius (R_S). The derived structures, for varying R/R_S, are illustrated by a series of graphs. A result of particular interest which emerges is that the ellipticity of the configuration, for varying radius but constant mass and angular momentum, exhibits a very pronounced maximum at $R/R_S \sim 2{\cdot}4$.

1. INTRODUCTION

The equations which govern equilibrium configurations in slow uniform rotation, in the framework of general relativity, have been derived by Hartle* (1967); they have been used by Hartle & Thorne (1968) for constructing models for slowly rotating neutron stars. In this paper, we shall use Hartle's equations to delineate the structure of slowly rotating masses in which the energy density ϵ is a constant. Apart from the fact that the case ϵ = constant provides the simplest illustration of the effects of slow rotation in general relativity, it will enable us to study these effects under conditions which are more 'extreme' than any that one encounters with more normal equations of state. Thus, in the case of 'realistic' neutron-star models, the requirements of stability with respect to radial pulsations (*cf.* Chandrasekhar 1964) restrict the models to radii that exceed 2·5 times the Schwarzschild radius; at these radii, while the effects of general relativity are substantial they are by no means 'overwhelming'. In contrast, homogeneous models can occur stably (for γ, the 'ratio of the specific heats', tending to infinity) down to 9/8 of the Schwarzschild radius; and at this radius, the effects of general relativity are as strong as they can ever become under conditions of static hydrostatic equilibrium. As we shall see the results which emerge from the study of these homogeneous models are qualitatively different from those that have, hitherto, been deduced.

* The following misprints in Hartle's paper have been noted.
In equation (36) the exponent of the expression for u^t in square brackets should be $-\frac{1}{2}$.
In equation (117), on the right-hand side, replace $(a-M)$ by $(a-2M)$.
In equation (124), replace $1/R$ in the third, fourth and fifth terms by $1/R^2$.
In equation (130), on the right-hand side, replace $-4/3$ by $+2/3$.
In equation (138), on the right-hand side, replace 16 by 8.

2. HARTLE'S EQUATIONS

In this section, we shall assemble Hartle's equations in a form that will be convenient.

The starting point is a consideration of the metric that will describe a fluid configuration which is slowly rotating with a uniform angular velocity Ω and relate it to the metric which will describe the same configuration in the non-rotating state.

The metric that is most suitable for describing spherically symmetric static configurations is the standard Schwarzschild form*

$$ds^2 = -e^{2\nu_0}\,dt^2 + e^{2\lambda_0}\,dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\phi^2), \tag{1}$$

where

$$e^{2\lambda_0} = \frac{r}{r-2M}, \quad M = M(r) = 4\pi\int_0^r \epsilon_0 r^2\,dr,$$

$$2\frac{d\nu_0}{dr} + \frac{1}{r} = r\,e^{2\lambda_0}\left(\frac{1}{r^2} + 8\pi p_0\right), \tag{2}$$

and

$$\frac{dp_0}{dr} = -(\epsilon_0 + p_0)\frac{d\nu_0}{dr}. \tag{3}$$

In the foregoing equations, the subscript '0' distinguishes that the quantities refer to the non-rotating configuration.

When the configuration is set into slow rotation and the isobaric surfaces become slightly oblate spheroids, the form of the metric that is most suitable and the one chosen by Hartle is

$$\begin{aligned} ds^2 = &-e^{2\nu_0}[1 + 2h_0(r) + 2h_2(r)\,P_2(\cos\theta)]\,dt^2 \\ &+ e^{2\lambda_0}\left\{1 + \frac{e^{2\lambda_0}}{r}[2m_0(r) + 2m_2(r)\,P_2(\cos\theta)]\right\}dr^2 \\ &+ r^2[1 + 2k_2(r)\,P_2(\cos\theta)]\{d\theta^2 + [d\phi - \omega(r)\,dt]^2\sin^2\theta\,d\phi^2\}, \end{aligned} \tag{4}$$

where $P_2(\cos\theta)$ denotes Legendre's polynomial of order 2 and h_0, h_2, m_0, m_2 and k_2 are quantities of order Ω^2 and functions of the radial coordinate r only, and ω is a further radial function (describing the so-called dragging of the inertial frame) which is of order Ω.

It will be observed that the chosen metric (4) is consistent with the form,

$$ds^2 = -e^{2\nu}(dx^0)^2 + e^{2\psi}(dx^1 - \omega\,dx^0)^2 + e^{2\mu_2}(dx^2)^2 + e^{2\mu_3}(dx^3)^2, \tag{5}$$

which one generally assumes for stationary axisymmetric systems in general relativity. Indeed, by making the identifications $dx^0 = dt$, $dx^1 = d\phi$, $dx^2 = dr$ and $dx^3 = d\theta$, and the substitutions

$$e^{\nu} = e^{\nu_0}[1 + h_0(r) + h_2(r)\,P_2(\cos\theta)],$$

$$e^{\psi} = r\sin\theta[1 + k_2(r)\,P_2(\cos\theta)],$$

$$e^{\mu_2} = e^{\lambda_0}\left\{1 + \frac{e^{2\lambda_0}}{r}[m_0(r) + m_2(r)\,P_2(\cos\theta)]\right\},$$

and

$$e^{\mu_3} = r[1 + k_2(r)\,P_2(\cos\theta)], \tag{6}$$

* We are using units in which $c = G = 1$.

(which are correct to the first order in Ω^2) in the equations that have been written down for the general metric (5) (in Chandrasekhar & Friedman 1972, Section II, for example) we can obtain the equations governing the field and the fluid appropriate for the metric (4). Thus, in the case $\epsilon = \epsilon(p)$, the equation of hydrodynamic equilibrium

$$p_{,\alpha} = (\epsilon+p)\left\{\log\frac{e^{-\nu}}{(1-V^2)^{1/2}}\right\}_{,\alpha} \quad (\alpha = r, \theta), \tag{7}$$

where

$$V = e^{\psi-\nu}(\Omega-\omega), \tag{8}$$

can be integrated to give

$$\nu+\tfrac{1}{2}\log(1-V^2)+P = \text{constant}, \tag{9}$$

where

$$P = \log(\epsilon+p)-\int^{\epsilon}\frac{d\epsilon}{\epsilon+p}, \tag{10}$$

or, equivalently,

$$(\epsilon+p)\,dP = dp. \tag{11}$$

Expanding P, appropriately for slow rotation, in the form*

$$P = P_0(r)+\delta P_0(r)+\delta P_2(r)\,P_2(\cos\theta), \tag{12}$$

and noting that

$$V = e^{-\nu_0}(\Omega-\omega)\,r\sin\theta+O(\Omega^2), \tag{13}$$

we obtain from equation (9) the pair of equations

$$h_0(r)-\tfrac{1}{3}r^2\,e^{-2\nu_0}\varpi^2+\delta P_0 = C = \text{constant of } O(\Omega^2) \tag{14}$$

and

$$h_2(r)+\tfrac{1}{3}r^2\,e^{-2\nu_0}\varpi^2+\delta P_2 = 0, \tag{15}$$

where we have written

$$\varpi = \Omega-\omega. \tag{16}$$

Considering next the various field equations appropriate for the metric (5) and linearizing them about the known spherically symmetric solutions (2) and (3), we obtain the following equations, where we have indicated, in each case, the component of the field equation from which it arises.

$$\frac{1}{r^3}\frac{d}{dr}\left(r^4 j\frac{d\varpi}{dr}\right)+4\frac{dj}{dr}\varpi = 0, \quad (G^{01}) \tag{17}$$

$$\frac{dm_0}{dr} = 4\pi r^2(\epsilon+p)\frac{d\epsilon}{dp}\delta P_0+\frac{1}{12}r^4 j^2\left(\frac{d\varpi}{dr}\right)^2-\frac{1}{3}r^3\varpi^2\frac{dj^2}{dr}, \quad (G^{00}) \tag{18}$$

$$\begin{aligned}\frac{dh_0}{dr} &= -\frac{d}{dr}\delta P_0+\frac{1}{3}\frac{d}{dr}(r^2\,e^{-2\nu_0}\varpi^2)\\ &= m_0\,e^{4\lambda_0}\left(\frac{1}{r^2}+8\pi p_0\right)-\frac{1}{12}e^{2\lambda_0}r^3 j^2\left(\frac{d\varpi}{dr}\right)^2+4\pi r\,e^{2\lambda_0}(\epsilon+p)\,\delta P_0, \quad (G^{22})\end{aligned} \tag{19}$$

$$\frac{d}{dr}(h_2+k_2) = h_2\left(\frac{1}{r}-\frac{d\nu_0}{dr}\right)+\frac{m_2}{r-2M}\left(\frac{1}{r}+\frac{d\nu_0}{dr}\right), \quad (G^{23}) \tag{20}$$

$$h_2+\frac{m_2}{r-2M} = \frac{1}{6}r^4 j^2\left(\frac{d\varpi}{dr}\right)^2-\frac{1}{3}r^3\varpi^2\frac{dj^2}{dr}, \quad (G^{11}-G^{33}) \tag{21}$$

* It is awkward that 'P_2' occurs in this equation with two different meanings: as δP_2 and as $P_2(\cos\theta)$. However, since P_2 as the Legendre function $P_2(\cos\theta)$ occurs only in equations (4), (6), (12) and (72) the awkwardness is not a serious one.

and

$$\frac{2}{r}\left(1-\frac{2M}{r}\right)\frac{dh_2}{dr}+2\left(1-\frac{2M}{r}\right)\left(\frac{d\nu_0}{dr}+\frac{1}{r}\right)\frac{dk_2}{dr}-\frac{2m_2}{r^2}\left(2\frac{d\nu_0}{dr}+\frac{1}{r}\right)$$
$$-\frac{6h_2}{r^2}-\frac{4k_2}{r^2}-8\pi(\epsilon+p)\,\delta P_2-\frac{1}{6}r^2j^2\left(\frac{d\varpi}{dr}\right)^2=0, \quad (G^{22}) \qquad (22)$$

where

$$j=e^{-(\lambda_0+\nu_0)}. \qquad (23)$$

Letting

$$v_2=h_2+k_2, \qquad (24)$$

and making use of equations (15) and (21), we can reduce equations (20) and (22) to the forms

$$\frac{dv_2}{dr}=-2\frac{d\nu_0}{dr}h_2+\left(\frac{1}{r}+\frac{d\nu_0}{dr}\right)\left[\frac{1}{6}r^4j^2\left(\frac{d\varpi}{dr}\right)^2-\frac{1}{3}r^3\varpi^2\frac{dj^2}{dr}\right], \qquad (25)$$

and

$$\begin{aligned}\frac{dh_2}{dr}=&-\frac{2v_2}{r(r-2M)\,d\nu_0/dr}\\&+\left\{-2\frac{d\nu_0}{dr}+\frac{r}{2(r-2M)\,d\nu_0/dr}\left[8\pi(\epsilon+p)-\frac{4M}{r^3}\right]\right\}h_2\\&+\frac{1}{6}\left[r\frac{d\nu_0}{dr}-\frac{1}{2(r-2M)\,d\nu_0/dr}\right]r^3j^2\left(\frac{d\varpi}{dr}\right)^2\\&-\frac{1}{3}\left[r\frac{d\nu_0}{dr}+\frac{1}{2(r-2M)\,d\nu_0/dr}\right]r^2\varpi^2\frac{dj^2}{dr}.\end{aligned} \qquad (26)$$

The foregoing equations apply to the interior of the fluid configuration. Outside the configuration, in the vacuum,

$$\epsilon=p=0, \quad j=1 \quad \text{and} \quad M(r)=M_0 \qquad (27)$$

and the resulting equations can be solved explicitly to give

$$\varpi=\Omega-\frac{2J}{r^3}, \quad m_0=\delta M-\frac{J^2}{r^3}, \qquad (28)$$

$$h_0=-\frac{\delta M}{r-2M_0}+\frac{J^2}{r^3(r-2M_0)}, \qquad (29)$$

$$h_2=J^2\left(\frac{1}{M_0r^3}+\frac{1}{r^4}\right)+KQ_2{}^2\left(\frac{r}{M_0}-1\right), \qquad (30)$$

and

$$v_2=-\frac{J^2}{r^4}+K\frac{2M_0}{[r(r-2M_0)]^{1/2}}Q_2{}^1\left(\frac{r}{M_0}-1\right), \qquad (31)$$

where δM and K are constants, J is the angular momentum of the configuration, and $Q_n{}^m$ is the associated Legendre function of the second kind for the argument, $(r/M_0)-1$, specified.

The boundary conditions with respect to which equations (17), (18), (19), (25) and (26) must be solved are

m_0, h_0, h_2 and v_2 vanish at the origin while ϖ is finite; and at the boundary of the configuration, all of these functions join continuously with their exterior solutions (28)–(31). (32)

3. THE EQUATIONS GOVERNING SLOW ROTATION OF HOMOGENEOUS CONFIGURATIONS

The structure of spherically symmetric static distributions of matter when ϵ = constant follow from the solutions of equations (2) and (3) appropriate for this case. These solutions are well known and can be found in any standard textbook. Measuring the radial coordinate r in the unit*

$$\alpha = \left(\frac{3c^4}{8\pi G\epsilon}\right)^{1/2}, \tag{33}$$

the required solutions are

$$e^{\lambda_0} = \frac{1}{y}, \quad e^{\nu_0} = \tfrac{1}{2}(3y_1 - y), \quad \text{and} \quad \frac{p}{\epsilon} = \frac{y-y_1}{3y_1-y}, \tag{34}$$

where

$$r = (1-y^2)^{1/2}, \quad y_1{}^2 = 1 - R^2/\alpha^2, \tag{35}$$

and R is the radius of the configuration. Also, corresponding to the solution (34)

$$j = \frac{2y}{3y_1-y} \quad \text{and} \quad \frac{2M(r)}{r} = 1 - e^{-2\lambda_0} = 1-y^2. \tag{36}$$

It will be observed that the foregoing solutions require that

$$y_1 > \tfrac{1}{3}. \tag{37}$$

This last condition, as may be readily verified, is equivalent to the requirement,

$$R > \frac{9}{8} R_S = \frac{9}{8} \cdot \frac{2GM}{c^2}, \tag{38}$$

to which we referred in Section 1.†

The equations governing homogeneous configurations in slow uniform rotation can now be obtained by substituting the solutions (34)–(36) in the equations of Section 2. In writing the resulting equations, it is convenient to use

$$x = 1-y = 1-(1-r^2)^{1/2} \tag{39}$$

as the independent variable. We find that equations (17), (18), (19), (25) and (26)

* Strictly, we should write $\alpha = (3/8\pi\epsilon)^{1/2}$ since we have already adopted units in which $c = G = 1$.

† Quite generally, it may be noted that according to equation (36) $R/R_S = (1-y_1{}^2)^{-1}$.

can be reduced to the forms

$$x[2\kappa+(2-\kappa)\,x-x^2]\,\frac{d^2\varpi}{dx^2}+[5\kappa+(3-5\kappa)\,x-4x^2]\,\frac{d\varpi}{dx}-4(\kappa+1)\,\varpi = 0, \tag{40}$$

$$\frac{dm_0}{dx} = \frac{(1-x)\,x^{3/2}(2-x)^{3/2}}{(\kappa+x)^2}\left[\frac{1}{3}\,x(2-x)\left(\frac{d\varpi}{dx}\right)^2+\frac{8(\kappa+1)}{3(\kappa+x)}\,\varpi^2\right], \tag{41}$$

$$\begin{aligned}\frac{d}{dx}\,\delta P_0 = {} & -\frac{\kappa+1}{(1-x)(\kappa+x)}\,\delta P_0-\frac{2+(\kappa+1)(1-x)-3(1-x)^2}{(\kappa+x)(1-x)^2\,x^{3/2}(2-x)^{3/2}}\,m_0\\ & +\frac{8x(2-x)}{3(\kappa+x)^2}\,\varpi\,\frac{d\varpi}{dx}+\frac{x^2(2-x)^2}{3(1-x)(\kappa+x)^2}\left(\frac{d\varpi}{dx}\right)^2\\ & -\frac{8}{3}\cdot\frac{1-(\kappa+1)(1-x)}{(\kappa+x)^3}\,\varpi^2,\end{aligned} \tag{42}$$

$$\begin{aligned}\frac{dv_2}{dx} = {} & -\frac{2h_2}{\kappa+x}\\ & +\frac{2x^2(2-x)^2}{3(\kappa+x)^3}\,[1+(\kappa+1)(1-x)-2(1-x)^2]\left[\left(\frac{d\varpi}{dx}\right)^2+\frac{4(\kappa+1)}{x(\kappa+x)(2-x)}\,\varpi^2\right],\end{aligned} \tag{43}$$

and

$$\begin{aligned}\frac{dh_2}{dx} = {} & \frac{(1-x)^2+(\kappa+1)(1-x)-2}{x(2-x)(\kappa+x)}\,h_2-\frac{2(\kappa+x)}{x^2(2-x)^2}\,v_2\\ & +\frac{1}{3}\,[2x^2(2-x)^2-(\kappa+x)^2]\,\frac{x(2-x)}{(\kappa+x)^3}\left(\frac{d\varpi}{dx}\right)^2\\ & +\frac{4}{3}\,(\kappa+1)[2x^2(2-x)^2+(\kappa+x)^2]\,\frac{\varpi^2}{(\kappa+x)^4},\end{aligned} \tag{44}$$

where we have measured

ϖ in the unit ϖ_c, the value of ϖ at the centre, m_0 in the unit $\alpha^3\varpi_c^2$

and

δP_0, δP_2, h_2, k_2 and v_2 in the unit $\alpha^2\varpi_c^2$; (45)

and we have also written

$$\kappa = 3y_1-1. \tag{46}$$

Equations (40)–(44) must be supplemented by equations (14), (15) and (21) which, in the present context, take the forms

$$h_0-\frac{4x(2-x)}{3(\kappa+x)^2}\,\varpi^2+\delta P_0 = \gamma = \text{constant}, \tag{47}$$

$$h_2+\frac{4x(2-x)}{3(\kappa+x)^2}\,\varpi^2+\delta P_2 = 0, \tag{48}$$

and

$$h_2+\frac{m_2}{(1-x)^2\,x^{1/2}(2-x)^{1/2}} = \frac{2x^3(2-x)^3}{3(\kappa+x)^2}\left[\left(\frac{d\varpi}{dx}\right)^2+\frac{4(\kappa+1)\,\varpi^2}{x(2-x)(\kappa+x)}\right]. \tag{49}$$

Besides, we have equations (28)–(31) which determine the functions outside the configuration; these equations now take the forms

$$\varpi = \Omega - \frac{2J}{r^3}, \tag{50}$$

$$m_0 = \delta M - \frac{J^2}{r^3}, \quad h_0 = -\frac{m_0}{r-(1-y_1^2)^{3/2}}, \tag{51}$$

$$h_2 = \left[\frac{2}{(1-y_1^2)^{3/2}} + \frac{1}{r}\right] \frac{J^2}{r^3} + KQ_2^2, \tag{52}$$

and

$$v_2 = -\frac{J^2}{r^4} + K \frac{(1+y_1^2)^{3/2}}{r^{1/2}[r-(1-y_1^2)^{3/2}]^{1/2}} Q_2^1 \tag{53}$$

where

Ω is measured in the unit ϖ_c,

J is measured in the unit $\alpha^3\varpi_c$,

and

$$\delta M \text{ is measured in the unit } \alpha^3\varpi_c^2; \tag{54}$$

and the argument of the associated Legendre functions is $[2r/(1-y_1^2)^{3/2}]-1$.

At the boundary of the configuration, where the solutions of equations (40)–(44) must match those given by equations (50)–(53), we must have

$$\varpi_1 = \Omega - \frac{2J}{(1-y_1^2)^{3/2}}, \tag{55}$$

$$(m_0)_1 = \delta M - \frac{J^2}{(1-y_1^2)^{3/2}}, \quad (h_0)_1 = -\frac{(m_0)_1}{y_1^2(1-y_1^2)^{1/2}}, \tag{56}$$

$$(h_2)_1 = \frac{3-y_1^2}{(1-y_1^2)^3} J^2 + KQ_2^2\left(\frac{1+y_1^2}{1-y_1^2}\right), \tag{57}$$

and

$$(v_2)_1 = -\frac{J^2}{(1-y_1^2)^2} + K\frac{1-y_1^2}{y_1} Q_2^1\left(\frac{1+y_1^2}{1-y_1^2}\right), \tag{58}$$

where the subscript '1' distinguishes the value of function at the boundary where $x = x_1 = 1-y_1$.

Returning to equations (40)–(44), we observe that equation (40) for ϖ does not involve any of the other quantities; accordingly, it can be integrated, for any assigned value of $y_1 = \frac{1}{3}(1+\kappa)$, independently of the others. Near the origin ($x = 0$) it has the behaviour

$$\varpi = 1 + \frac{4(\kappa+1)}{5\kappa} x + O(x^2), \tag{59}$$

remembering that we are measuring ϖ in the unit ϖ_c, its value at the centre. The solution of equation (40), with the behaviour (59) at $x = 0$, when integrated out to the boundary, will, according to equation (55), determine the angular momentum (in the unit $\alpha^3\varpi_c$) that is to be associated with an assigned Ω (in the unit ϖ_c).

In terms of the solution for ϖ, equations (41) and (42) can be integrated for m_0 and δP_0. Near the origin they have the behaviours

$$m_0 = \frac{32(\kappa+1)\sqrt{2}}{15\kappa^3} x^{5/2} + O(x^{7/2})$$

and

$$\delta P_0 = \frac{8}{3\kappa^2}x + \mathrm{O}(x^2). \tag{60}$$

The solution of equations (41) and (42), with these initial behaviours, can be integrated to the boundary. The value of m_0 at the boundary will determine, according to equation (56), the value of δM, the increase in M_0 consequent to the rotation. Similarly, the solution for δP_0, together with equations (47) and (56), will determine h_0 and the value of the constant C.

The boundary conditions on h_2 and v_2 require that they vanish at the centre and further, that they join continuously with the exterior solutions (52) and (53). On this account, it will be convenient to treat equations (43) and (44) in the following manner.

Expressing the solutions of equations (43) and (44) as the sum of a particular integral and a complementary function (distinguished by the superscripts P and C, respectively) we write

$$h_2 = h_2{}^{(\mathrm{P})} + A h_2{}^{(\mathrm{C})} \quad \text{and} \quad v_2 = v_2{}^{(\mathrm{P})} + A v_2{}^{(\mathrm{C})}, \tag{61}$$

where A is a constant and the complementary functions are solutions of the homogeneous equations

$$\frac{dv_2{}^{(\mathrm{C})}}{dx} = -\frac{2h_2{}^{(\mathrm{C})}}{\kappa + x}, \tag{62}$$

and

$$\frac{dh_2{}^{(\mathrm{C})}}{dx} = \frac{(1-x)^2 + (\kappa+1)(1-x) - 2}{x(2-x)(\kappa+x)} h_2{}^{(\mathrm{C})} - \frac{2(\kappa+x)}{x^2(2-x)^2} v_2{}^{(\mathrm{C})}. \tag{63}$$

Near the origin, the solutions of equations (43) and (44) are found to have the behaviours

$$h_2{}^{(\mathrm{P})} = ax + \mathrm{O}(x^2) \quad \text{and} \quad v_2{}^{(\mathrm{P})} = bx^2 + \mathrm{O}(x^3), \tag{64}$$

where a and b are constants related in the manner

$$\kappa^3 b + \kappa^2 a = 8y_1; \tag{65}$$

while the solutions of the homogeneous equations (62) and (63) have the behaviours

$$h_2{}^{(\mathrm{C})} = -\kappa B x + \mathrm{O}(x^2) \quad \text{and} \quad v_2{}^{(\mathrm{C})} = Bx^2 + \mathrm{O}(x^3) \tag{66}$$

where B is a constant.

The particular integrals $h_2{}^{(\mathrm{P})}$ and $v_2{}^{(\mathrm{P})}$ can be found by integrating equations (43) and (44) out from the origin with the initial behaviours specified in equations (64) and (65) for an arbitrarily assigned value of the constant a. Similarly the complementary functions $h_2{}^{(\mathrm{C})}$ and $v_2{}^{(\mathrm{C})}$ can be found by integrating equations (62) and (63) with the initial behaviours specified in equations (66) for an arbitrarily assigned value of the constant B. The constant A (in the superposition (61)) and the constant K (in equations (52) and (53)) can then be determined from the required continuity of h_2 and v_2 with the exterior solutions (52) and (53). With h_2 and v_2 thus determined, the solutions for k_2 and m_2 follow from equations (24) and (49).

(a) *The case* $y_1 = \frac{1}{3}$

Even though the pressure distribution in the static non-rotating configuration has a singularity at $r = 0$ when

$$y_1 \to \tfrac{1}{3} \quad \text{and} \quad \kappa \to 0, \tag{67}$$

the equations governing the slowly rotating configurations have finite forms in this limit. But the behaviour of ϖ near the origin, given by equation (59), is divergent. However, considering the equations for $\kappa = 0$, *ab initio*, we find that they allow non-singular solutions. Thus, the equation for ϖ, in this limit, namely,

$$x^2(2-x)\frac{d^2\varpi}{dx^2}+x(3-4x)\frac{d\varpi}{dx}-4\varpi = 0, \tag{68}$$

allows a solution having the behaviour

$$\varpi = Ax^n+O(x^{n+1}), \tag{69}$$

where

$$n = \frac{\sqrt{33}}{4}-\frac{1}{4}, \tag{70}$$

and A is a constant*. A similar discussion of the remaining equations yields the behaviours

$$m_0 = \frac{8\sqrt{2}}{3}\cdot\frac{n^2+4}{4n-1}x^{2n-1/2}+O(x^{2n+1/2}),$$

$$\delta P_0 = \frac{4(2n^3+5n^2-6n-9)}{3n(4n-1)}x^{2n-1}+O(x^{2n}),$$

$$v_2^{(P)} = \frac{32(3n^3-n^2+6n-3)}{3(8n^2-2n-3)}x^{2n-1}+O(x^{2n}),$$

$$h_2^{(P)} = \frac{4(14n^3-13n^2+36n-30)}{3(8n^2-2n-3)}x^{2n-1}+O(x^{2n}),$$

and

$$h_2^{(C)} = -\tfrac{1}{4}Bx^{1/2}+O(x^{3/2}), \quad v_2^{(C)} = Bx^{1/2}+O(x^{3/2}), \tag{71}$$

where B is a constant.† With the foregoing initial behaviours, the relevant equations can be integrated and the solution completed in the manner already described.

(b) *The equation for the isobaric surfaces*

Writing the equation for the isobaric surfaces in the form,

$$r(p) = r_0+\xi_0(r_0)+\xi_2(r_0)\,P_2(\cos\theta), \tag{72}$$

where r_0 is the radius of the spherical surface in the non-rotating configuration on which the pressure is p, we conclude from the definitions (11) and (12) that

$$\delta P_0 = -\left(\frac{1}{\epsilon+p}\frac{dp}{dr}\right)_0\xi_0(r_0) \quad\text{and}\quad \delta P_2 = -\left(\frac{1}{\epsilon+p}\frac{dp}{dr}\right)_0\xi_2(r_0). \tag{73}$$

Alternatively, by making use of equation (3), we can also write

$$\delta P_0 = \frac{d\nu_0}{dr}\xi_0(r) \quad\text{and}\quad \delta P_2 = \frac{d\nu_0}{dr}\xi_2(r), \tag{74}$$

where we have not distinguished between r and r_0 as the distinction is no longer meaningful.

* A solution with $n = -0{\cdot}25-\sqrt{33}/4$ is also possible; but we ignore this possibility as inadmissible.

† The units in which the various quantities are now measured are those specified in equation (45) with ϖ_c replaced by the constant A in the solution (69).

In the present context, equations (74) give

$$\xi_0 = \frac{(1-x)(\kappa+x)}{x^{1/2}(2-x)^{1/2}}\,\delta P_0 \quad \text{and} \quad \xi_2 = \frac{(1-x)(\kappa+x)}{x^{1/2}(2-x)^{1/2}}\,\delta P_2. \tag{75}$$

Making use of equation (48), we can rewrite the foregoing expression for ξ_2 in the form

$$\xi_2 = -\frac{(1-x)(\kappa+x)}{x^{1/2}(2-x)^{1/2}}\left[h_2+\frac{4x(2-x)}{3(\kappa+x)^2}\,\varpi^2\right]. \tag{76}$$

The ellipticity of the isobaric surfaces is given by (*cf.* Thorne 1971)

$$\epsilon(r) = -\frac{3}{2}\left[\frac{\xi_2(r)}{r}+v_2(r)-h_2(r)\right], \tag{77}$$

or, according to equation (76)

$$\epsilon(r) = \frac{3(1-x)(\kappa+x)}{2x(2-x)}\left[h_2+\frac{4x(2-x)}{3(\kappa+x)^2}\,\varpi^2\right]-\frac{3}{2}(v_2-h_2). \tag{78}$$

In particular, the ellipticity of the bounding surface is given by

$$\epsilon(y_1) = \frac{3y_1^2}{1-y_1^2}\left\{h_2(y_1)+\frac{1-y_1^2}{3y_1^2}\left[\Omega-\frac{2J}{(1-y_1^2)^{3/2}}\right]^2\right\}-\frac{3}{2}[v_2(y_1)-h_2(y_1)], \tag{79}$$

where we have substituted for ϖ_1 its value given in equation (55). (Note that we have measured ξ_0 and ξ_2 in the unit $\alpha^3\varpi_c^2$ and ϵ in the unit $\alpha^2\varpi_c^2$.)

4. NUMERICAL RESULTS AND ILLUSTRATIONS

The equations derived in Section 3 were integrated for various initially assigned values of $R/R_S = (1-y_1^2)^{-1}$, where $R_S(=2GM/c^2)$ denotes the Schwarzschild radius. The integrations were carried out using the method of Nordsieck (1962).

In Table I, we list some of the deduced integral properties of the models as well as quantities that characterize the bounding surface. The principal results of the integrations are further illustrated in Figs. 1–8.

Both in the table and in the figures, dimensionless variables are used and the units in which the various quantities are expressed are given at the bottom of Table I and in the caption to each of the figures. One may obtain the physical parameters appropriate to a configuration of given mass (M), radius (R), and angular velocity (Ω), from the information provided in the table and in the figures, as follows. (i) Calculate the Schwarzschild radius $R_S = 2GM/c^2$ from the known mass and express the radius R in the unit R_S; R/R_S is the principal argument that is used. (ii) From Fig. 2 or from Table I, obtain (by reading or by interpolation) the value of I appropriate to R/R_S. The angular momentum J of the configuration in CGS units is then given by $I(R_S{}^3c^2/G)\,\Omega$. (iii) With the deduced value of J and the known value of R_S, evaluate (in CGS units) the units (listed at the bottom of Table I) in which the various quantities are expressed. (iv) From the table or from the graphs, the value of any desired quantity, in the units in which they are expressed, can be inferred; and with the values of the units known, the inferred values can be converted into CGS units.

While adequate descriptions accompany the figures, we may draw attention to the following specific features.

TABLE I

*Integral and surface properties of slowly rotating homogeneous configurations in general relativity**

R/R_S	I	I/MR^2	$\delta M/M$	Q	ϖ_1	ξ_0	ξ_2	ϵ
1·125	5·057 (−1)	7·991 (−1)	3·442 (0)	2·002 (0)	5·7270 (−1)	−4·7390 (−1)	−9·2667 (−1)	5·2307 (0)
1·15	5·105 (−1)	7·720 (−1)	3·454 (0)	2·076 (0)	6·4391 (−1)	−4·2403 (−1)	−1·3942 (0)	6·0898 (0)
1·2	5·248 (−1)	7·289 (−1)	3·412 (0)	2·169 (0)	7·4818 (−1)	−2·7505 (−1)	−2·0869 (0)	6·7278 (0)
1·3	5·657 (−1)	6·695 (−1)	3·225 (0)	2·441 (0)	8·5741 (−1)	+1·2203 (−1)	−3·5565 (0)	7·9708 (0)
1·4	6·171 (−1)	6·297 (−1)	2·993 (0)	2·813 (0)	8·9174 (−1)	+5·8490 (−1)	−5·0402 (0)	9·0327 (0)
1·5	6·758 (−1)	6·007 (−1)	2·757 (0)	3·272 (0)	8·8714 (−1)	+1·0740 (0)	−6·4841 (0)	9·8930 (0)
1·6	7·406 (−1)	5·786 (−1)	2·533 (0)	3·806 (0)	8·6205 (−1)	+1·5681 (0)	−7·8651 (0)	1·0569 (+1)
1·7	8·106 (−1)	5·610 (−1)	2·327 (0)	4·406 (0)	8·2652 (−1)	+2·0553 (0)	−9·1734 (0)	1·1088 (+1)
1·8	8·856 (−1)	5·467 (−1)	2·140 (0)	5·064 (0)	7·8622 (−1)	+2·5287 (0)	−1·0407 (+1)	1·1478 (+1)
1·9	9·653 (−1)	5·348 (−1)	1·971 (0)	5·773 (0)	7·4440 (−1)	+2·9847 (0)	−1·1566 (+1)	1·1761 (+1)
2·0	1·049 (0)	5·245 (−1)	1·819 (0)	6·525 (0)	7·0294 (−1)	+3·4213 (0)	−1·2655 (+1)	1·1960 (+1)
2·5	1·534 (0)	4·909 (−1)	1·259 (0)	1·081 (1)	5·2376 (−1)	+5·3097 (0)	−1·7183 (+1)	1·2131 (+1)
3·0	2·123 (0)	4·718 (−1)	9·163 (−1)	1·567 (1)	3·9700 (−1)	+6·7734 (0)	−2·0549 (+1)	1·1658 (+1)
4·0	3·604 (0)	4·505 (−1)	5·440 (−1)	2·636 (1)	2·4623 (−1)	+8·8387 (0)	−2·5151 (+1)	1·0300 (+1)
5·0	5·487 (0)	4·390 (−1)	3·588 (−1)	3·773 (1)	1·6625 (−1)	+1·0206 (+1)	−2·8126 (+1)	9·0297 (0)
10·0	2·091 (+1)	4·182 (−1)	9·493 (−2)	9·785 (1)	4·5821 (−2)	+1·3235 (+1)	−3·4564 (+1)	5·3517 (0)
20·0	8·177 (+1)	4·089 (−1)	2·437 (−2)	2·217 (2)	1·1980 (−2)	+1·4900 (+1)	−3·8031 (+1)	2·8966 (0)
35·0	2·481 (+2)	4·051 (−1)	8·046 (−3)	4·086 (2)	3·9848 (−3)	+1·5644 (+1)	−3·9569 (+1)	1·7106 (0)
50·0	5·043 (+2)	4·034 (−1)	3·960 (−3)	5·960 (2)	1·9668 (−3)	+1·5948 (+1)	−4·0192 (+1)	1·2131 (0)
100·0	2·009 (+3)	4·018 (−1)	9·950 (−4)	1·221 (3)	4·9585 (−4)	+1·6305 (+1)	−4·0926 (+1)	6·1574 (−1)

* The units in which the various quantities are tabulated are as follows.

R (radius) in the unit $R_S = 2GM/c^2$; I (moment of inertia) in the unit $R_S{}^3c^2/G$; $\delta M/M$ (change in mass) in the unit $G^2J^2/R_S{}^4c^6$; Q (quadrupole moment) in the unit G^2J^2/R_Sc^4; ξ_0 ($l = 0$ deformation) in the unit $G^2J^2/R_S{}^3c^6$; ξ_2 ($l = 2$ deformation) in the unit $G^2J^2/R_S{}^3c^6$; ϵ (ellipticity) in the unit $G^2J^2/R_S{}^4c^6$; $\varpi_1 = (\Omega - \omega)_1$ in the unit $GJ/R_S{}^3c^2$.

The integer in parenthesis following each entry is the power of ten by which the entry must be multiplied.

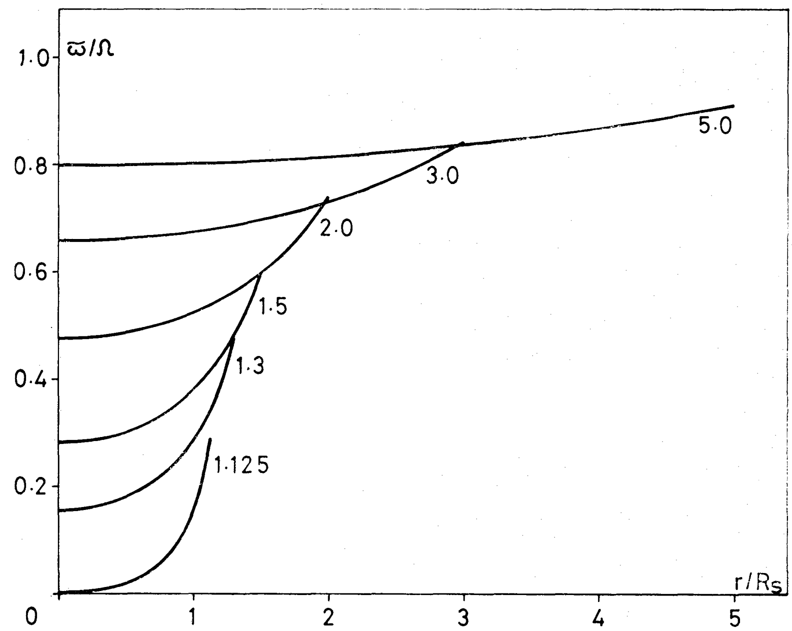

FIG. 1. *The angular velocity* $\varpi = (\Omega - \omega)$ *relative to the local inertial frame:* ϖ/Ω *is plotted aginst* r/R_S *for several values of* R/R_S. *The curves are labelled by the values of* R/R_S *to which they belong.*

It will be observed from Fig. 2 that I/MR^2 (where $I = J/\Omega$ defines the relativistic generalization, for slowly rotating systems, of the Newtonian concept of the moment of inertia) tends to the Newtonian limit 0·4 as $R/R_S \to \infty$, while it takes the value $\sim 0{\cdot}8$ for $R/R_S = 9/8$.

Figs 3, 4, 5 and 6 illustrate the behaviour of the functions $\xi_0(R)/R, -\xi_2(R)/R$ and $\epsilon(R)$ which describe the deformation of the bounding surface caused by the rotation (*cf.* equations (72) and (77)), *for constant mass and angular momentum.*

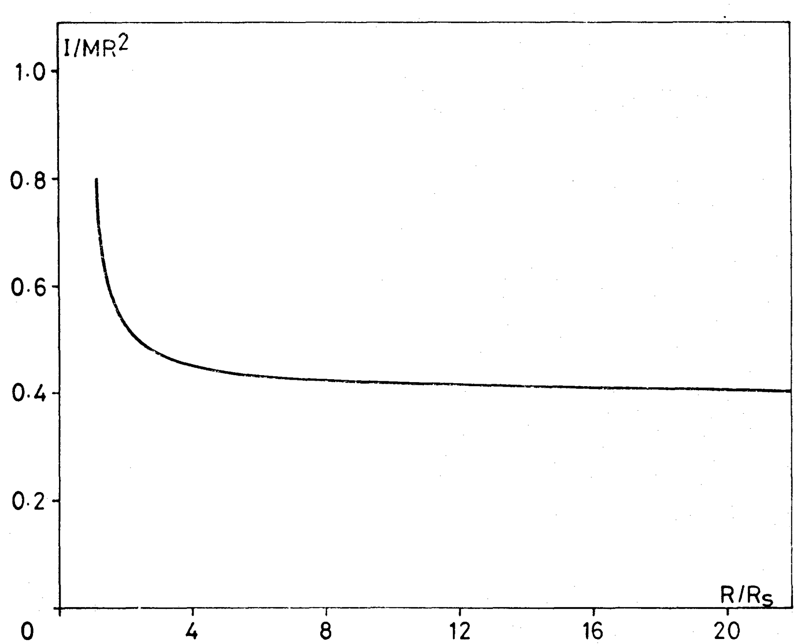

FIG. 2. *The moment of inertia:* I/MR^2 *is plotted against* R/R_S.

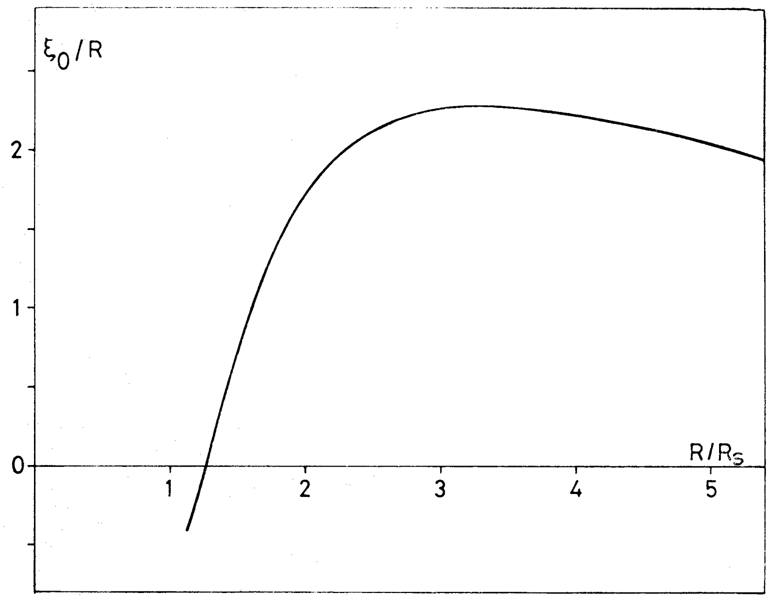

FIG. 3. *The $l = 0$ deformation of the boundary: ξ_0/R is plotted against R/R_S; ξ_0/R is measured in the unit $G^2J^2/R_S{}^4c^2$.*

These functions are not monotonic in R/R_S and have maxima at $R/R_S \sim 3{\cdot}3$, $2{\cdot}7$ and $2{\cdot}4$, respectively.

The variation of the ellipticity of the isobaric surfaces through the configuration is illustrated in Fig. 8. In the Newtonian limit $\epsilon(R)$ is, of course, constant through the configuration. Note that in the limit $R/R_S \rightarrow 9/8$ the ellipticity of the isobaric surfaces close to the centre tends to zero.

In many ways, the most interesting phenomenon disclosed by this study is the reversal in the behaviour of the ellipticity as the object contracts keeping its mass and angular momentum fixed. The underlying cause for this behaviour would

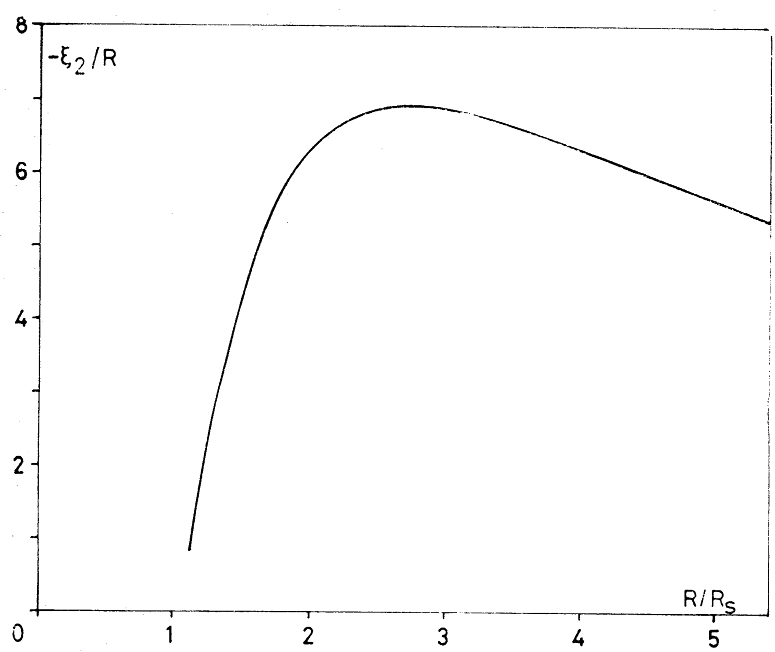

FIG. 4. *The $l = 2$ deformation of the boundary: $-\xi_2/R$ is plotted against R/R_S; ξ_2/R is measured in the unit $G^2J^2/R_S{}^4c^2$.*

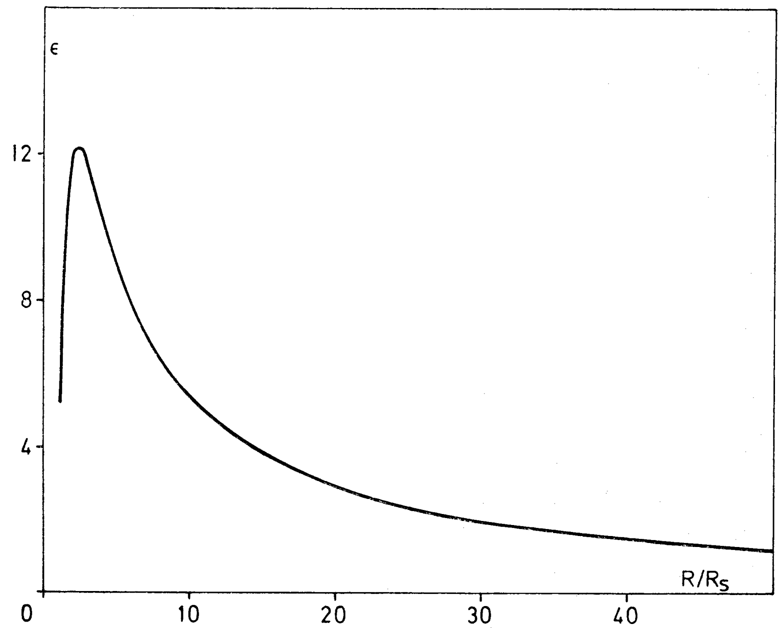

FIG. 5. *The ellipticity of the boundary: ϵ is plotted against R/R_S; ϵ is measured in the unit $G^2J^2/R_S{}^4c^2$.*

appear to be the following. In general relativity, the primary quantity that determines the magnitude of the centrifugal effects of rotation is not Ω but $\varpi = \Omega - \omega$. During contraction, ϖ (the angular velocity relative to the inertial frame) first increases but this trend reverses when $R/R_S \sim 1{\cdot}4$; thus, in Fig. 7, it will be observed that the curve for $R/R_S = 1{\cdot}3$ lies entirely below that for $R/R_S = 1{\cdot}4$ and the curve for $R/R_S = 9/8$ lies below all the curves for $9/8 < R/R_S < 1{\cdot}4$. Consequently, during the advanced stages of contraction, the contribution to the ellipticity by the second term $(= \varpi_1{}^2)$ on the right-hand side of equation (79) decreases.

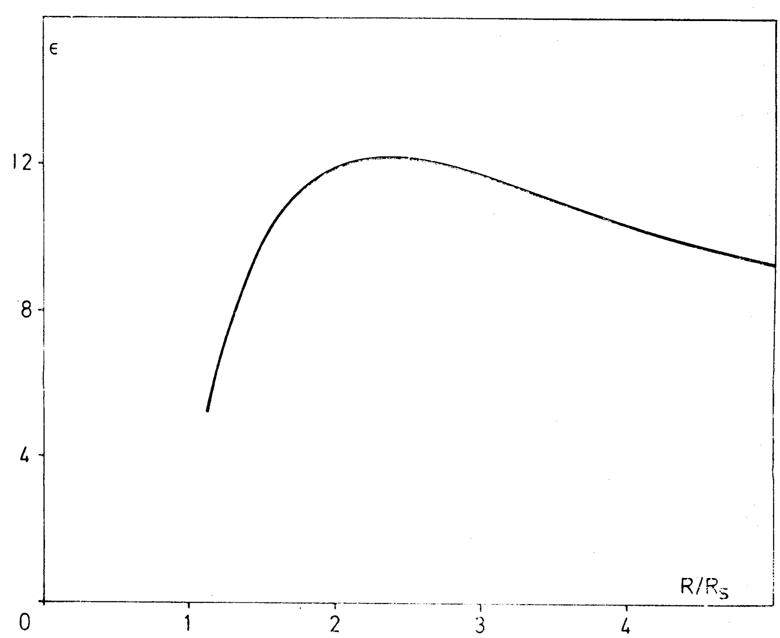

FIG. 6. *The ellipticity of the boundary: ϵ is plotted against R/R_S; ϵ is measured in the unit $G^2J^2/R_S{}^4c^2$.*

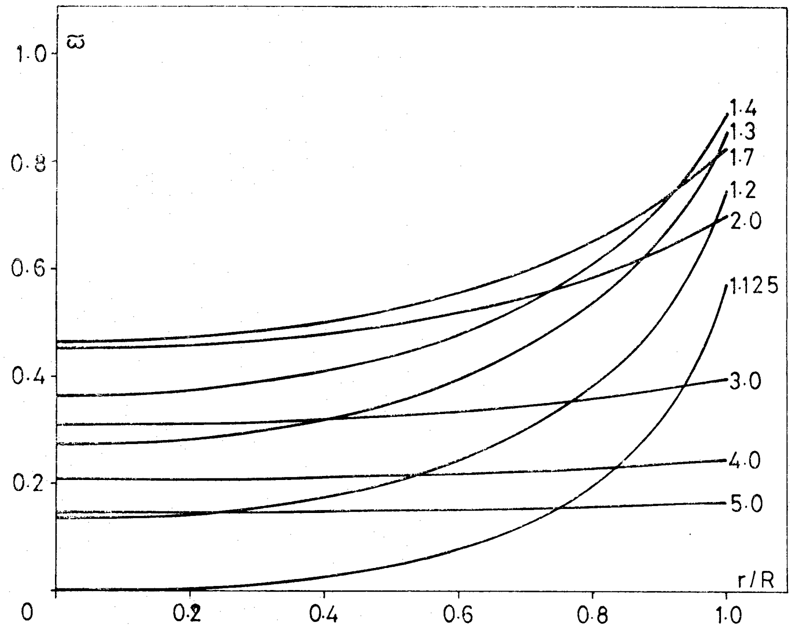

FIG. 7. *The variation of ϖ $(=\Omega-\omega)$ through the configuration: ϖ is plotted against r/R for several values of R/R_S; ϖ is measured in the unit $GJ/R_S{}^3c^2$. The curves are labelled by the values of R/R_S to which they belong.*

That the maximum in the ellipticity does not occur at $R/R_S \sim 1{\cdot}4$, but at $R/R_S \simeq 2{\cdot}4$, is due to the fact that the centrifugal force depends not only on ϖ but on other factors as well. These other factors are included in the term in h_2 (y_1) in equation (79) and the contribution to the ellipticity by this term monotonically decreases as the contraction proceeds. The maximum in the ellipticity occurs when the rates of change of the two terms which contribute to it just balance.

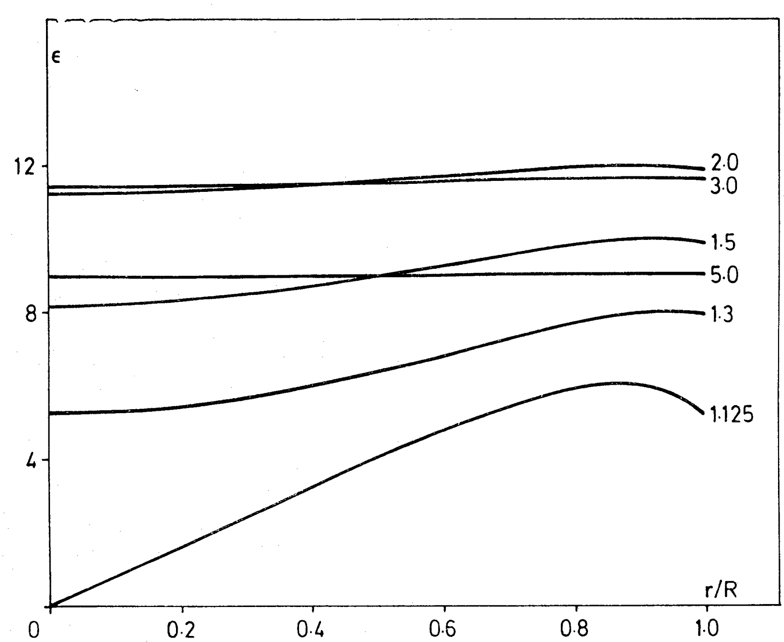

FIG. 8. *The variation of the ellipticity of the isobaric surfaces through the configuration: $\epsilon(r)$ is plotted against r/R for values of R/R_S = 1·125, 1·3, 1·5, 2·0, 3·0 and 5·0; ϵ is measured in the unit $G^2J^2/R_S{}^4c^2$. The curves are labelled by the values of R/R_S to which they belong.*

The value of the quadrupole moment $Q = 2{\cdot}002$ for the configuration of minimum radius, $R/R_S = 1{\cdot}125$, is noteworthy for the following reason. It is known (*cf.* Thorne 1971; see remarks following equation (3.70) on page 275) that the Kerr metric expanded to the first order in J^2 (equivalently Ω^2) corresponds to $Q = 2$. Consequently, the metric external to a slowly rotating configuration of minimum radius agrees with the Kerr metric to the requisite order to one part in a thousand.

A question related to the remarks in the foregoing paragraph concerns whether an ergosphere can occur external to a rotating configuration. For the models under consideration, the question is equivalent to asking whether there is a surface outside the boundary of the configuration on which g_{00} vanishes. Using the relevant equations given in the previous sections, we find that the condition for g_{00} to vanish outside the configuration is

$$-(R-1)+\left(\frac{\Omega R_S}{c}\right)^2 I^2 \left\{\frac{\delta M}{M}-\frac{2}{R^3}+\frac{4}{R^3}\sin^2\theta \right.$$
$$-2(R-1)\left[\frac{2R+1}{R^4}+5(Q-2)\,Q_2{}^2(2R-1)\right] P_2(\cos\theta)$$
$$\left. -\frac{1}{R}\left[\xi_0+\xi_2 P_2(\cos\theta)\right]\right\} > 0, \tag{80}$$

where R is measured in the unit R_S and the rest of the quantities I, $\delta M/M$, Q, ξ_0 and ξ_2 have the values listed in Table I. In particular, on the equator the condition is equivalent to

$$-(R-1)+\left(\frac{\Omega R_S}{c}\right)^2 I^2 \left\{\frac{\delta M}{M}+\frac{2}{R^3}-\frac{1}{R}(\xi_0-\tfrac{1}{2}\xi_2)\right.$$
$$\left. +(R-1)\left[\frac{2R+1}{R^4}+5(Q-2)\,Q_2{}^2(2R-1)\right]\right\}$$
$$= -(R-1)\left[1-\left(\frac{\Omega R_S}{c}\right)^2 f\right] > 0. \tag{81}$$

For the first few models listed in Table I the factor f has the following values:

$$\left.\begin{matrix} f = & 10{\cdot}46 & 8{\cdot}56 & 6{\cdot}16 & 3{\cdot}74 & 2{\cdot}45 & 1{\cdot}60 \\ R/R_S = & 1{\cdot}125 & 1{\cdot}15 & 1{\cdot}2 & 1{\cdot}3 & 1{\cdot}4 & 1{\cdot}5 \end{matrix}\right\}. \tag{82}$$

From the foregoing values it would appear that there is just a possibility that the configuration of minimum radius when rapidly rotating might develop an ergosphere near the equatorial plane.

5. CONCLUDING REMARKS

The behaviour of adiabatically contracting slowly rotating homogeneous masses, illustrated in Figs 5 and 6, combined with the known divergence of the solution for the deformed Maclaurin spheroids for an eccentricity $\epsilon = 0{\cdot}985$ (Chandrasekhar 1967, 1971) raises the serious question of whether stable disc-like objects are at all possible in the framework of general relativity. However, before one accepts the conclusions derived from these considerations pertaining to homogeneous masses, one must examine if the allowance for compressibility and inhomogeneity will affect the qualitative character of the behaviours as distinct from their quantitative aspects. In any event, it would appear that a study of the slow adiabatic contraction of rotating objects, in general relativity, may have some surprises in store.

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