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ON THE STABILITY OF AXISYMMETRIC SYSTEMS TO AXISYMMETRIC PERTURBATIONS IN GENERAL RELATIVITY. II. A CRITERION FOR THE ONSET OF INSTABILITY IN UNIFORMLY ROTATING CONFIGURATIONS AND THE FREQUENCY OF THE FUNDAMENTAL MODE IN CASE OF SLOW ROTATION

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ABSTRACT

The theory developed in Paper I is applied to solve two problems in general relativity: to obtain a criterion for the onset of instability in a uniformly rotating configuration via a neutral mode of axisymmetric oscillation; and to obtain an exact and an explicit formula for the square of the frequency of the fundamental axisymmetric mode of oscillation of a configuration rotating uniformly but slowly.

I. INTRODUCTION

In this paper, we shall be concerned with establishing a criterion for the onset of instability in uniformly rotating configurations in general relativity. It is not difficult to envisage the form such a criterion will take in the case of slow rotation. We know that in the absence of rotation the onset of instability, in general relativity and in the Newtonian theory, is via a neutral mode of radial oscillation; and the criterion for the instability is a condition on a suitably averaged value of the adiabatic exponent γ. On the Newtonian theory, the effect of a slow rotation is to modify the condition on γ by terms of order Ω^2 where Ω denotes the angular velocity of rotation; and the radial mode of oscillation in the nonrotating case is replaced by an axisymmetric mode of oscillation in the rotating case. But in general relativity, a feature that is absent in the Newtonian theory and which can have a decisive effect is the emission of gravitational radiation by rotating objects when they become nonstationary. However, since the effects derived from gravitational radiation depend both on the distortion of the object from sphericity and on the amplitude of the oscillation, it would appear that these effects can be ignored in the limit of slow rotation. Accordingly, we may expect that, in the case of slow rotation, the onset of instability in general relativity will also be via a neutral mode of axisymmetric oscillation. If such neutral modes exist for slow rotation, then their persistence for increasing rotation may also be expected, at least for a while. In any event, it would appear useful to establish a criterion for the occurrence of a neutral mode of axisymmetric oscillation. It is the object of this paper to establish such a criterion.

A by-product of the investigation is an explicit and an exact formula for the square of the frequency of oscillation of a slowly rotating configuration which depends (as in the Newtonian theory) only on a knowledge of the Lagrangian displacement associated with the fundamental radial mode of oscillation of the nonrotating configuration and of the uniform ($l = 0$)-deformation caused by the rotation.

The mathematical theory developed in an earlier paper (Chandrasekhar and

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Friedman 1972; this paper will be referred to hereafter as Paper I) for dealing with the evolution of axisymmetric perturbations of rotating masses does not exclude the possibility of gravitational radiation. Why then, it may be asked, is it necessary to concern oneself with quasi-stationary deformations and neutral modes in order just to avoid the effects derived from gravitational radiation? The answer is that the solution to the general problem requires the resolution of a number of subtle questions and it contributes to one's understanding to resolve these same questions, first, in the simpler contexts. The general problem, allowing for gravitational radiation, is considered in the fourth paper of this series (now nearing completion).

II. THE BASIC EQUATIONS

The metric appropriate to a *stationary* axisymmetric configuration has the form

$$ds^2 = -e^{2\nu}(dt)^2 + e^{2\psi}(d\phi - \omega dt)^2 + e^{2\mu_2}(dx^2)^2 + e^{2\mu_3}(dx^3)^2\,, \qquad (1)$$

where ν, ψ, ω, μ_2, and μ_3 are functions only of x^2 and x^3; and we have the freedom to impose a coordinate condition on μ_2 and μ_3. Also, in the stationary case, there can be no motions in the x^2- and the x^3-directions: only rotational motions specified by $\Omega = d\phi/dt$, in the ϕ-direction can prevail; and we are presently interested in the case Ω = constant. When such a system is perturbed, motions represented by v^α ($\alpha = 2, 3$) will ensue in the x^2- and the x^3-directions. Simultaneously, the distribution of Ω (whether initially constant or not) and all the other functions describing the field and the fluid will be subject to changes. In addition, the coordinate condition imposed on μ_2 and μ_3 will be violated; and, finally, even the *form* of the metric will be altered by the emergence of further nondiagonal components. Indeed, as we have seen in detail in Paper I (§ II), the metric in the nonstationary case must be of the form

$$ds^2 = -e^{2\nu}(dt)^2 + e^{2\psi}(d\phi - q_{2,0}dx^2 - q_{3,0}dx^3 - \omega dt)^2 + e^{2\mu_2}(dx^2)^2 + e^{2\mu_3}(dx^3)^2\,, \qquad (2)$$

where $q_{\alpha,0}$, like v^α, is a quantity of the first order of smallness and is a direct result of the perturbation.

The equations that govern the evolution of perturbed configurations have been written down in Paper I (Part III). We do not strictly need the full generality of these equations for ascertaining the conditions when a uniformly rotating configuration, of a given mass M and angular momentum J and described in terms of a metric of the form (1), can be subject to a quasi-stationary deformation, without violating any of the conditions of equilibrium,[1] so that it continues to be described by a metric of the same form (1). Nevertheless, it will be convenient, both for our present and later purposes, to establish certain basic relations of general validity; and the analysis in this and in the following two sections is not restricted to quasi-stationary deformations.

We shall now assemble the equations of the problem. The notation will be the same as in Paper I; the various symbols will have the same meanings and no attempt will be made to redefine them.

Among the equations which govern the departures from equilibrium of a system that is initially static or stationary, we distinguish two classes: *initial-value equations* and *dynamical equations*. Initial-value equations are those that are first order in the time derivatives. Dynamical equations are those that are second order in the time derivatives. Initial-value equations can be directly integrated with respect to the time when they are expressed in terms of a Lagrangian displacement. In contrast, dynamical equations lead to the characteristic-value problems that determine the normal modes of oscillation of the system.

For the problem we are presently considering, the initial-value equations are (1)

[1] In § V, we shall formulate precisely what we mean by the phrase "without violating any of the conditions of equilibrium."

the equation expressing the conservation of baryon number (I, eq. [127]),

$$\frac{\Delta N}{N} = -\frac{1}{u^0\sqrt{-g}}(\xi^\alpha u^0\sqrt{-g})_{,\alpha} - \delta[\log(u^0\sqrt{-g})]$$

$$= -\frac{1}{u^0\sqrt{-g}}(\xi^\alpha u^0\sqrt{-g})_{,\alpha} - \frac{V\delta V}{1-V^2} - \delta(\psi + \mu_2 + \mu_3)\,; \tag{3}$$

(2) the adiabatic condition (I, eq. [128])

$$\frac{\Delta p}{p} = \gamma\frac{\Delta N}{N} \quad \text{and} \quad \Delta\left(\frac{\epsilon + p}{N}\right) = \gamma p\frac{\Delta N}{N^2}\,; \tag{4}$$

(3) the equation expressing the conservation of angular momentum (I, eq. [133])

$$\frac{\delta V}{V(1-V^2)} = -\frac{\gamma p}{\epsilon + p}\frac{\Delta N}{N} - \delta\psi - \xi^\alpha(\log u_1)_{,\alpha}\,; \tag{5}$$

and finally (4) the (0, 2)- and the (0, 3)-components of the linearized field equations (I, eqs. [144] and [145])

$$8\pi\sqrt{-g}\frac{\epsilon + p}{1 - V^2}\xi^2 = e^{\mu_3-\mu_2}[(e^\beta\delta\psi)_{,2} - (e^\beta)_{,2}\delta\mu_3 + e^\beta\delta\mu_{3,2} - 2e^\beta\nu_{,2}\delta\psi$$
$$+ e^\beta(\psi + \mu_3)_{,2}\delta(\mu_3 - \mu_2)] + \tfrac{1}{2}Q\omega_{,3}\,, \tag{6}$$

and

$$8\pi\sqrt{-g}\frac{\epsilon + p}{1 - V^2}\xi^3 = e^{\mu_2-\mu_3}[(e^\beta\delta\psi)_{,3} - (e^\beta)_{,3}\delta\mu_2 + e^\beta\delta\mu_{2,3} - 2e^\beta\nu_{,3}\delta\psi$$
$$+ e^\beta(\psi + \mu_2)_{,3}\delta(\mu_2 - \mu_3)] - \tfrac{1}{2}Q\omega_{,2}\,, \tag{7}$$

where

$$Q = e^{3\psi+\nu-\mu_2-\mu_3}(q_{2,3} - q_{3,2})\,. \tag{8}$$

Alternative forms of equations (6) and (7) are (cf. eq. [147] and [148])

$$(\delta\psi + \delta\mu)_{,2} - \nu_{,2}(\delta\psi + \delta\mu) + \psi_{,2}(\delta\psi - \delta\mu)$$
$$= e^{2\mu_2}\left(8\pi\frac{\epsilon + p}{1 - V^2}\xi^2 - \frac{Q}{2\sqrt{-g}}\omega_{,3}\right) - \delta\tau_{,2} - (2\mu_3 + \psi - \nu)_{,2}\delta\tau \tag{9}$$

and

$$(\delta\psi + \delta\mu)_{,3} - \nu_{,3}(\delta\psi + \delta\mu) + \psi_{,3}(\delta\psi - \delta\mu)$$
$$= e^{2\mu_3}\left(8\pi\frac{\epsilon + p}{1 - V^2}\xi^3 + \frac{Q}{2\sqrt{-g}}\omega_{,2}\right) + \delta\tau_{,3} + (2\mu_2 + \psi - \nu)_{,3}\delta\tau\,, \tag{10}$$

where

$$\delta\mu = \tfrac{1}{2}\,\delta(\mu_3 + \mu_2) \quad \text{and} \quad \delta\tau = \tfrac{1}{2}\,\delta(\mu_3 - \mu_2)\,. \tag{11}$$

Since the equation

$$\delta\{e^{\mu_2+\mu_3}[G^{(0)(0)} - 8\pi T^{(0)(0)}]\} = 0 \tag{12}$$

can be derived from equations (6) and (7) (cf. I, eq. [167]), we may regard the (0, 0)-component of the field equation, also, as an initial value equation.

The (1, 2)- and the (1, 3)-components of the linearized field-equations play a double role in this theory: they provide initial-value equations for $\delta\omega_{,\alpha}$ while their integrability condition leads to a dynamical equation for Q. Thus (I, eqs. [151]–[153])

$$\delta\omega_{,2} - q_{2,00} = 16\pi(\epsilon + p)u^0u_1e^{-2\psi+2\nu+2\mu_2}\xi^2$$
$$- \omega_{,2}(3\delta\psi - \delta\nu - \delta\mu_2 + \delta\mu_3) - e^{-3\psi+\nu+\mu_2-\mu_3}Q_{,3} \tag{13}$$

and

$$\delta\omega_{,3} - q_{3,00} = 16\pi(\epsilon + p)u^0 u_1 e^{-2\psi+2\nu+2\mu_3}\xi^3 - \omega_{,3}(3\delta\psi - \delta\nu + \delta\mu_2 - \delta\mu_3) + e^{-3\psi+\nu+\mu_3-\mu_2}Q_{,2}\,; \tag{14}$$

and the elimination of $\delta\omega$ from these equations gives

$$\begin{aligned}&(e^{-3\psi+\nu-\mu_2+\mu_3}Q_{,2})_{,2} + (e^{-3\psi+\nu+\mu_2-\mu_3}Q_{,3})_{,3}\\ &= e^{-3\psi-\nu+\mu_2+\mu_3}Q_{,00}\\ &\quad - [\omega_{,2}(3\delta\psi - \delta\nu - \delta\mu_2 + \delta\mu_3)]_{,3} + [\omega_{,3}(3\delta\psi - \delta\nu + \delta\mu_2 - \delta\mu_3)]_{,2}\\ &\quad + 16\pi\{[(\epsilon + p)u^0 u_1 e^{-2\psi+2\nu+2\mu_2}\xi^2]_{,3} - [(\epsilon + p)u^0 u_1 e^{-2\psi+2\nu+2\mu_3}\xi^3]_{,2}\}\,. \end{aligned} \tag{15}$$

We next turn to the remaining dynamical equations. In writing these equations, we shall separate the time and the space variables and seek solutions which have a dependence on time given by

$$e^{i\sigma t}\,, \tag{16}$$

where σ is a characteristic-value parameter to be determined. This time dependence will occur as a factor in all the equations. We shall suppose that this common factor has been removed and that all quantities (such as ξ^α, ΔN, etc.) which appear in the equations from now on represent the space-dependent amplitude-functions; thus it will be assumed, for example, that the chosen Lagrangian displacement is of the form

$$\xi^\alpha(x^2, x^3)e^{i\sigma t}\,. \tag{17}$$

The pulsation equation (I, eq. [139]) is a dynamical equation. It is given by

$$\begin{aligned}-\sigma^2(\epsilon + p)(u^0)^2\sqrt{-g}\,e^{2\mu_\alpha}\xi^\alpha = &\underbrace{-\,u^0\sqrt{-g}\,\frac{\partial}{\partial x^\alpha}\left(\frac{\gamma p}{u^0}\frac{\Delta N}{N}\right)}_{\text{I}}\\ &\underbrace{+\,(\epsilon + p)\sqrt{-g}\,\frac{\partial}{\partial x^\alpha}\frac{\Delta u^0}{u^0}}_{\text{II}}\underbrace{+\,\sqrt{-g}\,\frac{\Delta N}{N}\frac{\partial p}{\partial x^\alpha}}_{\text{III}}\underbrace{-\,(\epsilon + p)u^0 u_1\sqrt{-g}\left(\frac{\partial\Delta\Omega}{\partial x^\alpha} - \frac{\partial^2 q_\alpha}{\partial t^2}\right)}_{\text{IV}}.\end{aligned} \tag{18}$$

Besides this pulsation equation, we must include, at most, two of the linearized field-equations given in Paper I, § XII; "at most two," since equations (6), (7), (13), and (14) already account for four equations; and there can be no more than six linearly independent field-equations. As the remaining dynamical field-equations, we may take the (1, 1)- and the [(2, 2) + (3, 3)]-components of the field equations.

Our problem then is to solve equation (18), consistently with the initial-value equations (3)–(7), equations (13) and (14), the remaining dynamical equations, and the appropriate boundary conditions. The boundary conditions are that ξ^α vanishes at the origin and remains bounded and continuous over its domain; that Δp vanishes on the boundary, and that all the remaining field-variables (such as $\delta\psi$, $\delta\nu$, etc.) vanish sufficiently rapidly at infinity. While it is not strictly necessary, we shall assume that ϵ (in addition to p) vanishes on the boundary of the stationary configuration: a consequence of this assumption is that δp will also vanish on the boundary—a fact which enables us to avoid some formal (unessential) complications.

The problem to which we are thus led is a characteristic-value problem for σ^2.

III. A FORMULA FOR σ^2

First we define a trial displacement $\bar{\xi}^\alpha$ as one which satisfies the same boundary conditions as are demanded of a proper ξ^α but is arbitrary otherwise. And we also define associated *barred variations*, $\bar{\Delta}N$, $\bar{\delta}\psi$, etc., and require that they satisfy the same boundary conditions as are demanded of the proper unbarred variations and are consistent with the initial-value equations (3)–(7) and equations (13) and (14) as initial-value equations for $\delta\omega_{,\alpha}$.

We shall return in Paper IV (in preparation) to the question of the extent to which the requirements on the barred variations suffice to specify them uniquely. For the present, we shall only assume that such variations can be defined consistently with the initial-value equations for any chosen trial displacement.

We now multiply the pulsation equation (18) by a trial displacement $\bar{\xi}^\alpha$, sum over α, and integrate over all 3-space. The left-hand side of the equations gives

$$-\sigma^2 \int\!\!\int\!\!\int (\epsilon + p)(u^0)^2 \sqrt{-g}\, \Sigma_\alpha\, e^{2\mu_\alpha}\xi^\alpha\bar{\xi}^\alpha dx\,. \tag{19}$$

This expression is manifestly symmetric in the barred and the unbarred quantities. Our object is to bring the result of the integration (over x^1, x^2, and x^3) and summation (over $\alpha = 2, 3$) of the terms on the right-hand side of the pulsation equation to a similar manifestly symmetric form. In order to achieve this symmetry numerous integrations by parts and substitutions from the field equations (satisfied only by the unbarred variations) and the initial-value equations (satisfied by both the barred and the unbarred variations) are required. These reductions are far too long even to attempt giving some details. We shall have to be content with essentially writing down the results of the reductions, but two general observations should be made. *First*, in view of the possibility of gravitational radiation, some of the volume integrals may not converge when extended to infinity. On this account, the integrations will be confined, in the first instance, to a sphere of a sufficiently large radius R. We shall let $R \to \infty$ if all the integrals converge as will be the case for quasi-stationary deformations considered in § V; but the discussion of the convergence in the general case is postponed to Paper IV. *Second*, the integrated parts which result from the various integrations by parts (with respect to x^2 and x^3) require careful scrutiny for their survival and their convergence. In most instances, they vanish by virtue of the regularity conditions at the poles, the conditions $\epsilon = p = \Delta\epsilon = \Delta p = \delta p = 0$ that we have imposed on the boundary of the configuration, and on the vanishing of the other variations (barred and unbarred) sufficiently rapid as $R \to \infty$. Nevertheless, the surface integrals that survive on R will be retained and exhibited. Also, care is needed when the integrations by parts are with respect to the polar angle θ when using a system of spherical polar coordinates.

Considering first the terms I, II, and III in equation (18), we find after some integrations by parts and making use only of equations (3) and (4) (both with respect to the barred and the unbarred quantities),

$$\begin{aligned}\text{I} + \text{II} + \text{III}: \int\!\!\int\!\!\int \sqrt{-g}\Big[&-\gamma p \frac{\Delta N \bar{\Delta} N}{N^2} - \frac{1}{\epsilon + p}\xi^\alpha p_{,\alpha}\bar{\xi}^\beta \epsilon_{,\beta} \\ &+ \left(\frac{\bar{\Delta}N}{N}\xi^\alpha + \frac{\Delta N}{N}\bar{\xi}^\alpha\right) p_{,\alpha} - (\epsilon + p)\bar{\xi}^\alpha \delta\nu_{,\alpha} \\ &- \delta p\bar{\delta}(\log u^0\sqrt{-g}) + (\epsilon + p)\bar{\xi}^\alpha \left(\frac{V\delta V}{1 - V^2}\right)_{,\alpha}\Big] dx\,.\end{aligned} \tag{20}$$

It is found that the integrated parts which arise during the course of these reductions all vanish.

The reduction of the term IV in equation (18) requires the use of equations (5), (13), and (14) for the unbarred variations. We find

$$\text{IV:}\ \int\!\!\int\!\!\int \sqrt{-g}\Big\{-16\pi(\epsilon+p)^2\frac{V^2}{(1-V^2)^2}\sum_\alpha e^{2\mu_\alpha}\xi^\alpha\bar{\xi}^\alpha$$
$$+2(\epsilon+p)u^0u_1[\delta\psi\bar{\xi}^\alpha\omega_{,\alpha}+\tfrac{1}{2}(\bar{\xi}^2\omega_{,2}-\bar{\xi}^3\omega_{,3})\delta(\mu_3-\mu_2)]$$
$$+(\epsilon+p)\frac{V^2}{1-V^2}\bar{\xi}^\alpha\delta(\psi-\nu)_{,\alpha}-(\epsilon+p)u^0u_1\Big[(\Omega-\omega)\frac{\delta V}{V}\Big]_{,\alpha}\bar{\xi}^\alpha$$
$$+(\epsilon+p)u^0u_1e^{-3\psi+\nu}[e^{\mu_2-\mu_3}\bar{\xi}^2Q_{,3}-e^{\mu_3-\mu_2}\bar{\xi}^3Q_{,2}]\Big\}dx\,; \qquad (21)$$

and the integrated parts which arise during the course of these reductions also vanish.

Next, we combine the results (20) and (21), simplify, and rearrange, making use of the initial-value equation (5) (both with respect to the barred and the unbarred quantities). We find

I + II + III + IV :

$$\int\!\!\int\!\!\int \Big[\!\!\Big[\sqrt{-g}\Big\{-\gamma p\Big(1+\frac{\gamma p V^2}{\epsilon+p}\Big)\frac{\Delta N\bar{\Delta}N}{N^2}-\frac{1}{\epsilon+p}\xi^\alpha p_{,\alpha}\bar{\xi}^\beta\epsilon_{,\beta}$$
$$+\Big(\frac{\bar{\Delta}N}{N}\xi^\alpha+\frac{\Delta N}{N}\bar{\xi}^\alpha\Big)[p_{,\alpha}-\gamma pV^2(\log u_1)_{,\alpha}]-\frac{V}{1-V^2}(\delta p\bar{\delta}V+\bar{\delta}p\delta V)$$
$$-16\pi(\epsilon+p)^2\frac{V^2}{(1-V^2)^2}\sum_\alpha e^{2\mu_\alpha}\xi^\alpha\bar{\xi}^\alpha-(\epsilon+p)V^2\xi^\alpha(\log u_1)_{,\alpha}\bar{\xi}^\beta(\log u_1)_{,\beta}$$
$$+(\epsilon+p)V^2\delta\psi\bar{\delta}\psi\Big\}$$
$$+\sqrt{-g}\Big\{-(\epsilon+p)\delta\psi\bar{\delta}(\psi+\mu_2+\mu_3)-\delta\psi\bar{\delta}\epsilon-\delta p\delta(\psi+\mu_2+\mu_3)$$
$$+2(\epsilon+p)u^0u_1[\delta\psi\bar{\xi}^\alpha\omega_{,\alpha}+\tfrac{1}{2}(\bar{\xi}^2\omega_{,2}-\bar{\xi}^3\omega_{,3})\delta(\mu_3-\mu_2)]$$
$$+\frac{\epsilon+p}{1-V^2}\bar{\xi}^\alpha\delta(\psi-\nu)_{,\alpha}\Big\}$$
$$+\sqrt{-g}\{(\epsilon+p)u^0u_1e^{-3\psi+\nu}(e^{\mu_2-\mu_3}\bar{\xi}^2Q_{,3}-e^{\mu_3-\mu_2}\bar{\xi}^3Q_{,2})\}\Big]\!\!\Big]dx\,. \qquad (22)$$

It will be observed that the terms included in the first braces in the integrand are manifestly symmetric in the barred and the unbarred quantities.[2] To bring the remaining

[2] The symmetry of the term $\xi^\alpha p_{,\alpha}\bar{\xi}^\beta\epsilon_{,\beta}$ follows from the fact that by virtue of the equation (I, eq. [81])

$$p_{,\alpha}=(\epsilon+p)(\log u^0)_{,\alpha}\,,$$

the surfaces of constant p, constant ϵ, and constant u^0 all coincide and that therefore $p\equiv p(u^0)$ and $\epsilon\equiv\epsilon(u^0)$. Consequently,

$$\xi^\alpha p_{,\alpha}\bar{\xi}^\beta\epsilon_{,\beta}=\frac{dp}{du^0}\frac{d\epsilon}{du^0}\xi^\alpha u^0{}_{,\alpha}\bar{\xi}^\beta u^0{}_{,\beta}\,;$$

and the symmetry *is* manifest.

terms to a similar symmetric form we must consider the term

$$\int\int\int \sqrt{-g}\,\frac{\epsilon + p}{1 - V^2}\,\xi^\alpha \delta(\psi - \nu)_{,\alpha} dx\,. \tag{23}$$

The reduction of this term is a particularly "difficult" one: it uses the linearized versions of the (0, 0)-, (1, 1)-, [(2, 2) + (3, 3)]-, and [(2, 2) − (3, 3)]-components of the field equations (given, respectively, by I, eqs. [157], [158], [161], and [162]) for the unbarred quantities, as well as the initial-value equations (6) and (7) (with respect to the barred quantities); and several integrations by parts are also involved. Eventually, we find

$$\begin{aligned}
&\int\int\int \sqrt{-g}\,\frac{\epsilon + p}{1 - V^2}\,\xi^\alpha \delta(\psi - \nu)_{,\alpha} dx \\
&= \int\int\int \Big[\!\Big[-\frac{1}{4\pi} e^{3\psi-\nu} X(\delta\psi\bar{\delta}\psi + \tfrac{1}{2}\delta\tau\bar{\delta}\tau) + \frac{1}{2\pi} U\delta\tau\bar{\delta}\tau \\
&\quad + \frac{1}{2\pi}(W - \tfrac{1}{4}e^{3\psi-\nu}Y)(\delta\psi\bar{\delta}\tau + \bar{\delta}\psi\delta\tau) \\
&\quad + \frac{1}{4\pi} e^{\psi+\nu}(e^{\mu_3-\mu_2}\delta\psi_{,2}\bar{\delta}\psi_{,2} + e^{\mu_2-\mu_3}\delta\psi_{,3}\bar{\delta}\psi_{,3}) \\
&\quad - \frac{\sigma^2}{4\pi} e^{-2\nu}\sqrt{-g}(\delta\mu\bar{\delta}\mu + \delta\psi\bar{\delta}\mu + \bar{\delta}\psi\delta\mu - \delta\tau\bar{\delta}\tau) \\
&\quad - \frac{1}{4\pi} e^{\beta}\{[(e^{\mu_3-\mu_2}\delta\mu_{3,2})_{,2} + (e^{\mu_2-\mu_3}\delta\mu_{2,3})_{,3}]\bar{\delta}\psi + [(e^{\mu_3-\mu_2}\bar{\delta}\mu_{3,2})_{,2} + (e^{\mu_2-\mu_3}\bar{\delta}\mu_{2,3})_{,3}]\delta\psi\} \\
&\quad + \frac{1}{4\pi} e^{\beta+\mu_3-\mu_2}\{[\bar{\delta}\mu_{3,2}\beta_{,2} + \bar{\delta}\psi_{,2}(\beta + 2\mu_3)_{,2}]\delta\tau + [\delta\mu_{3,2}\beta_{,2} + \delta\psi_{,2}(\beta + 2\mu_3)_{,2}]\bar{\delta}\tau\} \\
&\quad - \frac{1}{4\pi} e^{\beta+\mu_2-\mu_3}\{[\bar{\delta}\mu_{2,3}\beta_{,3} + \bar{\delta}\psi_{,3}(\beta + 2\mu_2)_{,3}]\delta\tau + [\delta\mu_{2,3}\beta_{,3} + \delta\psi_{,3}(\beta + 2\mu_2)_{,3}]\bar{\delta}\tau\} \\
&\quad + 2(\epsilon + p)u^0u_1\sqrt{-g}[\bar{\delta}\psi\xi^\alpha\omega_{,\alpha} + (\xi^2\omega_{,2} - \xi^3\omega_{,3})\bar{\delta}\tau] \\
&\quad + \sqrt{-g}[4p\bar{\delta}\mu(\delta\psi + \delta\mu) + 2\delta p\bar{\delta}\mu - 2\delta\mu\bar{\delta}\psi(\epsilon - p) - \delta(\epsilon - p)\bar{\delta}\psi] \\
&\quad - \frac{1}{8\pi}[(\omega_{,2}Q_{,3} - \omega_{,3}Q_{,2})\bar{\delta}\psi + (\omega_{,2}Q_{,3} + \omega_{,3}Q_{,2})\bar{\delta}\tau] \\
&\quad + \frac{1}{16\pi}\bar{Q}[\omega_{,3}\delta(\psi - \nu)_{,2} - \omega_{,2}\delta(\psi - \nu)_{,3}]\Big]\!\Big] dx\,.
\end{aligned} \tag{24}$$

On examining the integrated parts which arise during the course of the reductions leading to equation (24) we find that some of them can survive. We consider these terms separately in § III-*a*) below.

Now combining the results (22) and (24) and rearranging, we finally obtain

$$-\sigma^2 \int\int\int \Big\{ \sqrt{-g}\Big[(\epsilon + p)(u^0)^2 \sum_\alpha e^{2\mu_\alpha}\xi^\alpha\bar{\xi}^\alpha - \frac{e^{-2\nu}}{4\pi}(\delta\mu\bar{\delta}\mu + \delta\psi\bar{\delta}\mu + \bar{\delta}\psi\delta\mu - \delta\tau\bar{\delta}\tau)\Big] + \frac{1}{16\pi}e^{-3\psi-\nu+\mu_2+\mu_3}Q\bar{Q}\Big\} dx$$

$$= \int\int\int \Big[\!\Big[\sqrt{-g}\Big\{ -\gamma p\left(1 + \frac{\gamma p V^2}{\epsilon + p}\right)\frac{\Delta N \bar{\Delta} N}{N^2} - \frac{1}{\epsilon + p}\xi^\alpha p_{,\alpha}\bar{\xi}^\beta \epsilon_{,\beta}$$

$$+ \left(\frac{\Delta N}{N}\bar{\xi}^\alpha + \frac{\bar{\Delta} N}{N}\xi^\alpha\right)[p_{,\alpha} - \gamma p V^2(\log u_1)_{,\alpha}] - (\epsilon + p)V^2\xi^\alpha(\log u_1)_{,\alpha}\bar{\xi}^\beta(\log u_1)_{,\beta}$$

$$- \frac{V}{1 - V^2}(\delta p\bar{\delta}V + \bar{\delta}p\delta V) + (\epsilon + p)V^2\delta\psi\bar{\delta}\psi - 16\pi(\epsilon + p)^2\frac{V^2}{(1 - V^2)^2}\sum_\alpha e^{2\mu_\alpha}\xi^\alpha\bar{\xi}^\alpha$$

$$- [(\epsilon + p)\delta\psi\bar{\delta}\psi + 2(\epsilon - p)(\delta\psi\bar{\delta}\mu + \bar{\delta}\psi\delta\mu) + \delta\epsilon\bar{\delta}\psi + \bar{\delta}\epsilon\delta\psi - 4p\delta\mu\bar{\delta}\mu]$$

$$+ 2(\epsilon + p)u^0 u_1[\delta\psi\bar{\xi}^\alpha\omega_{,\alpha} + \bar{\delta}\psi\xi^\alpha\omega_{,\alpha} + \delta\tau(\bar{\xi}^2\omega_{,2} - \bar{\xi}^3\omega_{,3}) + \bar{\delta}\tau(\xi^2\omega_{,2} - \xi^3\omega_{,3})]\Big\}$$

$$- \frac{1}{4\pi}e^{3\psi-\nu}X(\delta\psi\bar{\delta}\psi + \tfrac{1}{2}\delta\tau\bar{\delta}\tau) + \frac{1}{2\pi}(W - \tfrac{1}{4}e^{3\psi-\nu}Y)(\delta\psi\bar{\delta}\tau + \bar{\delta}\psi\delta\tau)$$

$$+ \frac{1}{2\pi}U\delta\tau\bar{\delta}\tau + \frac{1}{4\pi}e^\beta(e^{\mu_3-\mu_2}\delta\psi_{,2}\bar{\delta}\psi_{,2} + e^{\mu_2-\mu_3}\delta\psi_{,3}\bar{\delta}\psi_{,3})$$

$$- \frac{1}{4\pi}e^\beta\{[(e^{\mu_3-\mu_2}\delta\mu_{3,2})_{,2} + (e^{\mu_2-\mu_3}\delta\mu_{2,3})_{,3}]\bar{\delta}\psi + [(e^{\mu_3-\mu_2}\bar{\delta}\mu_{3,2})_{,2} + (e^{\mu_2-\mu_3}\bar{\delta}\mu_{2,3})_{,3}]\delta\psi\}$$

$$+ \frac{1}{4\pi}e^{\mu_3-\mu_2+\beta}\{[\beta_{,2}\bar{\delta}\mu_{3,2} + (\beta + 2\mu_3)_{,2}\bar{\delta}\psi_{,2}]\delta\tau + [\beta_{,2}\delta\mu_{3,2} + (\beta + 2\mu_3)_{,2}\delta\psi_{,2}]\bar{\delta}\tau\}$$

$$- \frac{1}{4\pi}e^{\mu_2-\mu_3+\beta}\{[\beta_{,3}\bar{\delta}\mu_{2,3} + (\beta + 2\mu_2)_{,3}\bar{\delta}\psi_{,3}]\delta\tau + [\beta_{,3}\delta\mu_{2,3} + (\beta + 2\mu_2)_{,3}\delta\psi_{,3}]\bar{\delta}\tau\}$$

$$+ (\epsilon + p)u^0 u_1 e^{-3\psi+\nu}\sqrt{-g}[e^{\mu_2-\mu_3}(\bar{\xi}^2 Q_{,3} + \xi^2\bar{Q}_{,3}) - e^{\mu_3-\mu_2}(\bar{\xi}^3 Q_{,2} + \xi^3\bar{Q}_{,2})]$$

$$- \frac{1}{8\pi}[\delta\psi(\omega_{,2}\bar{Q}_{,3} - \omega_{,3}\bar{Q}_{,2}) + \bar{\delta}\psi(\omega_{,2}Q_{,3} - \omega_{,3}Q_{,2}) + \delta\tau(\omega_{,2}\bar{Q}_{,3} + \omega_{,3}\bar{Q}_{,2}) + \bar{\delta}\tau(\omega_{,2}Q_{,3} + \omega_{,3}Q_{,2})]$$

$$- \frac{1}{16\pi}e^{-3\psi+\nu}(e^{\mu_2-\mu_3}Q_{,3}\bar{Q}_{,3} + e^{\mu_3-\mu_2}Q_{,2}\bar{Q}_{,2})\Big]\!\Big] dx\,. \tag{25}$$

We observe that in the form to which we have now reduced the equation, it *is* manifestly symmetric in the barred and the unbarred quantities. Clearly a variational principle is implied; but we postpone a precise statement of it to § IV below.

a) The Terms That Survive among the Integrated Parts

In examining the results of the integrations by parts that were carried out during the course of the reductions leading to equation (24), we shall restrict our considerations to the important case when a system of spherical polar coordinates ($x^1 = \phi$, $x^2 = r$,

$x^3 = \theta$) is used and one sets (cf. I, eq. [82])

$$e^{\psi} = re^{\tau+\zeta}\sin\theta\,. \tag{26}$$

In this case

$$e^{\psi} = e^{\psi}\delta\psi = e^{\psi}\psi_{,2} = 0 \quad \text{at} \quad \theta = 0, \quad \text{and} \quad \theta = \pi\,; \tag{27}$$

but $e^{\psi}\psi_{,3}$ does not vanish at the poles.

In view of the relations (27) and the further fact that $\sqrt{-g}$ vanishes at the poles (on account of $\sin\theta$ being a factor), we conclude from equation (6) that

$$Q = 0 \quad \text{at} \quad \theta = 0 \quad \text{and} \quad \theta = \pi\,. \tag{28}$$

Similarly, from equation (7) it follows that

$$e^{\psi}\psi_{,3}(\delta\psi - \delta\mu_3) = 0\,, \quad \text{or} \quad \delta\psi = \delta\mu_3 \quad \text{at} \quad \theta = 0 \quad \text{and} \quad \theta = \pi\,. \tag{29}$$

Returning to integrated parts resulting from the various integrations by parts, we find that those carried out with respect to $x^2(=\theta)$ in the reduction to equation (24) leave the terms

$$\begin{aligned}
&-\frac{1}{8\pi}\int_0^{2\pi}\int_0^R d\phi dx^2\{e^{\beta+\mu_2-\mu_3}[(\psi+\nu)_{,3}\delta\psi\bar{\delta}\mu_2 - \delta\psi\bar{\delta}\mu_{2,3} + \bar{\delta}\psi\delta\psi_{,3} \\
&\qquad + (\psi+\mu_2)_{,3}\bar{\delta}\psi\delta(\mu_2-\mu_3)]\}_0^{\pi} \\
&+\frac{1}{8\pi}\int_0^{2\pi}\int_0^R d\phi dx^2\{e^{\beta+\mu_2-\mu_3}[-(\nu+\mu_2)_{,3}\bar{\delta}\psi\delta(\mu_2-\mu_3) - \bar{\delta}\psi\delta\nu_{,3} + \delta\psi\bar{\delta}\mu_{2,3} - (\psi+\nu)_{,3}\delta\psi\bar{\delta}\mu_2 \\
&\qquad - \delta(\psi+\nu)_{,3}\bar{\delta}\mu_2 - (\psi+\nu)_{,3}\delta\mu_2\bar{\delta}\mu_2 + (\psi+\nu)_{,3}\bar{\delta}\mu_2\delta\mu_3]\}_0^{\pi}\,,
\end{aligned} \tag{30}$$

where the vanishing of Q at the poles has been used. Making use of the relations (27) and (29), the foregoing terms reduce to the manifestly symmetric form

$$-\frac{1}{8\pi}\int_0^{2\pi}\int_0^R d\phi dx^2[e^{\beta+\mu_2-\mu_3}\psi_{,3}(\delta\mu_2\bar{\delta}\mu_2 - \delta\mu_3\bar{\delta}\mu_3 + \delta\mu_2\bar{\delta}\mu_3 + \bar{\delta}\mu_2\delta\mu_3)]_0^{\pi}\,; \tag{31}$$

and this integral must be added to the right-hand side of equation (25).

Examining next the integrated parts that result from the integrations with respect to $x^2(=r)$, we find that the terms which do not vanish on the boundary of the configuration (by virtue of ϵ, p, and Δp vanishing here) leave the surface integral

$$\begin{aligned}
&-\frac{1}{8\pi}\int_0^{2\pi}\int_0^{\pi} d\phi dx^3\{e^{\mu_3-\mu_2+\beta}[2(\psi+\nu)_{,2}(\delta\psi\bar{\delta}\mu_3 + \delta\tau\bar{\delta}\mu_3 + \delta\tau\bar{\delta}\psi) \\
&\qquad + 4\mu_{3,2}\delta\tau\bar{\delta}\psi - 2\delta\psi\bar{\delta}\mu_{3,2} + \delta(\psi+\nu)_{,2}\bar{\delta}(\psi+\mu_3)] \\
&\qquad + \omega_{,3}\bar{Q}\delta(\psi-\tau) - \tfrac{1}{2}e^{-3\psi+\nu+\mu_3-\mu_2}Q_{,2}\bar{Q}\}_R\,.
\end{aligned} \tag{32}$$

We shall verify in § V that this surface integral vanishes as $R \to \infty$ for quasi-stationary deformations; but in the general case it does not, as we shall see in detail in Paper IV.

IV. THE VARIATIONAL PRINCIPLE

It will be recalled that in deriving equation (25) $\bar{\xi}^{\alpha}$ was a trial displacement and that $\bar{\xi}^{\alpha}$ and the associated barred variations were required to be consistent only with the initial-value equations (3)–(7), while ξ^{α} and the associated unbarred variations were taken to satisfy the dynamical equations as well. We now *formally* identify ξ^{α} and $\bar{\xi}^{\alpha}$

and the barred and the unbarred variations in equation (25); we obtain

$$-\sigma^2 \iiint \Big[\!\Big[\sqrt{-g}\Big\{(\epsilon+p)(u^0)^2 \sum_\alpha e^{2\mu_\alpha}(\xi^\alpha)^2 - \frac{e^{-2\nu}}{4\pi}[(\delta\mu)^2 + 2\delta\psi\delta\mu - (\delta\tau)^2]\Big\} + \frac{Q^2}{16\pi} e^{-3\psi-\nu+\mu_2+\mu_3}\Big]\!\Big] dx$$

$$= \iiint \Big[\!\Big[\sqrt{-g}\Big\{ -\gamma p\left(1+\frac{\gamma p V^2}{\epsilon+p}\right)\left(\frac{\Delta N}{N}\right)^2 - \frac{1}{\epsilon+p}\xi^\alpha p_{,\alpha}\xi^\beta \epsilon_{,\beta}$$

$$+ 2\frac{\Delta N}{N}\xi^\alpha[p_{,\alpha} - \gamma p V^2(\log u_1)_{,\alpha}] - (\epsilon+p)V^2[\xi^\alpha(\log u_1)_{,\alpha}]^2$$

$$- \frac{2V}{1-V^2}\delta p\delta V + (\epsilon+p)V^2(\delta\psi)^2 - 16\pi(\epsilon+p)^2\frac{V^2}{(1-V^2)^2}\sum_\alpha e^{2\mu_\alpha}(\xi^\alpha)^2$$

$$+ 4(\epsilon+p)u^0 u_1[\delta\psi\xi^\alpha\omega_{,\alpha} + \delta\tau(\xi^2\omega_{,2} - \xi^3\omega_{,3})]$$

$$- [(\epsilon+p)(\delta\psi)^2 + 4(\epsilon-p)\delta\psi\delta\mu + 2\delta\epsilon\delta\psi - 4p(\delta\mu)^2]\Big\}$$

$$- \frac{1}{4\pi}e^{3\psi-\nu}X[(\delta\psi)^2 + \tfrac{1}{2}(\delta\tau)^2] + \frac{1}{\pi}(W - \tfrac{1}{4}e^{3\psi-\nu}Y)\delta\psi\delta\tau$$

$$+ \frac{1}{2\pi}U(\delta\tau)^2 + \frac{1}{4\pi}e^{\beta}[e^{\mu_3-\mu_2}(\delta\psi_{,2})^2 + e^{\mu_2-\mu_3}(\delta\psi_{,3})^2]$$

$$- \frac{1}{2\pi}e^{\beta}[(e^{\mu_3-\mu_2}\delta\mu_{3,2})_{,2} + (e^{\mu_2-\mu_3}\delta\mu_{2,3})_{,3}]\delta\psi$$

$$+ \frac{1}{2\pi}e^{\beta}\{e^{\mu_3-\mu_2}[\beta_{,2}\delta\mu_{3,2} + (\beta+2\mu_3)_{,2}\delta\psi_{,2}] - e^{\mu_2-\mu_3}[\beta_{,3}\delta\mu_{2,3} + (\beta+2\mu_2)_{,3}\delta\psi_{,3}]\}\delta\tau$$

$$+ 2(\epsilon+p)u^0 u_1 e^{-3\psi+\nu}\sqrt{-g}(e^{\mu_2-\mu_3}\xi^2 Q_{,3} - e^{\mu_3-\mu_2}\xi^3 Q_{,2})$$

$$- \frac{1}{4\pi}[\delta\psi(\omega_{,2}Q_{,3} - \omega_{,3}Q_{,2}) + \delta\tau(\omega_{,2}Q_{,3} + \omega_{,3}Q_{,2})]$$

$$- \frac{1}{16\pi}e^{-3\psi+\nu}[e^{\mu_2-\mu_3}(Q_{,3})^2 + e^{\mu_3-\mu_2}(Q_{,2})^2]\Big]\!\Big] dx\,, \tag{33}$$

where for brevity we have not explicitly written out the terms (31) and (32) which may survive the integrations by parts.

We will now consider equation (33) together with the terms (31) and (32) as a *formula* for σ^2 in which ξ^α is a trial displacement and the variations ΔN, $\delta\psi$, etc., are chosen consistently with the initial-value equations. Suppose now that we evaluate σ^2, successively, with the aid of two trial displacements ξ^α and $\xi^\alpha + \frac{1}{2}\delta\xi^\alpha$ and the associated variations, ΔN, $\delta\psi$, $\delta\mu_2$, etc., and $\Delta N + \frac{1}{2}\Delta^2 N$, $\delta\psi + \frac{1}{2}\delta^2\psi$, $\delta\mu_2 + \frac{1}{2}\delta^2\mu_2$, etc. In other words, we consider the effect on σ^2, given by equation (33), of an (arbitrary) increment $\frac{1}{2}\delta\xi^\alpha$ in a selected trial displacement and the corresponding increments in the other quantities which are consistent with the initial-value equations. Let the effect be an increment $\delta\sigma^2$. We can write down an expression for $\delta\sigma^2$ directly from equation (33) and the terms (31) and (32) by subjecting it to the desired variation. We start with this expression for $\delta\sigma^2$ and essentially trace backward the reductions that led to equation (25) starting from the pulsation equation. The only difference is that we are not now entitled to use any of the dynamical equations. We find

$$
\begin{aligned}
-\delta\sigma^2 \int\!\!\int\!\!\int \Big[\!\!\Big[& \sqrt{-g}\Big\{(\epsilon+p)(u^0)^2 \sum_\alpha e^{2\mu_\alpha}(\xi^\alpha)^2 - \frac{e^{-2\nu}}{4\pi}[(\delta\mu)^2 + 2\delta\psi\delta\mu - (\delta\tau)^2]\Big\} \\
& + \frac{1}{16\pi} e^{-3\psi-\nu+\mu_2+\mu_3} Q^2 \Big]\!\!\Big] dx \\
= \int\!\!\int\!\!\int \Big[\!\!\Big[& \delta\xi^\alpha \Big\{\sigma^2[(\epsilon+p)(u^0)^2 e^{2\mu_\alpha}\xi^\alpha - (\epsilon+p)u^0 u_1 q_\alpha] - u^0\left(\frac{\gamma p}{u^0}\frac{\Delta N}{N}\right)_{,\alpha} \\
& + (\epsilon+p)\left(\frac{\Delta u^0}{u^0}\right)_{,\alpha} + \frac{\Delta N}{N} p_{,\alpha} - (\epsilon+p)u^0 u_1 \Delta\Omega_{,\alpha}\Big\}\sqrt{-g} \\
& - \frac{1}{8\pi}\delta\{\sqrt{-g}[G^{(2)(2)} + G^{(3)(3)} - 16\pi p]\}\delta^2\mu - \frac{1}{8\pi}\delta\{\sqrt{-g}[G^{(2)(2)} - G^{(3)(3)}]\}\delta^2\tau \\
& - \frac{1}{8\pi} e^{\beta}\delta\{e^{\mu_2+\mu_3}[G^{(1)(1)} - 8\pi T^{(1)(1)}]\}\delta^2\psi \\
& - \frac{1}{16\pi}\{\sigma^2 e^{-3\psi-\nu+\mu_2+\mu_3} Q + (e^{-3\psi+\nu-\mu_2+\mu_3} Q_{,2})_{,2} + (e^{-3\psi+\nu+\mu_2-\mu_3} Q_{,3})_{,3} \\
& \quad + [\omega_{,2}(3\delta\psi - \delta\nu + \delta\mu_3 - \delta\mu_2)]_{,3} - [\omega_{,3}(3\delta\psi - \delta\nu - \delta\mu_3 + \delta\mu_2]_{,2} \\
& \quad - 16\pi[(\epsilon+p)u^0 u_1 e^{-2\psi+2\nu+2\mu_2}\xi^2]_{,3} + 16\pi[(\epsilon+p)u^0 u_1 e^{-2\psi+2\nu+2\mu_3}\xi^3]_{,2}\}\delta Q \Big]\!\!\Big] dx\,,
\end{aligned}
\tag{34}
$$

where it will be noted that the first group of terms in braces, in the integrand on the right-hand side, which are contracted with $\delta\xi^\alpha$, constitutes the terms of the pulsation equation (18).

From equation (34) we can draw the following inference.

If the selected trial displacement ξ^α and the associated variations are such that all the dynamical equations are also satisfied, then $\delta\sigma^2 = 0$ for arbitrary infinitesimal increments in ξ^α and in the associated quantities which are consistent only with the initial-value equations; conversely, if besides equations (3), (4), (5), (9), (10), (13), and (14), the field equations necessary to complete the set are satisfied and if $\delta\sigma^2 = 0$ for all arbitrary infinitesimal variations in ξ^α (and associated variations in the other quantities which are compatible with the initial-value equations), then the pulsation equation (18) will be satisfied by the selected ξ^α and σ^2 given by equation (33) and the terms (31) and (32) will be a characteristic value of the problem. (See note added in proof on page 767.)

V. THE CONDITION FOR THE OCCURRENCE OF A NEUTRAL MODE OF DEFORMATION

We return now to the principal question to which this paper is addressed, namely, the condition that a uniformly rotating configuration will allow a neutral mode of deformation. Precisely, the question concerns the existence (or otherwise) of a non-trivial Lagrangian displacement ξ^α such that all the equations governing equilibrium, linearized about a particular solution of the same equations, as well as certain necessary equations of constraint, namely those that ensure the constancy of baryon number, entropy, and angular momentum, are all simultaneously satisfied. When such a displacement exists, we say that the equilibrium configuration considered allows a neutral mode of deformation. When a configuration allows such a mode of deformation, then we may expect (if Newtonian analogies prevail in general relativity) that it will be on the verge of either dynamical or secular stability.

In the present context, the equations governing equilibrium have been written down in Paper I (Part II); they are the equations of hydrostatic equilibrium (I, eqs. [79] and [81]) and four linearly independent field equations, say, I, equations (70), (72), (73), and (74). The equations that we must consider, to answer the question concerning the occurrence of a neutral mode of deformation, are the linearized versions of these *same* equations together with equations (3), (4), and (5) which ensure the constancy of the baryon number, entropy, and angular momentum of each fluid element as it is displaced. One observation concerning the linearized equations should be made: since the form of the metric appropriate to a stationary state allows us the freedom to impose a coordinate condition on μ_2 and μ_3, we must retain the same freedom in the linearized versions of the equilibrium equations; in particular, we should have the freedom to put $\delta\mu_2 = \delta\mu_3$ corresponding to the choice of a cylindrical-polar or a spherical-polar system of coordinates for describing the stationary state.

Consider first the linearized version of the (0, 1)-component of the field equation given by I, equation (73). It is apparent from I, equation (159) that the resulting linearized equation can be written in the form

$$\begin{aligned}&\{e^{3\psi-\nu+\mu_3-\mu_2}[\delta\omega_{,2} + \omega_{,2}(3\delta\psi - \delta\nu - \delta\mu_2 + \delta\mu_3)]\\ &\quad - 16\pi(\epsilon + p)u^0u_1(\sqrt{-g})\xi^2\}_{,2}\\ &+ \{e^{3\psi-\nu+\mu_2-\mu_3}[\delta\omega_{,3} + \omega_{,3}(3\delta\psi - \delta\nu + \delta\mu_2 - \delta\mu_3)]\\ &\quad - 16\pi(\epsilon + p)u^0u_1(\sqrt{-g})\xi^3\}_{,3} = 0\,. \end{aligned} \tag{35}$$

Accordingly, there exists a function $Q(x^2, x^3)$ such that

$$\begin{aligned}&e^{3\psi-\nu+\mu_3-\mu_2}[\delta\omega_{,2} + \omega_{,2}(3\delta\psi - \delta\nu - \delta\mu_2 + \delta\mu_3)]\\ &\quad - 16\pi(\epsilon + p)u^0u_1(\sqrt{-g})\xi^2 = -Q_{,3}\end{aligned} \tag{36}$$

and

$$\begin{aligned}&e^{3\psi-\nu+\mu_2-\mu_3}[\delta\omega_{,3} + \omega_{,3}(3\delta\psi - \delta\nu + \delta\mu_2 - \delta\mu_3)]\\ &\quad - 16\pi(\epsilon + p)u^0u_1(\sqrt{-g})\xi^3 = +Q_{,2}\,.\end{aligned} \tag{37}$$ [3]

We observe that these equations are of exactly the same form as the *initial-value equations* (13) and (14) if we set $q_{\alpha,00} = 0$, as would indeed seem appropriate for the quasi-stationary case we are presently considering. However, the definition of Q in terms of $(q_{2,3} - q_{3,2})$, as in equation (8), has no relevance in the present context. Also, we are now entitled to put $\delta\mu_2 = \delta\mu_3$ in equations (36) and (37) which we cannot do in the non-stationary case.

The elimination of $\delta\omega$ from equations (36) and (37) leads to the equation (cf. eq. [15])

$$\begin{aligned}&(e^{-3\psi+\nu+\mu_3-\mu_2}Q_{,2})_{,2} + (e^{-3\psi+\nu+\mu_2-\mu_3}Q_{,3})_{,3}\\ &= -[\omega_{,2}(3\delta\psi - \delta\nu + \delta\mu_3 - \delta\mu_2)]_{,3} + [\omega_{,3}(3\delta\psi - \delta\nu + \delta\mu_2 - \delta\mu_3)]_{,2}\\ &\quad + 16\pi\{[(\epsilon + p)u^0u_1e^{-2\psi+2\nu+2\mu_2}\xi^2]_{,3} - [(\epsilon + p)u^0u_1e^{-2\psi+2\nu+2\mu_3}\xi^3]_{,2}\}\,,\end{aligned} \tag{38}$$

where we can put $\delta\mu_2 = \delta\mu_3$, if we so desire.

Since the relations (36) and (37) are of the same form as I, equations (151) and (152) (except that now $q_{\alpha,00} = 0$), the lemma stated in Paper I as equations (154) and (155) is now valid if the terms in $q_{\alpha,00}$ are suppressed. On this account, the linearized versions of the field equations as they are written down in Paper I, § XII, also continue to be valid if the terms in the time derivatives are similarly suppressed. In these resulting field equations we are again entitled to put $\delta\mu_2 = \delta\mu_3$, if we so desire.

[3] Equations (36) and (37) provide a generalization of an earlier result due to Ernst (1967, eq. [5]).

Now we have already verified (cf. I, eq. [167]) that the equation

$$\Big\{+\tfrac{1}{2}Q\omega_{,3} + e^{\psi+\nu+\mu_3-\mu_2}[\delta\psi_{,2} + (\psi-\nu)_{,2}\delta\psi + \delta\mu_{3,2} - (\nu-\mu_3)_{,2}\delta\mu_3 - (\psi+\mu_3)_{,2}\delta\mu_2] - 8\pi\sqrt{-g}\,\frac{\epsilon+p}{1-V^2}\,\xi^2\Big\}_{,2}$$
$$+ \Big\{-\tfrac{1}{2}Q\omega_{,2} + e^{\psi+\nu+\mu_2-\mu_3}[\delta\psi_{,3} + (\psi-\nu)_{,3}\delta\psi + \delta\mu_{2,3} - (\nu-\mu_2)_{,3}\delta\mu_2 - (\psi+\mu_2)_{,3}\delta\mu_3] - 8\pi\sqrt{-g}\,\frac{\epsilon+p}{1-V^2}\,\xi^3\Big\}_{,3} = 0 \quad (39)$$

is the same as the equation

$$\delta\{e^{\mu_2+\mu_3}[G^{(0)(0)} - 8\pi T^{(0)(0)}]\} = 0\,. \quad (40)$$

Hence there exists a function $P(x^2, x^3)$ such that

$$+\tfrac{1}{2}Q\omega_{,3} + e^{\psi+\nu+\mu_3-\mu_2}[\delta\psi_{,2} + (\psi-\nu)_{,2}\delta\psi + \delta\mu_{3,2} - (\nu-\mu_3)_{,2}\delta\mu_3 - (\psi+\mu_3)_{,2}\delta\mu_2] - 8\pi\sqrt{-g}\,\frac{\epsilon+p}{1-V^2}\,\xi^2 = -\tfrac{1}{2}P_{,3} \quad (41)$$

and

$$-\tfrac{1}{2}Q\omega_{,2} + e^{\psi+\nu+\mu_2-\mu_3}[\delta\psi_{,3} + (\psi-\nu)_{,3}\delta\psi + \delta\mu_{2,3} - (\nu-\mu_2)_{,3}\delta\mu_2 - (\psi+\mu_2)_{,3}\delta\mu_3] - 8\pi\sqrt{-g}\,\frac{\epsilon+p}{1-V^2}\,\xi^3 = +\tfrac{1}{2}P_{,2}\,. \quad (42)$$

Except for the replacement of $Q\omega_{,3}$ by $Q\omega_{,3} + P_{,3}$ equation (41) is the same as I, equation (143). Accordingly, we may now write (cf. I, eqs. [147] and [148]; also eqs. [9] and [10] of this paper)

$$(\delta\psi+\delta\mu)_{,2} - \nu_{,2}(\delta\psi+\delta\mu) + \psi_{,2}(\delta\psi-\delta\mu) = e^{2\mu_2}\left(8\pi\,\frac{\epsilon+p}{1-V^2}\,\xi^2 - \frac{Q\omega_{,3}+P_{,3}}{2\sqrt{-g}}\right) - \delta\tau_{,2} - (2\mu_3+\psi-\nu)_{,2}\delta\tau \quad (43)$$

and

$$(\delta\psi+\delta\mu)_{,3} - \nu_{,3}(\delta\psi+\delta\mu) + \psi_{,3}(\delta\psi-\delta\mu) = e^{2\mu_3}\left(8\pi\,\frac{\epsilon+p}{1-V^2}\,\xi^3 + \frac{Q\omega_{,2}+P_{,2}}{2\sqrt{-g}}\right) + \delta\tau_{,3} + (2\mu_2+\psi-\nu)_{,3}\delta\tau\,. \quad (44)$$

We can eliminate $(\delta\psi+\delta\mu)$ from equations (43) and (44) to obtain the following equation for P in the case $\delta\tau = 0$:

$$[e^{-\psi-2\nu+\mu_3-\mu_2}(P_{,2}+Q\omega_{,2})]_{,2} + [e^{-\psi-2\nu+\mu_2-\mu_3}(P_{,3}+Q\omega_{,3})]_{,3} = 16\pi\left[\left(e^{-\nu+2\mu_2}\,\frac{\epsilon+p}{1-V^2}\,\xi^2\right)_{,3} - \left(e^{-\nu+2\mu_3}\,\frac{\epsilon+p}{1-V^2}\,\xi^3\right)_{,2}\right] + 2\,\frac{\partial[e^{-\nu}(\delta\psi-\delta\mu),\psi]}{\partial(x^2,x^3)}\,. \quad (45)$$

Alternatively, we can eliminate $(\delta\psi - \delta\mu)$ to obtain the equation (also in the case $\delta\tau = 0$)

$$\psi_{,3}e^{-\nu+2\mu_2}\left(8\pi\,\frac{\epsilon+p}{1-V^2}\,\xi^2 - \frac{Q\omega_{,3}+P_{,3}}{2\sqrt{-g}}\right) - \psi_{,2}e^{-\nu+2\mu_3}\left(8\pi\,\frac{\epsilon+p}{1-V^2}\,\xi^3 + \frac{Q\omega_{,2}+P_{,2}}{2\sqrt{-g}}\right) = \psi_{,3}[e^{-\nu}(\delta\psi+\delta\mu)]_{,2} - \psi_{,2}[e^{-\nu}(\delta\psi+\delta\mu)]_{,3}\,. \quad (46)$$

Equations (36), (37), (43), and (44) replace the linearized versions of the (0, 1)- and the (0, 0)-components of the field equations. These equations may be completed by adding the linearized versions of, say, the [(2, 2) + (3, 3)]- and the (1, 1)-components of the field equations.

Finally, we have the linearized version of the equations of hydrostatic equilibrium. In the case of an initial uniform rotation, the equation takes the form (cf. eq. [18])

$$-u^0\left(\frac{\gamma p}{u^0}\frac{\Delta N}{N}\right)_{,\alpha}+(\epsilon+p)\left(\frac{\Delta u^0}{u^0}\right)_{,\alpha}+\frac{\Delta N}{N}p_{,\alpha}-(\epsilon+p)u^0u_1\Delta\Omega_{,\alpha}=0\,. \quad (47)$$

a) A Variational Base for Determining the Occurrence of a Neutral Mode of Deformation

As in § III, we now define a trial displacement $\bar{\xi}^\alpha$ as one which satisfies the same boundary conditions as are demanded of a proper solution, but is arbitrary otherwise. And we also define associated barred variations $\bar{\Delta}N$, $\bar{\delta}\psi$, etc., and require that they satisfy the same boundary conditions as are demanded of the proper variations and are further consistent with equations (3), (4), (5), (36), (37), (43), and (44). We then multiply the equation of hydrostatic equilibrium (47) by $\bar{\xi}^\alpha\sqrt{-g}$, sum over α, and integrate over the whole of 3-space (we do not expect any divergence at infinity in the present case as, indeed, we shall verify).

It is clear that by reductions exactly paralleling those adopted in § III, we can proceed to transform the result of the integration to a form that is manifestly symmetric in the barred and the unbarred quantities. The fact that in equations (43) and (44) we have $Q\omega_{,\alpha}+P_{,\alpha}$ in place of $Q\omega_{,\alpha}$ in equations (9) and (10) results in the only additional term

$$\frac{1}{16\pi}\int\!\!\int\!\!\int[\bar{P}_{,3}\delta(\psi-\nu)_{,2}-\bar{P}_{,2}\delta(\psi-\nu)_{,3}]dx\,; \quad (48)$$

and this term arises in the reduction of the term (23) and is the counterpart of the terms in $\bar{Q}\omega_{,2}$ and $\bar{Q}\omega_{,3}$ in the last line of equation (24). But on integration by parts even this additional term (48) vanishes! Thus, we shall arrive at formally the same expression as on the right-hand side of equation (25), the left-hand side now being zero.

By arguments analogous to those by which we established the variational principle in § IV, we can now formulate a necessary and a sufficient condition for the existence of a neutral mode of deformation. Restricting ourselves to the case when we can put $\delta\tau=0$, and rewriting equation (33) in the present context in the form

$$\begin{aligned}0=\int\!\!\int\!\!\int\Big[\!\!\Big[\sqrt{-g}\Big\{&-\gamma p\left(1+\frac{\gamma pV^2}{\epsilon+p}\right)\left(\frac{\Delta N}{N}\right)^2-\frac{1}{\epsilon+p}\xi^\alpha p_{,\alpha}\xi^\beta\epsilon_{,\beta}\\&+2\frac{\Delta N}{N}\xi^\alpha[p_{,\alpha}-\gamma pV^2(\log u_1)_{,\alpha}]-(\epsilon+p)V^2[\xi^\alpha(\log u_1)_{,\alpha}]^2\\&-\frac{2V}{1-V^2}\delta p\delta V+(\epsilon+p)V^2(\delta\psi)^2-16\pi(\epsilon+p)^2\frac{V^2}{(1-V^2)^2}\sum_\alpha e^{2\mu_\alpha}(\xi^\alpha)^2\\&-[(\epsilon+p)(\delta\psi)^2+4(\epsilon-p)\delta\psi\delta\mu+2\delta\epsilon\delta\psi-4p(\delta\mu)^2]\\&+4(\epsilon+p)u^0u_1\xi^\alpha\omega_{,\alpha}\delta\psi\Big\}-\frac{1}{4\pi}e^{3\psi-\nu}X(\delta\psi)^2\\&+\frac{1}{4\pi}e^\beta[e^{\mu_3-\mu_2}(\delta\psi_{,2})^2+e^{\mu_2-\mu_3}(\delta\psi_{,3})^2]\end{aligned}$$

$$- \frac{1}{2\pi} e^{\beta}[(e^{\mu_3-\mu_2}\delta\mu_{,2})_{,2} + (e^{\mu_2-\mu_3}\delta\mu_{,3})_{,3}]\delta\psi$$

$$+ 2(\epsilon + p)u^0 u_1 e^{-3\psi+\nu}\sqrt{-g}(e^{\mu_2-\mu_3}\xi^2 Q_{,3} - e^{\mu_3-\mu_2}\xi^3 Q_{,2})$$

$$- \frac{1}{4\pi}(\omega_{,2}Q_{,3} - \omega_{,3}Q_{,2})\delta\psi$$

$$- \frac{1}{16\pi} e^{-3\psi+\nu}[e^{\mu_2-\mu_3}(Q_{,3})^2 + e^{\mu_3-\mu_2}(Q_{,2})^2]\Big]\Big] dx\,, \tag{49}$$

we can state that a *necessary and a sufficient condition for the occurrence of a neutral mode of deformation is that if for some* ξ^α *and associated variations, consistent with equations (3), (4), (5), (36), (37), (43), and (44) and two additional field equations necessary to complete the set, the quantity on the right-hand side of equation (49) and its first variation vanish, simultaneously.*

It should be remarked that the condition (49) for the existence of a neutral mode of deformation may not be equivalent to the condition derived from equation (33) for the onset of dynamical instability. But it is clear that equation (49) provides a *sufficient condition* since equation (33), together with the constraint $\delta\tau = 0$, are formally equivalent to equation (49); and the imposition of the additional constraint does not invalidate the acceptability of the resulting trial functions. Whether the condition (33) is also a necessary one requires to be established, though in the case of slow rotation (see § VI below) an additional coordinate freedom that obtains for the ($l = 0$)-part of the perturbation makes the two conditions that follow from equations (33) and (49) equivalent.

b) The Asymptotic Behavior of the Perturbed Potentials

It remains to verify that all the volume integrals which appear in equation (49) converge and also that the surface integral (32) evaluated at $r = R$ vanishes as $R \to \infty$. For this purpose it is necessary to determine the asymptotic behaviors of the perturbed potentials $\delta\psi$, $\delta\mu$, $\delta\nu$, and Q as $r \to \infty$. We shall consider this problem in the system of coordinates in which the corresponding behaviors of the unperturbed potentials were determined in Paper I, § VII. The linearized versions of the equilibrium equations (I, eqs. [84]–[86]) that formed the basis of that treatment are

$$\nabla^2\delta\nu + \text{grad}\,\delta\nu\cdot\text{grad}\,(\eta + \zeta + \nu) + \text{grad}\,\nu\cdot\text{grad}\,(\delta\eta + \delta\zeta + \delta\nu)$$

$$= -2r^2\sin^2\theta e^{2\eta+2\zeta-2\nu}|\text{grad}\,\omega|^2(\delta\eta + \delta\zeta) + \frac{1}{r}S$$

$$+ 4\pi\delta\Big\{e^{2\eta-2\zeta}\Big[(\epsilon + p)\frac{1 + V^2}{1 - V^2} + 2p\Big]\Big\}\,, \tag{50}$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)(\delta\nu + \delta\eta - \delta\zeta) + 2\,\text{grad}\,\nu\cdot\text{grad}\,\delta\nu$$

$$= -3r^2\sin^2\theta e^{2\eta+2\zeta-\nu}|\text{grad}\,\omega|^2(\delta\eta + \delta\zeta) + \frac{3}{2r}S$$

$$+ 8\pi\delta\Big\{e^{2\eta-2\zeta}\Big[(\epsilon + p)\frac{V^2}{1 - V^2} + p\Big]\Big\}\,, \tag{51}$$

and

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\delta\zeta+\text{grad}\,(\eta+\zeta+\nu)\cdot\text{grad}\,(\delta\eta+\delta\zeta)$$

$$+\text{grad}\,(\eta+\zeta)\cdot\text{grad}\,\delta\nu+\text{grad}\,(\log r\sin\theta)\cdot\text{grad}\,(\delta\eta+\delta\zeta+\delta\nu)$$

$$=\tfrac{3}{2}r^2\sin^2\theta e^{2\eta+2\zeta-2\nu}|\text{grad}\,\omega|^2(\delta\eta+\delta\zeta)-\frac{3}{4r}S$$

$$-4\pi\delta\left\{e^{2\eta-2\zeta}\left[\frac{\epsilon+pV^2}{1-V^2}-(\epsilon+p)\right]\right\}, \tag{52}$$

where

$$S=\frac{e^{-\eta-\zeta-\nu}}{r\sin\theta}\left(\frac{\partial Q}{\partial r}\frac{\partial\omega}{\partial\theta}-\frac{\partial Q}{\partial\theta}\frac{\partial\omega}{\partial r}\right). \tag{53}$$

And equation (38) governing Q takes the form

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{e^{-3\eta-3\zeta+\nu}}{r^4\sin^4\theta}\frac{\partial Q}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{e^{-3\eta-3\zeta+\nu}}{r^4\sin^4\theta}\frac{\partial Q}{\partial\theta}\right)$$

$$=\frac{1}{r^2\sin\theta}\left[\frac{\partial\omega}{\partial\theta}\frac{\partial}{\partial r}(3\delta\eta+3\delta\zeta-\delta\nu)-\frac{\partial\omega}{\partial r}\frac{\partial}{\partial\theta}(3\delta\eta+3\delta\zeta-\delta\nu)\right]$$

$$+\frac{16\pi}{r^2\sin\theta}\left\{\frac{\partial}{\partial\theta}\left[\frac{(\epsilon+p)u^0u_1e^{-4\zeta+2\nu}}{r^2\sin^2\theta}\xi^2\right]-\frac{\partial}{\partial r}\left[\frac{(\epsilon+p)u^0u_1e^{-4\zeta+2\nu}}{r\sin^2\theta}\xi^3\right]\right\}. \tag{54}$$

In determining the asymptotic behaviors of $\delta\nu$, $\delta\eta$, and $\delta\zeta$ for $r\to\infty$ from equations (50)–(52), we may ignore the terms in ϵ and p on the right-hand sides (since these terms vanish outside the fluid) and substitute for the equilibrium potentials their known asymptotic forms (cf. I, eqs. [101] and [109]):

$$\eta=+\frac{M}{r}+\frac{1}{2}\frac{A(1+\cos 2\theta)-\frac{1}{2}M^2}{r^2}+O(r^{-3}), \tag{55}$$

$$\zeta=\frac{1}{2}\frac{A(1-\cos 2\theta)}{r^2}+O(r^{-3}), \tag{56}$$

$$\nu=-\frac{M}{r}+O(r^{-3}),\quad \eta+\zeta+\nu=(A-\tfrac{1}{4}M^2)\frac{1}{r^2}+O(r^{-3}), \tag{57}$$

and

$$\omega=\frac{2J}{r^3}-\frac{6JM}{r^4}+O(r^{-5}). \tag{58}$$

With the aid of the foregoing relations we find that

$$\frac{S}{r}\to\frac{6J}{r^6\sin\theta}\frac{\partial Q}{\partial\theta} \tag{59}$$

and

$$r^2\sin^2\theta|\text{grad}\,\omega|^2(\delta\eta+\delta\zeta)\to\frac{36J^2}{r^6}\sin^2\theta(\delta\eta+\delta\zeta)\quad(r\to\infty). \tag{60}$$

We shall presently verify that Q is of $O(r^{-3})$ while $\delta\eta+\delta\zeta$ is at most of $O(r^{-2})$. Accordingly, the terms in S and $|\text{grad}\,\omega|^2$ on the right-hand sides of equations (50)–(52) may also be ignored for the orders in which we shall be working; and substituting from equations

(55)–(52) for the remaining terms, we obtain

$$\nabla^2\delta\nu + \left[\frac{M}{r^2} + O(r^{-4})\right]\frac{\partial}{\partial r}(\delta\eta + \delta\zeta + \delta\nu) + \left[-2(A - \tfrac{1}{4}M^2)\frac{1}{r^3} + O(r^{-4})\right]\frac{\partial}{\partial r}\delta\nu = 0\,, \tag{61}$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)(\delta\eta - \delta\zeta + \delta\nu) + \left[\frac{2M}{r^2} + O(r^{-4})\right]\frac{\partial}{\partial r}\delta\nu = 0\,, \tag{62}$$

and

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\delta\zeta + \left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta}\right)(\delta\eta + \delta\zeta + \delta\nu) - \frac{M}{r^2}\frac{\partial}{\partial r}\delta\nu$$
$$- 2(A - \tfrac{1}{4}M^2)\frac{1}{r^3}\frac{\partial}{\partial r}(\delta\eta + \delta\zeta + \delta\nu) + O(r^{-4})\left[\frac{\partial}{\partial r}\delta\nu, \frac{\partial}{\partial r}(\delta\eta + \delta\zeta)\right] = 0\,. \tag{63}$$

Adding equation (62) to twice equation (63), we obtain

$$\left(\frac{\partial^2}{\partial r^2} + \frac{3}{r}\frac{\partial}{\partial r} + \frac{2\cot\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)(\delta\eta + \delta\zeta + \delta\nu)$$
$$- 2(A - \tfrac{1}{4}M^2)\frac{1}{r^3}\frac{\partial}{\partial r}(\delta\eta + \delta\zeta + \delta\nu) + O(r^{-4})\left[\frac{\partial}{\partial r}\delta\nu, \frac{\partial}{\partial r}(\delta\eta + \delta\zeta)\right] = 0\,. \tag{64}$$

In deducing the behaviors of the perturbed potentials from equations (61), (62), and (64), we first observe that the quasi-stationary deformations we are presently considering must leave the inertial mass of the configuration unchanged. By equations (55) and (57) this requirement implies that

$$\delta\nu = O(r^{-3}) \quad \text{and} \quad \delta\eta \quad \text{is at most} \quad O(r^{-2})\,. \tag{65}$$

Now two possibilities arise: either $\delta\eta$ and $\delta\zeta$ are of $O(r^{-2})$ or $\delta\eta$ and $\delta\zeta$ are at most of $O(r^{-3})$. It is convenient to consider these two cases separately.

Case i: *$\delta\eta$ and $\delta\zeta$ are of* $O(r^{-2})$. In this case the equations that determine the highest orders of $\delta\nu$, $\delta\eta$, and $\delta\zeta$ are

$$\nabla^2\delta\nu + \frac{M}{r^2}\frac{\partial}{\partial r}(\delta\eta + \delta\zeta) = 0\,, \tag{66}$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)(\delta\eta - \delta\zeta) = 0\,, \tag{67}$$

and

$$\left(\frac{\partial^2}{\partial r^2} + \frac{3}{r}\frac{\partial}{\partial r} + \frac{2\cot\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)(\delta\eta + \delta\zeta) = 0\,. \tag{68}$$

From these equations it can be readily deduced that

$$\delta\eta = \frac{a\cos^2\theta}{r^2}\,, \quad \delta\zeta = \frac{a\sin^2\theta}{r^2}\,, \quad \text{and} \quad \delta\nu = \frac{1}{3}\frac{aM}{r^3}\,, \tag{69}$$

where a is constant. In obtaining the solutions of equations (66)–(68) in the form (69), we have made use of the condition that $\delta\zeta$ must behave like $\sin^2\theta$ at the poles.

Case ii: *$\delta\eta$ and $\delta\zeta$ are both at most of* $O(r^{-3})$. In this case, it follows from equation (61) that the leading term in $\delta\nu$ must satisfy the equation

$$\nabla^2 \delta\nu = 0 ; \tag{70}$$

and the solution of this equation compatible with the requirement (65) is

$$\delta\nu = \frac{D}{r^3} P_2(\cos\theta) = \frac{D}{2r^3}(3\cos^2\theta - 1) = \frac{D}{4r^3}(3\cos 2\theta + 1) , \tag{71}$$

where D is a constant. Similarly, from equation (64) we conclude that

$$\delta\eta + \delta\zeta + \delta\nu = \frac{B}{r^4} P_2(\cos\theta\,|\,2) = \frac{B}{2r^4}(4\cos^2\theta - 1) = \frac{B}{2r^4}(2\cos 2\theta + 1) , \tag{72}$$

where B is a further constant and $P_m(\cos\theta\,|\,p)$ is the Gegenbauer polynomial of order m and index p determined as a solution of the equation

$$\frac{1}{\sin^p\theta}\frac{d}{d\theta}\left(\sin^p\theta\frac{dP_m}{d\theta}\right) + m(m+p)P_m = 0 . \tag{73}$$

(The Legendre polynomials coincide with the Gegenbauer polynomials of index 1.)

Now letting

$$\delta\eta - \delta\zeta + \delta\nu = \frac{g(\theta)}{r^4} , \tag{74}$$

we find that equation (62) gives

$$\frac{d^2 g}{d\theta^2} + 16 g = \tfrac{3}{2} MD(3\cos 2\theta + 1) ; \tag{75}$$

and the solution of this equation compatible with the requirement of symmetry about the equatorial plane is

$$g = C\cos 4\theta + \tfrac{3}{8} MD(\cos 2\theta + \tfrac{1}{4}) , \tag{76}$$

where C is a constant. Accordingly,

$$\delta\eta - \delta\zeta + \delta\nu = \frac{1}{r^4}[C\cos 4\theta + \tfrac{3}{8} MD(\cos 2\theta + \tfrac{1}{4})] . \tag{77}$$

From the solutions (72) and (77) we now find that

$$\delta\zeta = \frac{1}{2r^4}[\tfrac{1}{2}B(2\cos 2\theta + 1) - C\cos 4\theta - \tfrac{3}{8} MD(\cos 2\theta + \tfrac{1}{4})] . \tag{78}$$

And the condition that $\delta\zeta$ vanishes at the poles gives

$$C = \tfrac{3}{2}B - \tfrac{15}{32} MD . \tag{79}$$

Inserting this value of C in equations (77) and (78) and combining with the results of case i, we obtain the solutions

$$\delta\nu = \frac{1}{r^3}[\tfrac{1}{3}aM + \tfrac{1}{2}D(3\cos^2\theta - 1)] + O(r^{-4}) , \tag{80}$$

$$\delta\eta = \frac{a}{r^2}\cos^2\theta - \frac{D}{2r^3}(3\cos^2\theta - 1) + \frac{1}{r^4}[\tfrac{1}{2}B(12\cos^4\theta - 10\cos^2\theta + 1) + \tfrac{3}{8} MD(5\cos^2\theta - 1)\sin^2\theta] + O(r^{-5}) , \tag{81}$$

and

$$\delta\zeta = \left\{\frac{a}{r^2} + \frac{1}{r^4}[B(6\cos^2\theta - 1) - \tfrac{3}{8}MD(5\cos^2\theta - 1)]\right\}\sin^2\theta + O(r^{-5}) . \quad (82)$$

Finally, turning to equation (54) governing Q, and substituting for the various quantities their asymptotic forms, we have to consider

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{1}{r^2\sin^4\theta}\frac{\partial Q}{\partial r}\right) + \frac{1}{r^6\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin^3\theta}\frac{\partial Q}{\partial\theta}\right) = \frac{72DJ}{r^9}\cos\theta \cdot \quad (83)$$

and the solution of this equation appropriate to the problem is

$$Q = \frac{9DJ}{r^3}\sin^4\theta\cos\theta + \frac{\text{constant}}{r^{m-8}}P_m(\cos\theta\,|-3) . \quad (84)^4$$

With the foregoing asymptotic forms for the perturbed potentials we verify that each of the terms in the surface integral (32) vanishes as $R \to \infty$; and also that all the terms that appear in the integrand of equation (49) lead to convergent integrals. The demonstration of the validity of the criterion for the occurrence of a neutral mode of deformation is thus completed.

VI. THE REDUCTION OF THE VARIATIONAL EXPRESSION FOR σ^2 IN CASE OF SLOW ROTATION

From the fact that the variational expression for σ^2 holds generally, it can be concluded that in the case of slow rotation, when its effects can be considered as a perturbation on the spherical configuration, an *exact* formula for σ^2 can be obtained by using for the trial displacement in the variational expression the proper solution that belongs to the radial mode of oscillation of the spherical nonrotating configuration and evaluating the expression consistently to order Ω^2. A formal proof of this conclusion can be given along the lines of Lebovitz's proof (1970) in the corresponding Newtonian context. As Lebovitz has pointed out, the result stated is analogous to the one in quantum mechanics, namely, that the first-order correction to an eigenvalue due to a perturbation is given by the diagonal matrix-element of the perturbed operator evaluated with the eigenfunction of the unperturbed operator and "its validity can be traced to the same reason: the symmetry of the unperturbed operator." The same arguments apply, almost in all details, to our present problem; and we shall not repeat them.

Once the conclusion of the preceding paragraph is accepted, it follows at once that the terms representing the ($l = 2$)-deformation of the spherical configuration due to the rotation will not contribute to the variational integral: they will simply vanish on integration over the polar angle. Only the terms representing the ($l = 0$)-deformation will contribute to σ^2.

Now Hartle (1967; see eqs. [67]–[70]), who has developed the theory of slowly rotating configurations in general relativity, has shown that a metric of the form

$$ds^2 = -e^{2\nu}(dt)^2 + r^2\sin^2\theta(d\phi - \omega dt)^2 + e^{2\lambda}(dr)^2 + r^2(d\theta)^2 \quad (85)$$

is adequate for deducing all the ($l = 0$)-changes caused by the rotation. In particular, we can write

$$\nu = \nu_{(S)} + \nu_{(\Omega)} \quad \text{and} \quad \lambda = \lambda_{(S)} + \lambda_{(\Omega)} , \quad (86)$$

distinguishing the corrections of $O(\Omega^2)$ due to the rotation from the (zero-order) "Schwarz-

[4] The solution of the homogeneous equation is $r^{-m+8}P_m(\cos\theta\,|-3)$; but regularity conditions at the poles would appear to allow the inclusion of only $m = 5$.

schild" values by the subscripts (Ω) and (S), respectively. In Hartle's notation

$$\nu_{(\Omega)} = h_0 \quad \text{and} \quad \lambda_{(\Omega)} = \frac{m_0}{r - 2M(r)}, \tag{87}$$

where h_0 and m_0 are functions that can be readily evaluated (Hartle 1967, eqs. [97] and [98]).

The foregoing remarks with respect to the metric apply to the equilibrium configuration. But it is clear that the *form* of the metric is equally adequate for describing the perturbed time-dependent configuration so long as we are interested only in the ($l = 0$)-deformation.

It will be observed that with the identifications

$$e^{\psi} = r \sin\theta\,, \quad e^{\mu_3} = r\,, \quad \mu_2 = \lambda\,, \quad \text{and} \quad \nu = \nu\,, \tag{88}$$

the metric (85) is of the form that underlies our present considerations. In particular, to $O(\Omega^2)$,

$$\sqrt{-g} = r^2 e^{\lambda+\nu} \sin\theta = r^2 \sin\theta[1 + (\lambda + \nu)_{(\Omega)}] \exp[(\lambda + \nu)_{(S)}]\,,$$

$$u^0 = \frac{e^{-\nu}}{(1 - V^2)^{1/2}} = [1 - \nu_{(\Omega)} + \tfrac{1}{2}V^2] \exp[-\nu_{(S)}]\,,$$

$$V = r(\Omega - \omega) \sin\theta \exp[-\nu_{(S)}]\,, \quad \text{and} \quad u_1 = rV \sin\theta\,. \tag{89}$$

Turning to the time-dependent perturbed problem, we observe that for the chosen form of the metric the Eulerian changes in ψ and μ_3 vanish identically:

$$\delta\psi = \delta\mu_3 \equiv 0\,. \tag{90}$$

For the Eulerian changes in ν and λ, we may write

$$\delta\nu = \delta\nu_{(S)} + \delta\nu_{(\Omega)} \quad \text{and} \quad \delta\mu_2 = -\tfrac{1}{2}\delta\tau = \delta\lambda = \delta\lambda_{(S)} + \delta\lambda_{(\Omega)}\,, \tag{91}$$

in accordance with our (notational) convention of separating the $O(\Omega^2)$ corrections from the "Schwarzschild" values. In the present instance, $\delta\nu_{(S)}$ and $\delta\lambda_{(S)}$ are known from the theory of the radial oscillation of a spherical configuration (Chandrasekhar 1964; eqs. [36] and [41]).

In accordance, finally, with what we have stated earlier, we shall now insert for the trial displacement ξ^α in the variational expression (33) for σ^2, the proper Lagrangian displacement $\xi(r)$ belonging to the radial mode of oscillation of the spherical nonrotating configuration:

$$\xi^2 = \xi(r) \quad \text{and} \quad \xi^3 = 0\,. \tag{92}$$

Consider first the contribution (31) that survives the integrations by parts. In view of equations (88)–(90) it is, in the present context,

$$-\frac{1}{8\pi}\int_0^{2\pi}\int_0^{\infty} e^{\lambda+\nu}(\delta\lambda)^2[(\sin\theta)_{,\theta}]_0^{\pi} dr d\phi = \frac{1}{8\pi}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} e^{\lambda+\nu}(\delta\lambda)^2 \sin\theta\, dr d\theta d\phi$$

$$= \int\int\int \frac{\sqrt{-g}}{8\pi r^2}(\delta\lambda)^2 dx\,. \tag{93}$$

In this last form, the contribution can be incorporated with the rest of the expression (33) for σ^2 (which is again considerably simplified because of the vanishing of $\delta\psi$ and

$\delta\mu_3$). The variational expression for σ^2 thus becomes

$$\sigma^2 \int\int\int (\epsilon + p)(u^0)^2(\sqrt{-g})e^{2\lambda}\xi^2 dx$$

$$= \int\int\int \Big[\!\Big[\sqrt{-g}\Big\{ \gamma p\left(1 + \frac{\gamma p V^2}{\epsilon + p}\right)\left(\frac{\Delta N}{N}\right)^2 + \frac{1}{\epsilon + p} p_{,r}\epsilon_{,r}\xi^2$$

$$- 2\frac{\Delta N}{N}[p_{,r} - \gamma p V^2(\log u)_{,r}]\xi + (\epsilon + p)V^2[(\log u_1)_{,r}]^2\xi^2$$

$$- \left(p + \frac{1}{8\pi r^2}\right)(\delta\lambda)^2 + 2V\delta p\delta V + 16\pi(\epsilon + p)^2V^2e^{2\lambda}\xi^2$$

$$+ 2(\epsilon + p)u^0u_1\xi\delta\lambda\omega_{,r}\Big\} + \frac{1}{32\pi}e^{3\psi-\nu}X(\delta\lambda)^2 - \frac{1}{8\pi}U(\delta\lambda)^2\Big]\!\Big] dx\,. \tag{94}$$

Inserting the relations (cf. I, eqs. [68])

$$U = e^{\mu_3-\mu_2+\beta}(\beta_{,2}\mu_{3,2} + \psi_{,2}\nu_{,2}) = W$$

and

$$X = Y = e^{\mu_3-\mu_2}(\omega_{,2})^2\,, \tag{95}$$

which are now valid, in I, equations (74) and (75), we find

$$2U = -\tfrac{1}{2}e^{3\psi-\nu}X + 16\pi p\sqrt{-g} - 2[e^{\mu_2-\mu_3}(e^{\beta})_{,3}]_{,3}$$

$$= -\tfrac{1}{2}e^{3\psi-\nu}X + 16\pi p\sqrt{-g} + \frac{2\sqrt{-g}}{r^2}\,. \tag{96}$$

Making use of this last relation in equation (94) and separating the terms that are explicitly of $O(\Omega)^2$, we obtain

$$\sigma^2 \int\int\int (\epsilon + p)(u^0)^2(\sqrt{-g})e^{2\lambda}\xi^2 dx$$

$$= \int\int\int \sqrt{-g}\left[\gamma p\left(\frac{\Delta N}{N}\right)^2 + \frac{1}{\epsilon + p}p_{,r}\epsilon_{,r}\xi^2 - 2\frac{\Delta N}{N}p_{,r}\xi - 2\left(p + \frac{1}{8\pi r^2}\right)(\delta\lambda)^2\right]dx$$

$$+ \int\int\int \Big[\!\Big[V^2\sqrt{-g}_{(S)}\left\{\frac{\gamma^2 p^2}{\epsilon + p}\left(\frac{\Delta N}{N}\right)^2 + 2\gamma p\frac{\Delta N}{N}(\log u_1)_{,r}\xi + (\epsilon + p)[(\log u_1)_{,r}]^2\xi^2\right.$$

$$\left. + 2\delta p\frac{\delta V}{V} + 16\pi(\epsilon + p)^2e^{2\lambda}\xi^2 + 2(\epsilon + p)u^0u_1\omega_{,r}\frac{\delta\lambda\xi}{V^2}\right\}_{(S)}$$

$$+ \frac{1}{16\pi}r^4\sin^3\theta e^{-\lambda-\nu}(\omega_{,r})^2(\delta\lambda)^2\Big]\!\Big] dx\,. \tag{97}$$

It can now be verified that by ignoring the second integral, explicitly of $O(\Omega^2)$, on the right-hand side of equation (97) and evaluating the first integral with the appropriate Schwarzschild values, we recover the variational expression (Chandrasekhar 1964; eq. [61]) that gives σ^2 for the radial oscillations of a spherical configuration.

It is clear that in evaluating the quantities in braces in the second integral on the right-hand side of equation (97) it will suffice to use for $\sqrt{-g}$, $\Delta N/N$, ϵ, p, etc., their Schwarzschild values. But for these same quantities we must insert expressions that are correct to $O(\Omega^2)$ in evaluating the first integral on the right-hand side (as well as the integral on the left-hand side).

Turning now to the evaluation of the corrections $\Delta N_{(\Omega)}$ and $\delta\lambda_{(\Omega)}$, we first prove the following two lemmas.

LEMMA 1:

$$[\lambda_{(S)} + \nu_{(S)}]_{,r} = 4\pi r(\epsilon + p)_{(S)} \exp[2\lambda_{(S)}]$$

and

$$[\lambda_{(\Omega)} + \nu_{(\Omega)}]_{,r} = 4\pi r(\epsilon + p)_{(S)}\left[\frac{(\epsilon + p)_{(\Omega)}}{(\epsilon + p)_{(S)}} + 2\lambda_{(\Omega)} + V^2\right]. \tag{98}$$

Proof: Writing out equations I, (70) and (72), appropriately for the metric considered here, we obtain

$$e^{-2\lambda}\left(\frac{2\lambda_{,r}}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} = +\tfrac{1}{4}e^{2\psi-2\nu-2\lambda}(\omega_{,r})^2 + 8\pi\frac{\epsilon + pV^2}{1 - V^2}$$

and

$$e^{-2\lambda}\left(\frac{2\nu_{,r}}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} = -\tfrac{1}{4}e^{2\psi-2\nu-2\lambda}(\omega_{,r})^2 + 8\pi p\,. \tag{99}$$

Adding these two equations, we have

$$\frac{e^{-2\lambda}}{r}(\lambda + \nu)_{,r} = 4\pi\frac{\epsilon + p}{1 - V^2}. \tag{100}$$

Now separating the zero-order and the first-order terms in this last equation, we obtain the desired relations.

The first of the relations (98) for the Schwarzschild metric is, of course, well known.

LEMMA 2:

$$\delta\lambda_{(S)} = -[\lambda_{(S)} + \nu_{(S)}]_{,r}\xi$$

and

$$\delta\lambda_{(\Omega)} = -[\lambda_{(\Omega)} + \nu_{(\Omega)}]_{,r}\xi\,. \tag{101}$$

Proof: Equation (6) written out for the metric considered here gives

$$8\pi\frac{\epsilon + p}{1 - V^2}e^{\psi+\nu+\mu_2+\mu_3}\xi = e^{\psi+\nu+\mu_3-\mu_2}[-\delta\mu_2(\psi + \mu_3)_{,2}] = -\frac{2}{r}e^{\psi+\nu+\mu_3-\mu_2}\delta\lambda\,, \tag{102}$$

or

$$\delta\lambda = -4\pi r\frac{\epsilon + p}{1 - V^2}e^{2\lambda}\xi\,. \tag{103}$$

Separating the zero-order and the first-order terms in this last equation and making use of the relations of Lemma 1, we obtain equations (101). Again, the first of these equations for the Schwarzschild metric is known (Chandrasekhar 1964; eq. [36]).

Turning next to the evaluation of $\Delta N/N$ to $O(\Omega^2)$, we first consider $\delta V/V$ which not only occurs in the expression for it but is also needed in the evaluation of the second integral on the right-hand side of equation (103).

Since $\delta V/V$ always occurs with a factor V^2 in the relevant expressions, it will suffice to evaluate it with the zero-order values for the various quantities. Thus, we obtain from equation (5)

$$\frac{\delta V}{V} = -\left(\frac{\gamma p}{\epsilon + p}\frac{\Delta N}{N}\right)_{(S)} - \xi\left(\frac{1}{r} + \frac{1}{V}\frac{\partial V}{\partial r}\right), \tag{104}$$

or, substituting for V its value given in equation (89), we have

$$\frac{\delta V}{V} = -\left(\frac{\gamma p}{\epsilon + p}\frac{\Delta N}{N}\right)_{(S)} - \xi\left(\frac{2}{r} - \frac{d\nu_{(S)}}{dr} - \frac{1}{\Omega - \omega}\frac{d\omega}{dr}\right). \tag{105}$$

Finally, considering equation (3) for $\Delta N/N$ and rewriting it in the form

$$\frac{\Delta N}{N} = -V^2\frac{\delta V}{V} - \delta\lambda - \xi\frac{d}{dr}[\psi + \mu_2 + \mu_3 - \tfrac{1}{2}\log(1 - V^2)] - \frac{d\xi}{dr}, \tag{106}$$

we find, after some rearranging,

$$\frac{\Delta N}{N} = V^2\left[\left(\frac{\gamma p}{\epsilon + p}\frac{\Delta N}{N}\right)_{(S)} + \frac{1}{r}\xi\right] - \xi\left(\frac{2}{r} + \lambda_{,r}\right) - \frac{d\xi}{dr} - \delta\lambda\,. \tag{107}$$

Now substituting for λ and $\delta\lambda$ from Lemmas 1 and 2 and separating the zero-order and the first-order terms, we obtain

$$\left(\frac{\Delta N}{N}\right)_{(S)} = \xi\frac{d\nu_{(S)}}{dr} - \frac{d\xi}{dr} - \frac{2}{r}\xi \tag{108}$$

and

$$\left(\frac{\Delta N}{N}\right)_{(\Omega)} = V^2\left[\left(\frac{\gamma p}{\epsilon + p}\frac{\Delta N}{N}\right)_{(S)} + \frac{1}{r}\xi\right] + \xi\frac{d\nu_{(\Omega)}}{dr}\,. \tag{109}$$

We have now evaluated all the quantities that occur in equation (52) to the relevant orders and expressed them in terms of known functions; it accordingly becomes an explicit formula for the determination of σ^2 to $O(\Omega^2)$; and it is exact. As we have stated in Paper I, § I, an equivalent formula has been derived independently by Hartle and Thorne (1972).

In the near future we hope to investigate with the aid of equation (97) the stability of the slowly rotating neutron-star models that Hartle and Thorne (1968) have constructed.

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Note added in proof.—The variational principle stated in § IV can be formulated alternatively as follows. If we subject ξ^2, ξ^3, $\delta\tau$, and Q to arbitrary infinitesimal variations and require $\delta\sigma^2$ to vanish for such variations, then *all* the dynamical equations will be satisfied. Note that for any initial choice of ξ^2, ξ^3, $\delta\tau$, and Q, the initial-value equations (9) and (10) suffice to determine $\delta\psi$ and $\delta\mu$ so that all the quantities appearing in equation (33) can be evaluated consistently with the initial-value equations only. These and related matters will be considered in greater detail in Paper IV.

ERRATA FOR PAPER I

1. In the expression for g^{11} given in equation (4) on page 381, the last term, *read* $q_3{}^2e^{-2\mu_3}$ *instead of* $q_2{}^2e^{-2\mu_3}$.

2. In the expression for $G_{(A)(B)}$ in equation (17) on page 383 there are two misprints. In the second line the term in parentheses immediately preceding $\exp(-\mu_A - \mu_B)$ *read* $\mu_{C,B}\mu_{B,A}$ *instead of* $\mu_{C,A}\mu_{B,A}$; and in the third line, *read* $\exp(+2\psi - \mu_A - \mu_B - 2\mu_C)$ *instead of* $\exp(-2\psi - \mu_A - \mu_B - 2\mu_C)$.

3. In equation (126) on page 398, *read* $\delta(\psi + \nu + \mu_2 + \mu_3)$ *instead of* $\delta(\psi + \mu_2 + \mu_3)$.

4. In the footnote on page 399, *read* 1969 *instead of* 1969*b*.

5. On page 404 in the second line following equation (167), *read* 167 *instead of* 166.