

ON A CRITERION FOR THE OCCURRENCE OF A DEDEKIND-LIKE POINT OF BIFURCATION ALONG A SEQUENCE OF AXISYMMETRIC SYSTEMS. I. RELATIVISTIC THEORY OF UNIFORMLY ROTATING CONFIGURATIONS

S. CHANDRASEKHAR AND JOHN L. FRIEDMAN*

University of Chicago

Received 1973 April 16

ABSTRACT

Stationary nonaxisymmetric systems in general relativity are considered. It is shown that the theory of such systems can be developed along lines which closely parallel the theory of nonstationary axisymmetric systems. Equations are derived which govern small nonaxisymmetric departures from equilibrium of axisymmetric configurations of perfect fluid in uniform rotation. In terms of these equations, the condition that a uniformly rotating configuration will allow a quasi-stationary nonaxisymmetric deformation with a ϕ -dependence of the form $e^{im\phi}$ (where m is an integer greater than or equal to one) is obtained. A variational principle expressing this condition is also derived.

Subject headings: relativity — rotation

I. INTRODUCTION

The best known example in the Newtonian theory of a sequence of nonaxisymmetric configurations bifurcating from a sequence of axisymmetric configurations is that of the Jacobian sequence of triaxial ellipsoids from the Maclaurin sequence of uniformly rotating spheroids. But it is not as well known that at the same point where the Jacobian sequence bifurcates from the Maclaurin sequence, a second congruent sequence, the *Dedekind sequence*, also bifurcates. The Dedekind ellipsoids, in contrast to the Jacobian ellipsoids, are stationary in an inertial frame and maintain their ellipsoidal figure by virtue of internal motions of uniform vorticity. Related to this fact is the circumstance that while the Jacobian sequence bifurcates from the Maclaurin sequence by a deformation that is quasi-stationary in a frame of reference rotating with the angular velocity of the Maclaurin spheroid, the Dedekind sequence bifurcates by a deformation that is quasi-stationary in the inertial frame. (For an account of these matters see Chandrasekhar 1969, chapters 1, 5, 6, and 7, particularly, §§ 5, 34, 36, 44, 45, and 48*b*.)

The bifurcation of a sequence, such as the Jacobian or the Dedekind sequence, signals that at the point of bifurcation *secular instability* can set in in the parent axisymmetric sequence, i.e., an instability that will be manifested only if a *suitable* dissipative mechanism is operative. Thus, if dissipation by normal viscosity is operative, then the Maclaurin spheroid is unstable past the point of bifurcation by a mode that transforms it into a Jacobian ellipsoid (Chandrasekhar 1969, § 37). On the other hand, if dissipation by gravitational radiation-reaction is operative, by the inclusion in the equations of motion of the radiation-reaction terms of general-relativistic origin in the $2\frac{1}{2}$ -post-Newtonian approximation (Chandrasekhar and Esposito 1970), then the Maclaurin spheroid is unstable by the mode which transforms it into a Dedekind ellipsoid (Chandrasekhar 1970).

The foregoing remarks suggest that in general relativity we need be concerned

* Present address: Department of Physics, Yale University, New Haven, Connecticut.

only with quasi-stationary deformations that transform axisymmetric systems into Dedekind-like nonaxisymmetric systems in an inertial frame. In this general context, it is important to draw attention to the fact that in the Newtonian theory, the occurrence of points of bifurcation such as the ones described here, is by no means restricted to homogeneous masses: the recent studies by Ostriker and his associates (cf. Ostriker and Bodenheimer 1973 and the references listed in it) have demonstrated that axisymmetric sequences of differentially rotating compressible masses exhibit patterns of behavior remarkably similar to the classical sequences of homogeneous masses described in *Ellipsoidal Figures of Equilibrium*.

The present paper is devoted to establishing a criterion in the framework of general relativity for the occurrence of points along axisymmetric sequences where quasi-stationary nonaxisymmetric deformations may be possible. In this paper, we shall restrict ourselves to uniformly rotating fluid configurations. Extensions to differentially rotating systems and to vacuum solutions of Einstein's equations will be considered in later papers. In the paper following this one (Chandrasekhar and Lebovitz 1973) the Newtonian version of the theory applicable to differentially rotating systems is presented.

The present paper is very closely related, in its methods and its analysis, to the recent series by the authors on the stability of axisymmetric systems to modes of axisymmetric oscillations in general relativity (Chandrasekhar and Friedman 1972*a, b, c*, 1973; these papers will be referred to hereafter as Papers I, II, III, and IV, respectively); accordingly the analysis will be presented only with few details.

II. ON THE METRIC APPROPRIATE FOR DESCRIBING STATIONARY NONAXISYMMETRIC SYSTEMS; AND THE EQUATIONS THAT FOLLOW

The arguments of Paper I, § II in the context of nonstationary axisymmetric systems apply in all essentials in the present context of stationary but nonaxisymmetric systems: only the roles of the indices 0 and 1 need be interchanged. We choose then for the metric the form (cf. I, eq. [5])

$$ds^2 = -e^{2n}(dt - w d\phi - q_2 dx^2 - q_3 dx^3)^2 + e^{2p}(d\phi)^2 + e^{2\mu_2}(dx^2)^2 + e^{2\mu_3}(dx^3)^2, \quad (1)$$

where n , p , μ_2 , μ_3 , w , q_2 , and q_3 are functions of the spatial coordinates $x^1(=\phi)$, x^2 , and x^3 only and independent of t . While the metric involves seven functions, only six of them are independent since by arguments similar to those in Paper I, § II, it can be shown that the functions w , q_2 , and q_3 can occur in the field equations only in the combinations

$$w_{,2} - q_{2,1}, \quad w_{,3} - q_{3,1}, \quad \text{and} \quad q_{2,3} - q_{3,2}. \quad (2)$$

In considering quasi-stationary nonaxisymmetric deformations, we shall suppose that the various equations that are appropriate to the metric (1) are linearized about equations that are appropriate to the metric

$$ds^2 = -e^{2n}(dt - w d\phi)^2 + e^{2p}(d\phi)^2 + e^{2\mu_2}(dx^2)^2 + e^{2\mu_3}(dx^3)^2, \quad (3)$$

where n , p , w , μ_2 , and μ_3 are now independent of ϕ as well. In the framework of the metric (3) we have the further freedom to impose a coordinate condition on μ_2 and μ_3 (such as $\mu_2 = \mu_3$ or $e^{\mu_3} = x^2 e^{\mu_2}$). It should be further noted that for the linearized problem we have in mind, the notion of ϕ as a cyclic coordinate continues to be a valid one.

By letting

$$dt = -idx^4, \quad iw = q_1, \quad iq_2 = q_2, \quad iq_3 = q_3, \quad \text{and} \quad p = \mu_1, \quad (4)$$

we can rewrite the metric in the form

$$ds^2 = e^{2n} \left(dx^4 - \sum_{A=1}^3 q_A dx^A \right)^2 + \sum_{A=1}^3 e^{2\mu_A} (dx^A)^2, \quad (5)$$

where the capital Latin letters (as indices) are restricted to the values 1, 2, and 3. By comparison with I, equation (13), it is clear that in terms of a local tetrad-frame defined by the one-forms

$$\omega^{(A)} = e^{\mu_A} dx^A \quad (A = 1, 2, 3) \quad \text{and} \quad \omega^{(4)} = e^n \left(dx^4 - \sum_{A=1}^3 q_A dx^A \right), \quad (6)$$

we can immediately write down the expressions for the tetrad components of the Riemann and the Einstein tensors by letting $A, B,$ and C in I, equations (16) and (17), take the values 1, 2, and 3 and further replacing "1" by "4" and " ψ " by n . The explicit forms of the various required equations follow from them; but we shall not write them out.

a) The Equations Governing the Fluid

As in Paper I, we shall be concerned with a perfect fluid described by the standard form of the energy-momentum tensor and an equation of state relating the energy-density ϵ with the pressure p and the baryon number N that is conserved.

Since the chosen form of the metric is different from that in Paper I, we shall write down the equations governing the fluid explicitly.

With the definitions

$$\frac{d\phi}{dt} = \Omega, \quad \frac{dx^\alpha}{dt} = v^\alpha \quad (\alpha = 2, 3),$$

$$v^{(1)} = \frac{e^{p-n}\Omega}{1 - w\Omega - q_\alpha v^\alpha}, \quad v^{(\alpha)} = \frac{e^{\mu_\alpha - n} v^\alpha}{1 - w\Omega - q_\alpha v^\alpha},$$

and

$$V^2 = [v^{(1)}]^2 + [v^{(2)}]^2 + [v^{(3)}]^2, \quad (7)$$

we readily find that the contravariant and covariant components of the four-velocity are

$$u^0 = \frac{e^{-n}}{(1 - V^2)^{1/2} (1 - w\Omega - q_\alpha v^\alpha)}, \quad u^1 = \Omega u^0, \quad u^\alpha = u^0 v^\alpha, \quad (8)$$

and

$$u_0 = -\frac{e^n}{(1 - V^2)^{1/2}}, \quad u_1 = \frac{v^{(1)} e^p + w e^n}{(1 - V^2)^{1/2}}, \quad u_\alpha = \frac{v^{(\alpha)} e^{\mu_\alpha} + q_\alpha e^n}{(1 - V^2)^{1/2}}. \quad (9)$$

The corresponding tetrad components of the four-velocity are

$$u^{(0)} = \frac{1}{(1 - V^2)^{1/2}}, \quad u^{(1)} = \frac{v^{(1)}}{(1 - V^2)^{1/2}}, \quad \text{and} \quad u^{(\alpha)} = \frac{v^{(\alpha)}}{(1 - V^2)^{1/2}}. \quad (10)$$

The hydrodynamic equations now follow from the relation (I, eq. [32])

$$u^j u_{A;j} = -\frac{1}{\epsilon + p} (\delta^j_A + u^j u_A) p_{,j} \quad (A = 1, 2, 3). \quad (11)$$

We find

$$\begin{aligned} u^0 \frac{du_A}{dt} - \frac{1}{2}(u^0)^2 \left[\frac{\partial}{\partial x^A} (-e^{2n}) + 2\Omega \frac{\partial}{\partial x^A} (we^{2n}) + \Omega^2 \frac{\partial}{\partial x^A} (e^{2p} - w^2 e^{2n}) \right. \\ \left. + (v^2)^2 \frac{\partial}{\partial x^A} (e^{2\mu_2} - q_2^2 e^{2n}) + (v^3)^2 \frac{\partial}{\partial x^A} (e^{2\mu_3} - q_3^2 e^{2n}) \right. \\ \left. + 2v^\alpha \frac{\partial}{\partial x^A} (q_\alpha e^{2n}) + 2\Omega v^\alpha \frac{\partial}{\partial x^A} (-q_\alpha w e^{2n}) \right. \\ \left. + 2v^2 v^3 \frac{\partial}{\partial x^A} (-q_2 q_3 e^{2n}) \right] = -\frac{1}{\epsilon + p} \left(\frac{\partial p}{\partial x^A} + u^0 u_A \frac{dp}{dt} \right), \quad (12) \end{aligned}$$

where we have introduced the operator

$$\frac{d}{dt} = \Omega \frac{\partial}{\partial \phi} + v^\alpha \frac{\partial}{\partial x^\alpha}. \quad (13)$$

The equation ensuring the conservation of baryon number now takes the form

$$\frac{d}{dt} (Nu^0 \sqrt{-g}) + Nu^0 (\Omega_{,1} + v^\alpha{}_{,\alpha}) \sqrt{-g} = 0. \quad (14)$$

Since the metric coefficients are independent of t , we also have an equation that expresses the conservation of the "energy" u_0 per baryon:

$$\frac{d}{dt} \left(\frac{\epsilon + p}{N} u_0 \right) = \frac{d}{dt} \left[\frac{\epsilon + p}{N} \frac{e^n}{(1 - V^2)^{1/2}} \right] = 0. \quad (15)$$

III. THE EQUATIONS GOVERNING EQUILIBRIUM

The stationary axisymmetric system we consider is one in which there are no motions in the x^2 - and the x^3 -directions and only rotational motions (specified by Ω) prevail. Under these same circumstances we can set q_2 and q_3 equal to zero; and we also have the freedom to relate μ_2 and μ_3 by any coordinate condition that we may find convenient.

By a comparison of the forms of the metric chosen in Paper I (eq. [10]) and in the present paper (eq. [3]) to describe a stationary axisymmetric system, we infer that the functions introduced in the two forms are related in the manner

$$\begin{aligned} e^{p+n} &= e^{\psi+v} = e^\beta, \\ e^{2n} &= e^{2v} - \omega^2 e^{2\psi}, \quad e^{2p} = \frac{e^{2(\psi+v)}}{e^{2v} - \omega^2 e^{2\psi}}, \end{aligned}$$

and

$$w = -\frac{\omega e^{2\psi}}{e^{2v} - \omega^2 e^{2\psi}}. \quad (16)$$

The components of the four-velocity appropriate under the present stationary conditions are

$$\begin{aligned} v^{(1)} &= V = \frac{e^{\nu-n}\Omega}{1-w\Omega}, & v^\alpha &= v^{(\alpha)} = 0, \\ u^0 &= \frac{e^{-n}}{(1-V^2)^{1/2}(1-w\Omega)}, & u^1 &= \Omega u^0, & u^\alpha &= 0, \\ u_0 &= -\frac{e^n}{(1-V^2)^{1/2}}, & u_1 &= \frac{e^\nu V + e^n w}{(1-V^2)^{1/2}}, & u_\alpha &= 0, \\ u^{(0)} &= \frac{1}{(1-V^2)^{1/2}}, & u^{(1)} &= \frac{V}{(1-V^2)^{1/2}}, & \text{and } u^{(\alpha)} &= 0. \end{aligned} \quad (17)$$

We shall find the following relations among the foregoing quantities useful:

$$u^1 u_0 = \Omega u^0 u_0 = -\frac{\Omega}{(1-w\Omega)(1-V^2)} = -\frac{e^{n-\nu}V}{1-V^2},$$

and

$$\Omega(1 + \Omega u^0 u_1) = -u^1 u_0. \quad (18)$$

The relevant field-equations can now be written down by making the following replacements in I, equations (68)–(78):

$$\begin{aligned} \psi &\rightarrow n, & \nu &\rightarrow \rho, & \omega &\rightarrow -w, & V &\rightarrow -\frac{1}{V}, \\ u^0 &\rightarrow -iu^1, & u^1 &\rightarrow iu^0, & u_0 &\rightarrow iu_1, & \text{and } u_1 &\rightarrow -iu_0. \end{aligned} \quad (19)$$

And when $\Omega = \text{constant}$, the equation of hydrodynamic equilibrium is as before (I, eq. [81])

$$p_{,\alpha} = (\epsilon + p)(\log u^0)_{,\alpha}. \quad (20)$$

a) *The Asymptotic Forms of the Potentials*

In view of the known asymptotic behavior (I, eq. [109])

$$\omega \rightarrow \frac{2J}{r^3} \quad (r \rightarrow \infty), \quad (21)$$

we may conclude from the relations (16) that

$$e^n = e^\nu + O(r^{-4}), \quad e^\nu = e^\psi + O(r^{-3}),$$

and

$$w = -\frac{2J \sin^2 \theta}{r} + O(r^{-2}), \quad (22)$$

where the asymptotic forms are expressed in a frame in which (cf. I, eq. [82])

$$e^\nu = r e^{\eta+\zeta} \sin \theta, \quad e^{\mu_2} = e^{\eta-\zeta}, \quad \text{and } e^{\mu_3} = r e^{\eta-\zeta}. \quad (23)$$

The asymptotic behaviors of $n(\sim\nu)$, η , and ζ are therefore the same as given in Paper I, equations (101) and (102).

IV. THE EQUATIONS GOVERNING INFINITESIMAL NONAXISYMMETRIC DEFORMATIONS

We shall suppose that a configuration initially axisymmetric, stationary, and uniformly rotating with an angular velocity Ω is subjected to an infinitesimal non-axisymmetric deformation executed in a quasi-stationary manner (i.e., infinitely slowly). We wish to ascertain whether such a quasi-stationary deformation is possible consistently with the various equations and constraints that govern the problem. For this purpose, it is convenient to describe the deformation in terms of a Lagrangian displacement $\xi^\alpha(\phi, x^\alpha)$ which is the spatial displacement that an element of the fluid suffers relative to its location in the undeformed axisymmetric state.

Since the initial configuration is assumed to be axisymmetric and uniformly rotating, it is clear that the components ξ^α suffice to describe the deformed configuration provided we formally associate with ξ^α a "velocity" given by (cf. eq. [13])

$$v^\alpha = \Omega \xi^\alpha_{,1} = \xi^\alpha_{,t}. \quad (24)$$

In this manner of description, the dependence of the displacement on ϕ is considered in formally the same way as the dependence on time was considered in Papers I–III. Indeed, we may separate ϕ from the remaining space-variables x^α by seeking solutions whose dependence on ϕ is given by

$$e^{im\phi}, \quad (25)$$

where, in contrast to the time-dependent case, we must now require that m is an integer greater than or equal to 1.

We are naturally most interested in obtaining a condition that a deformation belonging to $m = 2$ is possible: the existence of such a deformation will, according to our earlier remarks, signal the onset of secular instability.

With the deformation described in terms of a Lagrangian displacement, we can distinguish between the Eulerian and the Lagrangian changes in a quantity that ensue as a result of the deformation. These changes can in turn be expressed as the result of the action of an Eulerian (δ) and a Lagrangian (Δ) operator which are related to

$$\Delta = \delta + \xi^\alpha \frac{\partial}{\partial x^\alpha}. \quad (26)$$

Turning next to the equations that the deformations must satisfy, we first observe that since the q_α 's vanish in the initial state, their appearance in the metric which describes the deformed state may be thought of as being *caused* by the deformation. The q_α 's, like the v^α 's, are therefore quantities of the first order of smallness. An important consequence of *both* v^α and q_α being quantities of the first order of smallness is that V and $v^{(1)}$ as *defined* in equation (7) differ from each other and from the *expression* for V in the undeformed state by quantities of the second order of smallness; they can therefore be ignored in a linearized theory such as the present. The relations given in equations (17), namely,

$$V = v^{(1)} = \frac{e^{p-n}}{1 - w\Omega}, \quad (27)$$

continues to hold *formally* in the deformed state as well—but only formally, since n , p , Ω , and w are all subject to *first order* changes.

A further consequence of the formal applicability of the definition (27) to the deformed state is that the remaining expressions for u^0 , u^1 , u_0 , u_1 , $u^{(0)}$, and $u^{(1)}$ given in equations (17) also continue to be applicable.

To emphasize that q_α , like v^α , is nonvanishing only in the deformed state, we shall write

$$q_{\alpha,1} \text{ in place of } q_\alpha \quad (28)$$

in all developments dealing with nonaxisymmetric deformations of systems that are initially axisymmetric.

a) The Equation Ensuring the Conservation of Baryon Number

From equation (14) it follows that (cf. I, eq. [127])

$$\frac{\Delta N}{N} = -\frac{\delta V}{V(1-V^2)} - \delta(n + \mu_2 + \mu_3) - \frac{1}{u^1 \sqrt{-g}} (u^1 \xi^\alpha \sqrt{-g})_{,\alpha}. \quad (29)$$

In terms of ΔN , the corresponding Lagrangian changes in the pressure (p) and in the energy density (ϵ) are given by

$$\frac{\Delta p}{p} = \gamma \frac{\Delta N}{N} \quad \text{and} \quad \frac{\Delta \epsilon}{\epsilon + p} = \frac{\Delta N}{N}. \quad (30)$$

b) The Equation Ensuring the Conservation of Energy

From equation (15) it follows that

$$\Delta \left(\frac{\epsilon + p}{N} u_0 \right) = 0. \quad (31)$$

On further simplification this equation gives (cf. I, eq. [133])

$$\frac{V \delta V}{1 - V^2} = -\frac{\gamma p}{\epsilon + p} \frac{\Delta N}{N} - \delta n - \xi^\alpha (\log u_0)_{,\alpha}. \quad (32)$$

With δV determined by equation (32), the redistribution of Ω that results from the deformation can be deduced from the relation (27). We find

$$\delta \Omega = -\Omega^2 \delta w + \Omega(1 - w\Omega) \left[\frac{\delta V}{V} - \delta(p - n) \right], \quad (33)$$

or, alternatively

$$\frac{\delta(w\Omega)}{1 - w\Omega} = \Omega \delta w + w\Omega \left[\frac{\delta V}{V} - \delta(p - n) \right]. \quad (34)$$

Finally, we may note the following two identities which can be derived from the foregoing relations:

$$\delta[(\epsilon + p)u^1 u_0 \sqrt{-g}] = -[(\epsilon + p)u^1 u_0 \xi^\alpha \sqrt{-g}]_{,\alpha},$$

and

$$[(\epsilon + p)\xi^\alpha \sqrt{-g}]_{,\alpha} = -\left[\delta \epsilon + (\epsilon + p)\delta(n + \mu_2 + \mu_3) + (\epsilon + p) \frac{\delta V}{V(1 - V^2)} \right] \sqrt{-g}. \quad (35)$$

c) *The "Pulsation" Equation*

The pulsation equation is no more than equation (12) linearized about the equilibrium equation (20). The required equation is obtained by applying the Lagrangian operator Δ to equation (12). We obtain

$$(\epsilon + p)[(u^1)^2 e^{2\mu_\alpha} \xi^\alpha_{,11} - u^1 u_0 q_{\alpha,11}] = \left[\frac{\Delta(\epsilon + p)}{\epsilon + p} + 2 \frac{\Delta u^0}{u^0} \right] \frac{\partial p}{\partial x^\alpha} - \frac{\partial}{\partial x^\alpha} \left(\gamma p \frac{\Delta N}{N} \right) - \frac{1}{2}(\epsilon + p)(u^0)^2 \frac{\partial}{\partial x^\alpha} \Delta \left(\frac{1}{u^0} \right)^2 - (\epsilon + p) u^0 u_1 \frac{\partial}{\partial x^\alpha} \Delta \Omega, \quad (36)$$

where we have made use of the relation

$$u_\alpha = u^1 e^{2\mu_\alpha} \xi^\alpha_{,1} - u_0 q_{\alpha,1} \quad (37)$$

which follows from equations (9) and the expression (24) for v^α in terms of ξ^α .

On further simplification, in which use is made of equations (30), (33), and (34), we find that equation (36) can be brought to the form

$$\begin{aligned} -m^2(\epsilon + p)(u^1)^2(\sqrt{-g})e^{2\mu_\alpha}\xi^\alpha = & -u^1\sqrt{-g} - g\frac{\partial}{\partial x^\alpha}\left(\frac{\gamma p}{u^1}\frac{\Delta N}{N}\right) \\ & + (\epsilon + p)\sqrt{-g} - g\frac{\partial}{\partial x^\alpha}\frac{\Delta u^1}{u^1} + \sqrt{-g}\frac{\Delta N}{N}\frac{\partial p}{\partial x^\alpha} \\ & + (\epsilon + p)\sqrt{-g} - g\frac{u^1 u_0}{\Omega^2}\frac{\partial}{\partial x^\alpha}\Delta\Omega \\ & + (\epsilon + p)u^1 u_0\sqrt{-g} - g q_{\alpha,11}, \end{aligned} \quad (38)$$

where we have inserted the ϕ -dependence of ξ^α in accordance with the assumption (25).

d) *The Linearized Versions of the (1, α)- and the (0, α)-Components of the Field Equations*

As in Paper I, § XI, the (1, α)- and the (0, α)-components of the field equations directly integrate to provide initial-value equations. Thus, the (1, α)-components give

$$\begin{aligned} -e^{p+n-\mu_2-\mu_3}[\delta n_{,2} + (n-p)_{,2}\delta n + \delta\mu_{3,2} - (p-\mu_3)_{,2}\delta\mu_3 - (n+\mu_3)_{,2}\delta\mu_2] \\ + \frac{1}{2}w_{,3}\Omega = 8\pi(\epsilon + p)\sqrt{-g} - g\frac{V^2}{1-V^2}\xi^2, \end{aligned} \quad (39)$$

$$\begin{aligned} -e^{p+n-\mu_2-\mu_3}[\delta n_{,3} + (n-p)_{,3}\delta n + \delta\mu_{2,3} - (p-\mu_2)_{,3}\delta\mu_2 - (n+\mu_2)_{,3}\delta\mu_3] \\ - \frac{1}{2}w_{,2}\Omega = 8\pi(\epsilon + p)\sqrt{-g} - g\frac{V^2}{1-V^2}\xi^3, \end{aligned} \quad (40)$$

where

$$\Omega = e^{3n+p-\mu_2-\mu_3}(q_{2,3} - q_{3,2}). \quad (41)$$

Again, it will be convenient to define the variables

$$\delta\mu = \frac{1}{2}\delta(\mu_3 + \mu_2) \quad \text{and} \quad \delta\tau = \frac{1}{2}\delta(\mu_3 - \mu_2). \quad (42)$$

In terms of these variables, equations (39) and (40) become

$$(\delta n + \delta \mu)_{,2} - p_{,2}(\delta n + \delta \mu) + n_{,2}(\delta n - \delta \mu) \\ = e^{2\mu_2} \left[-8\pi(\epsilon + p) \frac{V^2}{1 - V^2} \xi^2 + \frac{\Omega w_{,3}}{2\sqrt{-g}} \right] - \delta \tau_{,2} - (2\mu_3 + n - p)_{,2} \delta \tau,$$

and

$$(\delta n + \delta \mu)_{,3} - p_{,3}(\delta n + \delta \mu) + n_{,3}(\delta n - \delta \mu) \\ = e^{2\mu_3} \left[-8\pi(\epsilon + p) \frac{V^2}{1 - V^2} \xi^3 - \frac{\Omega w_{,2}}{2\sqrt{-g}} \right] + \delta \tau_{,3} + (2\mu_2 + n - p)_{,3} \delta \tau. \quad (43)$$

Considering next the $(0, \alpha)$ -components of the field equations, we obtain

$$\delta w_{,2} - q_{2,11} = 16\pi(\epsilon + p)u^1 u_0 e^{2p-2n+2\mu_2} \xi^2 \\ - w_{,2}(3\delta n - \delta p + \delta \mu_3 - \delta \mu_2) + e^{-3n+p+\mu_2-\mu_3} \Omega_{,3}, \\ \delta w_{,3} - q_{3,11} = 16\pi(\epsilon + p)u^1 u_0 e^{2p-2n+2\mu_3} \xi^3 \\ - w_{,3}(3\delta n - \delta p + \delta \mu_2 - \delta \mu_3) - e^{-3n+p+\mu_3-\mu_2} \Omega_{,2}. \quad (44)$$

These are initial-value equations for $\delta w_{,\alpha}$; and their integrability condition, namely,

$$(e^{-3n+p+\mu_3-\mu_2} \Omega_{,2})_{,2} + (e^{-3n+p+\mu_2-\mu_3} \Omega_{,3})_{,3} - m^2 e^{-3n-p+\mu_2+\mu_3} \Omega \\ = [w_{,2}(3\delta n - \delta p + \delta \mu_3 - \delta \mu_2)]_{,3} - [w_{,3}(3\delta n - \delta p + \delta \mu_2 - \delta \mu_3)]_{,2} \\ - 16\pi \{ [(\epsilon + p)u^1 u_0 e^{2p-2n+2\mu_2} \xi^2]_{,3} - [(\epsilon + p)u^1 u_0 e^{2p-2n+2\mu_3} \xi^3]_{,2} \}, \quad (45)$$

provides a dynamical equation for Ω .

e) The Linearized Versions of the Remaining Field Equations

The linearized versions of the remaining field equations can be obtained from the equations of Paper I, § XI, by making the replacements (19) and noting that, if in analogy with I, equations (156), we now define

$$\mathcal{S} = e^{-\beta}(\Omega_{,2} w_{,3} - \Omega_{,3} w_{,2}) \quad \text{and} \quad \mathcal{D} = -e^{-\beta}(\Omega_{,2} w_{,3} + \Omega_{,3} w_{,2}), \quad (46)$$

then in rewriting I, equations (157)-(169),¹ we must also let

$$S \rightarrow -\mathcal{S} \quad \text{and} \quad D \rightarrow -\mathcal{D}; \quad (47)$$

and it is also clear that

$$\sigma^2 \rightarrow -m^2. \quad (48)$$

The problem whose solution we now seek can be formulated precisely as follows.

The pulsation equation (38) and the linearized versions of the $(2, 2)$ - and the $(3, 3)$ -components of the field equations² together with equations (29), (30), (32), (43), and

¹ There is a misprint in I, eq. (169), which is noted in Paper IV (p. 495).

² Instead of the $(2, 2)$ - and the $(3, 3)$ -components of the field equations, we may choose *any* two of the field equations that remain after the elimination of the $(1, 1)$ -, $(0, 1)$ -, $(1, \alpha)$ -, and $(0, \alpha)$ -components (cf. the remarks in Paper II following eq. [18]).

(44) (as initial-value equations) and boundary conditions (that require that the ξ^α 's vanish at the origin, Δp vanishes on the boundary of the configuration, and the remaining field variables, such as δp , δn , etc., are all "well-behaved" at the origin and at infinity) provide a characteristic-value problem for m^2 . We should not, of course, expect that for an arbitrary initial configuration an allowed characteristic value for m will be an integer. *We are, however, interested precisely in the circumstances that will permit an integral characteristic value (≥ 1) for m .* In particular, if $m = 2$ is an allowed characteristic value, we should conclude that the initial configuration is susceptible to a Dedekind-like point of bifurcation and secular instability.

V. A FORMULA FOR m^2

As we have explained in § IV, we shall consider m^2 as a characteristic value parameter that is to be determined consistently with the dynamical equations, the initial-value equations, and the boundary conditions that govern the problem. Then defining a trial displacement $\bar{\xi}^\alpha$ and the associated barred variations compatible only with the initial-value equations and boundary conditions, and proceeding exactly as in Paper II § III, we can derive a formula for m^2 . The formula in question can in fact be directly written down by making the replacements (19), (47), and (48) in II, equation (25). We thus obtain

$$\begin{aligned}
 & -m^2 \int \int \int \left\{ \sqrt{-g} \left[(\epsilon + p)(u^1)^2 \sum_{\alpha} e^{2\mu_{\alpha}} \xi^{\alpha} \bar{\xi}^{\alpha} \right. \right. \\
 & \quad \left. \left. + \frac{e^{-2p}}{4\pi} (\delta\mu\bar{\delta}\mu + \delta n\bar{\delta}\mu + \bar{\delta}n\delta\mu - \delta\tau\bar{\delta}\tau) \right] \right. \\
 & \quad \left. - \frac{1}{16\pi} e^{-3n-p+\mu_2+\mu_3} \Omega\bar{\Omega} \right\} dx^2 dx^3 d\phi \\
 & = \int \int \int \left[\left[\sqrt{-g} \left\{ -\gamma p \left(1 + \frac{\gamma p}{V^2} \frac{1}{\epsilon + p} \right) \frac{\Delta N \bar{\Delta} N}{N^2} - \frac{1}{\epsilon + p} \xi^{\alpha} p_{,\alpha} \bar{\xi}^{\beta} \epsilon_{,\beta} \right. \right. \right. \\
 & \quad \left. \left. + \left(\frac{\Delta N}{N} \bar{\xi}^{\alpha} + \frac{\bar{\Delta} N}{N} \xi^{\alpha} \right) \left[p_{,\alpha} - \frac{\gamma p}{V^2} (\log u_0)_{,\alpha} \right] \right. \right. \\
 & \quad \left. \left. - \frac{\epsilon + p}{V^2} \xi^{\alpha} (\log u_0)_{,\alpha} \bar{\xi}^{\beta} (\log u_0)_{,\beta} \right. \right. \\
 & \quad \left. \left. - \frac{1}{V(1-V^2)} (\delta p \bar{\delta} V + \bar{\delta} p \delta V) + \frac{\epsilon + p}{V^2} \delta n \bar{\delta} n \right. \right. \\
 & \quad \left. \left. - 16\pi (\epsilon + p)^2 \frac{V^2}{(1-V^2)^2} \sum_{\alpha} e^{2\mu_{\alpha}} \xi^{\alpha} \bar{\xi}^{\alpha} \right. \right. \\
 & \quad \left. \left. - [(\epsilon + p)\delta n \bar{\delta} n + 2(\epsilon - p)(\delta n \bar{\delta}\mu + \bar{\delta} n \delta\mu) + \delta\epsilon \bar{\delta} n + \bar{\delta}\epsilon \delta n - 4p\delta\mu \bar{\delta}\mu] \right. \right. \\
 & \quad \left. \left. + 2(\epsilon + p)u^1 u_0 [\delta n \bar{\xi}^{\alpha} w_{,\alpha} + \bar{\delta} n \xi^{\alpha} w_{,\alpha} \right. \right. \\
 & \quad \quad \left. \left. + \delta\tau (\bar{\xi}^2 w_{,2} - \bar{\xi}^3 w_{,3}) + \bar{\delta}\tau (\xi^2 w_{,2} - \xi^3 w_{,3}) \right] \right. \\
 & \quad \left. \left. - (\epsilon + p)u^1 u_0 e^{-3n+p} [e^{\mu_2-\mu_3} (\bar{\xi}^2 \Omega_{,3} + \xi^2 \bar{\Omega}_{,3}) - e^{\mu_3-\mu_2} (\bar{\xi}^3 \Omega_{,2} + \xi^3 \bar{\Omega}_{,2})] \right\} \right.
 \end{aligned}$$

$$\begin{aligned}
& - \frac{1}{4\pi} e^{3n-p} \mathfrak{X}(\delta n \bar{\delta} n + \frac{1}{2} \delta \tau \bar{\delta} \tau) + \frac{1}{2\pi} (\mathfrak{Y} - \frac{1}{4} e^{3n-p} \mathfrak{Z})(\delta n \bar{\delta} \tau + \bar{\delta} n \delta \tau) \\
& + \frac{1}{2\pi} \mathfrak{U} \delta \tau \bar{\delta} \tau + \frac{1}{4\pi} e^{\beta} (e^{\mu_3 - \mu_2} \delta n_{,2} \bar{\delta} n_{,2} + e^{\mu_2 - \mu_3} \delta n_{,3} \bar{\delta} n_{,3}) \\
& - \frac{1}{4\pi} e^{\beta} \{ [(e^{\mu_3 - \mu_2} \delta \mu_{3,2})_{,2} + (e^{\mu_2 - \mu_3} \delta \mu_{2,3})_{,3}] \bar{\delta} n \\
& \quad + [(e^{\mu_3 - \mu_2} \bar{\delta} \mu_{3,2})_{,2} + (e^{\mu_2 - \mu_3} \bar{\delta} \mu_{2,3})_{,3}] \delta n \} \\
& + \frac{1}{4\pi} e^{\beta} \{ [(\beta_{,2} \bar{\delta} \mu_{3,2} + (\beta + 2\mu_3)_{,2} \bar{\delta} n_{,2}] \delta \tau \\
& \quad + [\beta_{,2} \delta \mu_{3,2} + (\beta + 2\mu_3)_{,2} \delta n_{,2}] \bar{\delta} \tau \} e^{\mu_3 - \mu_2} \\
& - \frac{1}{4\pi} e^{\beta} \{ [\beta_{,3} \bar{\delta} \mu_{2,3} + (\beta + 2\mu_2)_{,3} \bar{\delta} n_{,3}] \delta \tau \\
& \quad + [\beta_{,3} \delta \mu_{2,3} + (\beta + 2\mu_2)_{,3} \delta n_{,3}] \bar{\delta} \tau \} e^{\mu_2 - \mu_3} \\
& - \frac{1}{8\pi} (\mathfrak{C} \bar{\delta} n + \bar{\mathfrak{C}} \delta n + \mathfrak{D} \bar{\delta} \tau + \bar{\mathfrak{D}} \delta \tau) \\
& - \frac{1}{16\pi} e^{-3n+p} (e^{\mu_3 - \mu_2} \mathfrak{Q}_{,2} \bar{\mathfrak{Q}}_{,2} + e^{\mu_2 - \mu_3} \mathfrak{Q}_{,3} \bar{\mathfrak{Q}}_{,3}) \Big] dx^2 dx^3 d\phi \\
& - \frac{1}{8\pi} \int \int d\phi dx^2 \left[e^{\mu_2 - \mu_3 + \beta} [\delta(n+p)_{,3} \bar{\delta}(n+\mu_2) - 2\delta n \bar{\delta} \mu_{2,3} \right. \\
& \quad - 2(\beta + 2\mu_2)_{,3} \delta \tau \bar{\delta} n + 2\beta_{,3} \delta(n-\tau) \bar{\delta} \mu_2] \\
& \quad \left. + \bar{\mathfrak{Q}} w_{,2} \delta(n+\tau) - \frac{1}{2} e^{-3n+p+\mu_2-\mu_3} \mathfrak{Q} \bar{\mathfrak{Q}}_{,3} \right]_{[x^3]} \\
& - \frac{1}{8\pi} \int \int d\phi dx^3 \left[e^{\mu_3 - \mu_2 + \beta} [\delta(n+p)_{,2} \bar{\delta}(n+\mu_3) - 2\delta n \bar{\delta} \mu_{3,2} \right. \\
& \quad + 2(\beta + 2\mu_3)_{,2} \delta \tau \bar{\delta} n + 2\beta_{,2} \delta(n+\tau) \bar{\delta} \mu_3 \\
& \quad \left. - \bar{\mathfrak{Q}} w_{,3} \delta(n-\tau) - \frac{1}{2} e^{-3n+p+\mu_3-\mu_2} \mathfrak{Q} \bar{\mathfrak{Q}}_{,2} \right]_{[x^2]} \quad (49)
\end{aligned}$$

where the symbol

$$[[\cdots]]_{[x^\alpha]}$$

in the integrands of the surface integrals has the following meaning. For a fixed x^β ($\beta \neq \alpha$) let the appropriate limits of x^α be $x^\beta(1)$ and $x^\beta(2)$ and $x^\beta(2) \geq x^\beta(1)$; the symbol then stands for the difference in the values of the quantity enclosed by the double brackets at $x^\beta(2)$ and $x^\beta(1)$. Further abbreviations used in equation (49) are

$$\begin{aligned}
\mathfrak{X} &= e^{\mu_3 - \mu_2} (w_{,2})^2 + e^{\mu_2 - \mu_3} (w_{,3})^2, & \mathfrak{Y} &= e^{\mu_3 - \mu_2} (w_{,2})^2 - e^{\mu_2 - \mu_3} (w_{,3})^2, \\
\mathfrak{U} &= e^{\beta} [e^{\mu_3 - \mu_2} (\beta_{,2} \mu_{3,2} + p_{,2} n_{,2}) + e^{\mu_2 - \mu_3} (\beta_{,3} \mu_{2,3} + p_{,3} n_{,3})],
\end{aligned}$$

and

$$\mathfrak{B} = e^{\beta} [e^{\mu_3 - \mu_2} (\beta_{,2} \mu_{3,2} + \nu_{,2} n_{,2}) - e^{\mu_2 - \mu_3} (\beta_{,3} \mu_{2,3} + \nu_{,3} n_{,3})]. \quad (50)$$

In the system of spherical polar coordinates (r, θ, ϕ) used in Paper I, § VII and Paper II, § IIIb the first of the two surface integrals in equation (49) simplifies considerably. Thus, from equations (39) and (40), we conclude that at the limits of $x^3 (= \theta)$, namely, 0, and π ,

$$\mathfrak{Q} = 0 \quad \text{at} \quad \theta = 0 \quad \text{and} \quad \theta = \pi$$

and

$$(e^{\nu})_{,3} (\delta n + \delta \mu_2) = 0 \quad \text{or} \quad \delta n = -\delta \mu_2 \quad \text{at} \quad \theta = 0 \quad \text{and} \quad \theta = \pi. \quad (51)$$

By making use of these relations, it can be verified that the surface integral over ϕ and x^2 reduces to the following manifestly symmetric form:

$$\frac{1}{4\pi} \int \int d\phi dx^2 [e^{\mu_2 - \mu_3 + n} (e^{\nu})_{,3} \delta n \bar{\delta} n]_0^{\pi}. \quad (52)$$

By considering the asymptotic behavior of the perturbed potentials at infinity, we shall show in § a) below that the second surface integral (over ϕ and x^3) in equation (49) does not survive.

a) The Asymptotic Behavior of the Perturbed Potentials

Letting (cf. Paper I, eq. [8] and Paper II, § Vb)

$$e^{\nu} = r e^{\eta + \zeta} \sin \theta, \quad e^{\mu_2} = e^{\eta - \zeta - \delta \tau}, \quad \text{and} \quad e^{\mu_3} = r e^{\eta + \zeta + \delta \tau}, \quad (53)$$

we find that the linearized versions of the relevant equations in the vacuum, outside the fluid configuration, are

$$\begin{aligned} \nabla^2 \delta n - \frac{m^2 e^{-4\zeta}}{r^2 \sin^2 \theta} \delta n + \text{grad } \delta n \cdot \text{grad } (n + \zeta + n) + \text{grad } n \cdot \text{grad } \delta(\eta + \zeta + n) \\ + 2\delta\tau \left\{ n_{,22} + \frac{2}{r} n_{,2} + n_{,2}(\eta + \zeta + n)_{,2} \right. \\ \left. - \frac{1}{r^2} [n_{,33} + n_{,3} \cot \theta + n_{,3}(\eta + \zeta + n)_{,3}] \right\} \\ + 2 \left(n_{,2} \delta\tau_{,2} - \frac{1}{r^2} n_{,3} \delta\tau_{,3} \right) \\ = \frac{2e^{2n - 2(\eta + \zeta)}}{r^2 \sin^2 \theta} \left\{ |\text{grad } w|^2 \delta n + \frac{1}{2} \left[(w_{,r})^2 - \frac{1}{r^2} (w_{,\theta})^2 \right] \delta\tau \right\} + \frac{\mathfrak{G}}{r} (\delta R^{00}), \quad (54) \end{aligned}$$

$$\begin{aligned}
& \left(\frac{\partial^2}{\partial r^2} + \frac{3}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{2 \cot \theta}{r^2} \frac{\partial}{\partial \theta} \right) \delta(\eta + \zeta + n) + 2 \operatorname{grad}(\eta + \zeta + n) \cdot \operatorname{grad} \delta(\eta + \zeta + n) \\
& + 2 \left\{ \delta\tau_{,2} \left[(\eta + \zeta + n)_{,2} + \frac{1}{r} \right] - \frac{1}{r^2} \delta\tau_{,3} [(\eta + \zeta + n)_{,3} + \cot \theta] \right\} \\
& + 2\delta\tau \left\{ \left(\frac{\partial^2}{\partial r^2} + \frac{3}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} - \frac{2 \cot \theta}{r^2} \frac{\partial}{\partial \theta} \right) (\eta + \zeta + n) + \frac{2}{r^2} \right. \\
& \quad \left. + [(\eta + \zeta + n)_{,2}]^2 - \frac{1}{r^2} [(\eta + \zeta + n)_{,3}]^2 \right\} \\
& = \frac{2m^2 e^{-4\tau}}{r^2 \sin^2 \theta} \delta(\eta - \zeta + n) (\delta G^{22} + \delta G^{33}), \quad (55)
\end{aligned}$$

$$\begin{aligned}
& \delta(\eta + \zeta + n)_{,22} + 4 \left[(\eta + \zeta)_{,2} + \frac{1}{r} \right] \delta\zeta_{,2} + 2n_{,2} \delta(n - \eta + \zeta)_{,2} \\
& - \delta(\eta + \zeta + n)_{,2} \left[2(\eta - \zeta)_{,2} + \frac{1}{r} \right] \\
& - \frac{1}{r^2} \{ \delta(\eta + \zeta + n)_{,33} + 4[(\eta + \zeta)_{,3} + \cot \theta] \delta\zeta_{,3} \\
& \quad + 2n_{,3} \delta(n - \eta + \zeta)_{,3} - 2(\eta - \zeta)_{,3} \delta(\eta + \zeta + n)_{,3} \} \\
& - 4\delta\tau \left\{ \left[(\eta + \zeta + n)_{,2} + \frac{1}{r} \right] \left[(\eta - \zeta)_{,2} + \frac{1}{r} \right] + \left[(\eta + \zeta)_{,2} + \frac{1}{r} \right] n_{,2} \right. \\
& \quad \left. + \frac{1}{r^2} [(\eta + \zeta + n)_{,3} + \cot \theta] (\eta - \zeta)_{,3} + [(\eta + \zeta)_{,3} + \cot \theta] n_{,3} \right\} \\
& + \frac{2e^{2n-2(\eta+\zeta)}}{r^2 \sin^2 \theta} \left[(w_{,r})^2 - \frac{1}{r^2} (w_{,\theta})^2 \right] \delta n + \frac{\mathfrak{D}}{r} \\
& = -\frac{e^{2n-2(\eta+\zeta)}}{r^2 \sin^2 \theta} |\operatorname{grad} w|^2 \delta\tau - \frac{2m^2 e^{-4\tau}}{r^2 \sin^2 \theta} \delta\tau (\delta G^{22} - \delta G^{33}), \quad (56)
\end{aligned}$$

$$\begin{aligned}
& \delta(n + \eta - \zeta)_{,2} - \left[(\eta + \zeta)_{,2} + \frac{1}{r} \right] \delta(n + \eta - \zeta) + n_{,2} \delta(n - \eta + \zeta) \\
& = \frac{Qw_{,3}}{2r^2 \sin \theta} e^{-(\eta+\zeta+n)} - \delta\tau_{,2} - \left[(\eta + n - 3\zeta)_{,2} + \frac{1}{r} \right] \delta\tau (\delta R^{12}), \quad (57)
\end{aligned}$$

$$\begin{aligned}
& \delta(n + \eta - \zeta)_{,3} - [(\eta + \zeta)_{,3} + \cot \theta] \delta(n + \eta - \zeta) + n_{,3} \delta(n - \eta + \zeta) \\
& = -\frac{Qw_{,2}}{2 \sin \theta} e^{-(\eta+\zeta+n)} + \delta\tau_{,3} + [(\eta + n - 3\zeta)_{,3} - \cot \theta] \delta\tau (\delta R^{13}), \quad (58)
\end{aligned}$$

and

$$\begin{aligned} & \frac{1}{r^2} (e^{-3n+\eta+\zeta} r^2 \Omega_{,2})_{,2} + \frac{1}{r^2 \sin \theta} (e^{-3n+\eta+\zeta} \Omega_{,3} \sin \theta)_{,3} - \frac{m^2}{r^2 \sin^2 \theta} e^{-3n+\eta-3\zeta} \Omega \\ &= \frac{1}{r^2 \sin \theta} \{ [w_{,2}(3\delta n - \delta\eta - \delta\zeta + 2\delta\tau)]_{,3} - [w_{,3}(3\delta n - \delta\eta - \delta\zeta - 2\delta\tau)]_{,2} \}, \quad (59) \end{aligned}$$

where

$$\mathfrak{E} = \frac{e^{-(\eta+\zeta+n)}}{r \sin \theta} (\Omega_{,r} w_{,\theta} - \Omega_{,\theta} w_{,r}) \quad \text{and} \quad \mathfrak{D} = -\frac{e^{-(\eta+\zeta+n)}}{r \sin \theta} (\Omega_{,r} w_{,\theta} + \Omega_{,\theta} w_{,r}). \quad (60)$$

For determining the asymptotic behaviors of δn , $\delta\eta$, and $\delta\zeta$ as $r \rightarrow \infty$, we substitute in the foregoing equations the following known asymptotic forms of the equilibrium quantities:

$$n = -\frac{M}{r} + O(r^{-3}), \quad \eta = \frac{M}{r} + O(r^{-2}), \quad \zeta = O(r^{-2}),$$

and

$$\omega = -\frac{2J \sin \theta}{r} + O(r^{-2}) \quad \text{as} \quad r \rightarrow \infty. \quad (61)$$

Retaining only the dominant terms, we find that equations (54)–(58) reduce to³

$$\nabla^2 \delta n - \frac{m^2}{r^2 \sin^2 \theta} \delta n = 0, \quad (62)$$

$$\begin{aligned} & \left(\frac{\partial^2}{\partial r^2} + \frac{3}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{2 \cot \theta}{r^2} \frac{\partial}{\partial \theta} \right) \delta(\eta + \zeta + n) \\ & + \frac{4}{r^2} \delta\tau + \frac{2}{r} \delta\tau_{,2} - \frac{2 \cot \theta}{r^2} \delta\tau_{,3} = \frac{2m^2}{r^2 \sin^2 \theta} \delta(n + \eta - \zeta), \quad (63) \end{aligned}$$

$$\begin{aligned} & \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} - \frac{2 \cot \theta}{r^2} \frac{\partial}{\partial \theta} \right) \delta(\eta + \zeta + n) + \frac{4M}{r^2} \delta n_{,2} \\ & - \frac{4}{r^2} \delta\tau - 2 \left(\frac{1}{r} \frac{\partial}{\partial r} - \frac{\cot \theta}{r^2} \frac{\partial}{\partial \theta} \right) \delta(n + \eta - \zeta) = -\frac{2m^2}{r^2 \sin^2 \theta} \delta\tau, \quad (64) \end{aligned}$$

$$(\delta n + \delta\eta - \delta\zeta + \delta\tau)_{,2} - \frac{1}{r} (\delta n + \delta\eta - \delta\zeta - \delta\tau) + \frac{2M}{r^2} \delta n = 0, \quad (65)$$

and

$$(\delta n + \delta\eta - \delta\zeta - \delta\tau)_{,3} - (\delta n + \delta\eta - \delta\zeta - \delta\tau) \cot \theta = 0. \quad (66)$$

³ In deriving equations (62)–(66), it has further been assumed that decreases more rapidly than Qr^3 . This is the case for the particular solution (76) we shall find. The solutions we shall find form a consistent set; and it would appear that these are the solutions relevant for the problem on hand.

From equation (62), we may conclude that

$$\delta n = D \frac{P_l^m(\cos \theta)}{r^{l+1}}, \quad (67)$$

where D is a constant; and from equation (66) it follows that

$$\delta n + \delta \eta - \delta \zeta = \delta \tau. \quad (68)$$

Equation (65) now gives

$$\delta \tau_{,r} = -\frac{M}{r^2} \delta n = -MD \frac{P_l^m(\cos \theta)}{r^{l+3}}, \quad (69)$$

or

$$\delta \tau = \frac{MD}{l+2} \frac{P_l^m(\cos \theta)}{r^{l+2}}. \quad (70)$$

Next, adding equations (63) and (64) and making use of the relation (68), we find

$$\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} \right) \delta(n + \eta + \zeta) = 2(l+1)MD \frac{P_l^m(\cos \theta)}{r^{l+4}}. \quad (71)$$

The required solution of this equation is

$$\delta n + \delta \eta + \delta \zeta = \frac{2MD}{l+2} \frac{P_l^m(\cos \theta)}{r^{l+2}}. \quad (72)$$

Combining the results of equations (67), (68), (70), and (72), we finally obtain

$$\delta n = \frac{DP_l^m(\cos \theta)}{r^{l+1}}, \quad \delta \tau = \frac{MD}{l+2} \frac{P_l^m(\cos \theta)}{r^{l+2}},$$

$$\delta \zeta = \frac{MD}{2(l+2)} \frac{P_l^m(\cos \theta)}{r^{l+2}} \quad \text{and} \quad \delta \eta = -\frac{DP_l^m(\cos \theta)}{r^{l+1}} + \frac{3MD}{2(l+2)} \frac{P_l^m(\cos \theta)}{r^{l+2}}. \quad (73)$$

The case of greatest interest is $m = l$ when

$$P_l^l(\cos \theta) = \sin^l \theta. \quad (74)$$

In this case equation (59) gives

$$\begin{aligned} \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial}{\partial \theta} - \frac{l^2}{r^2 \sin^2 \theta} \right) \Omega \\ = -8(l+2) \frac{JD \sin^l \theta \cos \theta}{r^{l+2}}; \quad (75) \end{aligned}$$

and the required solution of this equation is

$$\Omega = -\frac{4JD \sin^l \theta \cos \theta}{r^{l+3}}. \quad (76)$$

It can now be verified that with the asymptotic forms of the unperturbed and the perturbed potentials given by equations (61), (73), and (75), the second surface-integral

which occurs in equation (49) makes no contribution as $r \rightarrow \infty$; and it also makes no contribution for $r \rightarrow 0$ in view of the regularity conditions here. Hence it vanishes identically.

VI. THE VARIATIONAL PRINCIPLE

Formally identifying ξ^α and $\bar{\xi}^\alpha$ and the barred and the unbarred quantities in equations (49) and (52), we obtain

$$\begin{aligned}
 & -m^2 \iiint \left[\sqrt{-g} \left\{ (\epsilon + p)(u^1)^2 \sum_{\alpha} e^{2\mu_{\alpha}} (\xi^{\alpha})^2 \right. \right. \\
 & \quad \left. \left. + \frac{e^{-2p}}{4\pi} [(\delta\mu)^2 + 2\delta n \delta\mu - (\delta\tau)^2] \right\} - \frac{1}{16\pi} e^{-3n-p+\mu_2+\mu_3} \Omega^2 \right] d\phi dx^2 dx^3 \\
 & = \iiint \left[\sqrt{-g} \left\{ -\gamma p \left(1 + \frac{\gamma p}{V^2} \frac{1}{\epsilon + p} \right) \left(\frac{\Delta N}{N} \right)^2 - \frac{1}{\epsilon + p} \xi^{\alpha} p_{,\alpha} \xi^{\beta} \epsilon_{,\beta} \right. \right. \\
 & \quad \left. \left. + 2 \frac{\Delta N}{N} \xi^{\alpha} \left[p_{,\alpha} - \frac{\gamma p}{V^2} (\log u_0)_{,\alpha} \right] - \frac{\epsilon + p}{V^2} [\xi^{\alpha} (\log u_0)_{,\alpha}]^2 \right. \right. \\
 & \quad \left. \left. - \frac{2}{V(1-V^2)} \delta p \delta V + \frac{\epsilon + p}{V^2} (\delta n)^2 - 16(\epsilon + p)^2 \frac{V^2}{(1-V^2)^2} \sum_{\alpha} e^{2\mu_{\alpha}} (\xi^{\alpha})^2 \right. \right. \\
 & \quad \left. \left. + 4(\epsilon + p) u^1 u_0 [\delta n \xi^{\alpha} w_{,\alpha} + \delta\tau (\xi^2 w_{,2} - \xi^3 w_{,3})] \right. \right. \\
 & \quad \left. \left. - [(\epsilon + p)(\delta n)^2 + 4(\epsilon - p)\delta n \delta\mu + 2\delta\epsilon \delta n - 4p(\delta\mu)^2] \right\} \right. \\
 & \quad \left. - \frac{1}{4\pi} e^{3n-p} \mathfrak{X} [(\delta n)^2 + \frac{1}{2}(\delta\tau)^2] + \frac{1}{\pi} (\mathfrak{B} - \frac{1}{4}e^{3n-p}\mathfrak{J}) \delta n \delta\tau \right. \\
 & \quad \left. + \frac{1}{2\pi} \mathfrak{U}(\delta\tau)^2 + \frac{1}{4\pi} e^{\beta} [e^{\mu_3-\mu_2} (\delta n_{,2})^2 + e^{\mu_2-\mu_3} (\delta n_{,3})^2] \right. \\
 & \quad \left. - \frac{1}{2\pi} e^{\beta} \{ (e^{\mu_3-\mu_2} \delta\mu_{3,2})_{,2} + (e^{\mu_2-\mu_3} \delta\mu_{2,3})_{,3} \} \delta n \right. \\
 & \quad \left. + \frac{1}{2\pi} e^{\beta} \{ e^{\mu_3-\mu_2} [\beta_{,2} \delta\mu_{3,2} + (\beta + 2\mu_3)_{,2} \delta n_{,2}] \right. \\
 & \quad \quad \left. - e^{\mu_2-\mu_3} [\beta_{,3} \delta\mu_{2,3} + (\beta + 2\mu_2)_{,3} \delta n_{,3}] \} \delta\tau \right. \\
 & \quad \left. - 2(\epsilon + p) u^1 u_0 e^{-3n+p} (e^{\mu_2-\mu_3} \xi^2 \Omega_{,3} - e^{\mu_3-\mu_2} \xi^3 \Omega_{,2}) \sqrt{-g} \right. \\
 & \quad \left. - \frac{1}{4\pi} (\mathfrak{C} \delta n + \mathfrak{D} \delta\tau) - \frac{1}{16\pi} e^{-3n+p} [e^{\mu_2-\mu_3} (\Omega_{,2})^2 + e^{\mu_3-\mu_2} (\Omega_{,3})^2] \right] d\phi dx^2 dx^3 \\
 & \quad + \frac{1}{4\pi} \iint d\phi dr [e^{\mu_2-\mu_3+n} (e^p)_{,3} (\delta n)^2]. \tag{77}
 \end{aligned}$$

By arguments exactly analogous to those used in Paper II, § IV, and Paper IV, § V, it can be shown that the foregoing equation provides a variational base for determining m^2 in the sense that if we assign ξ^α , τ , and Ω compatible only with the boundary conditions, determine ΔN , Δp , $\Delta\epsilon$, δV , δn , and $\delta\mu$ with the aid of the initial-value equations (29), (30), (32), and (43), and require that the first variation δm^2

vanishes for all arbitrary variations in ξ^α , τ , and Ω consistent only with the initial-value equations and boundary conditions, *then* all the dynamical equations of the problem will be satisfied and the chosen functions ξ^α , τ , and Ω will be proper solutions and m^2 as determined by equations (72) will be a true characteristic value.

It will be observed that ΔN and δV as given by equations (9) and (32) require a knowledge of δn and $\delta\mu$. It remains to show how the initial-value equations (43) suffice to determine them. For this latter purpose we first rewrite equations (43) in the forms

$$\begin{aligned} e^p[e^{-p}(\delta n + \delta\mu)]_{,2} + n_{,2}(\delta n - \delta\mu) &= \mathfrak{F}_2, \\ e^p[e^{-p}(\delta n + \delta\mu)]_{,3} + n_{,3}(\delta n - \delta\mu) &= \mathfrak{F}_3, \end{aligned} \quad (78)$$

where

$$\begin{aligned} \mathfrak{F}_2 &= e^{2\mu_2} \left[-8\pi(\epsilon + p) \frac{V^2}{1 - V^2} \xi^2 + \frac{\Omega w_{,3}}{2\sqrt{-g}} \right] - \delta\tau_{,2} - (2\mu_3 + n - p)_{,2} \delta\tau, \\ \mathfrak{F}_3 &= e^{2\mu_3} \left[-8\pi(\epsilon + p) \frac{V^2}{1 - V^2} \xi^3 - \frac{\Omega w_{,2}}{2\sqrt{-g}} \right] + \delta\tau_{,3} + (2\mu_2 + n - p)_{,3} \delta\tau. \end{aligned} \quad (79)$$

By the variational hypothesis, \mathfrak{F}_2 and \mathfrak{F}_3 may be considered as known functions. Accordingly, equations (78) provide standard quasi-linear differential equations for δn and $\delta\mu$; we can accordingly solve for them. We find (cf. Paper IV, eq. [60])

$$(\delta n + \delta\mu)_{\text{along } n = \text{constant}} = e^p \int_{n = \text{constant}} e^{-p} (\mathfrak{F}_2 dx^2 + \mathfrak{F}_3 dx^3), \quad (80)$$

where the line integral on the right-hand side is taken over curves of constant n . With $\delta n + \delta\mu$ determined in this fashion, $\delta n - \delta\mu$ follows from either of the two equations (73).

VII. CONCLUDING REMARKS

The present paper executes for the theory of nonaxisymmetric quasi-stationary deformations of uniformly rotating configurations what Papers I, II, and IV accomplished for the theory of axisymmetric oscillations. The two theories are remarkably similar. The present theory is, in fact, simpler since gravitational radiation is absent in view of the quasi-stationary conditions that have been assumed to prevail.

The extension of the theory developed in this paper to differentially rotating systems is particularly important since in its terms the question of the stable existence of disk-like objects in general relativity can be decided. The extension is presently being considered.

The specialization of the criterion derived in this paper, for the occurrence of a Dedekind-like point of bifurcation along a sequence of axisymmetric systems, to the vacuum solutions of Einstein's equations (such as the Kerr metric) in the manner of Paper III, while it is straightforward at a formal level, nevertheless, requires that certain delicate conceptual problems be first resolved. On this account the subject is postponed to a later paper.

The work presented in this paper was begun and largely completed while the authors were visiting members of the Department of Astrophysics, Oxford University (England) during 1972 January–June; and we are grateful for the excellent facilities provided to us by Professor D. E. Blackwell and Dr. D. W. Sciama; and S. C. is also indebted to All Souls College, Oxford, for a Visiting Fellowship during the same period.

The research reported in this paper has in part been supported by the National Science Foundation under grants GP-28342 and GP-34721X to the University of Chicago.

REFERENCES

- Chandrasekhar, S. 1969, *Ellipsoidal Figures of Equilibrium* (New Haven, Conn.: Yale University Press).
———. 1970, *Ap. J.*, **161**, 561.
Chandrasekhar, S., and Esposito, F. Paul. 1970, *Ap. J.*, **160**, 153.
Chandrasekhar, S., and Friedman, J. L. 1972a, *Ap. J.*, **175**, 379 (Paper I).
———. 1972b, *ibid.*, **176**, 745 (Paper II).
———. 1972c, *ibid.*, **177**, 745 (Paper III).
———. 1973, *ibid.*, **181**, 481 (Paper IV).
Chandrasekhar, S., and Lebovitz, Norman, 1973, *Ap. J.*, **185**, 19.
Ostriker, Jeremiah, P., and Bodenheimer, Peter. 1973, *Ap. J.*, **180**, 171.