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THE POST-NEWTONIAN EFFECTS OF GENERAL RELATIVITY ON THE EQUILIBRIUM OF UNIFORMLY ROTATING BODIES. VI. THE DEFORMED FIGURES OF THE JACOBI ELLIPSOIDS (*Continued*)

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ABSTRACT

The theory of the (deformed) post-Newtonian Jacobi ellipsoid developed in an earlier paper is specialized suitably to make a comparison with the Newtonian ellipsoid having the same angular momentum and baryon number.

I. INTRODUCTION

In an earlier paper[1] (Chandrasekhar 1967*b;* this paper will be referred to hereafter as Paper III) the deformed figures of the Jacobi ellipsoid, in the first post-Newtonian approximation to the equations of general relativity, were determined. Precisely, configurations with a specified density ρ, rotating with a constant angular velocity Ω, were considered; and it was shown how the figures of equilibrium can be obtained consistently with the equations of post-Newtonian hydrodynamics (Chandrasekhar 1965) by subjecting the Newtonian ellipsoids, with the same density ρ, to a Lagrangian displacement of the form

$$\xi = \frac{\pi G \rho a_1^2}{c^2} \sum_{i=1}^{5} S_i \xi^{(i)} , \tag{1}$$

where a_1 denotes the semimajor axis of the ellipsoid,

$$\xi^{(1)} = (x_1, 0, -x_3) , \quad \xi^{(2)} = (0, x_2, -x_3) , \quad \xi^{(3)} = \frac{1}{a_1^2} (\tfrac{1}{3}x_1^3, -x_1^2 x_2, 0) ,$$
$$\xi^{(4)} = \frac{1}{a_1^2} (0, \tfrac{1}{3}x_2^3, -x_2^2 x_3) , \quad \xi^{(5)} = \frac{1}{a_1^2} (-x_3^2 x_1, 0, \tfrac{1}{3}x_3^3) , \tag{2}$$

and S_i ($i = 1, \ldots, 5$) are constants to be determined. (Note that since the chosen displacement is divergence free, the coordinate volumes of the Newtonian and the post-Newtonian configurations have been assumed to be the same.)

On carrying through the analysis, it was found that while the constants S_3, S_4, and S_5 could be uniquely determined, only a linear relation between the constants S_1 and S_2 could be established (see the comments in Paper III following eq. [91]). As stated in Paper III, this indeterminacy in the solution reflects the freedom of choice we have in selecting a particular Jacobi ellipsoid (among neighboring ones) for comparison with the post-Newtonian configuration. But the indeterminacy can be removed if, following Bardeen (1971), we arrange that *the Newtonian and the post-Newtonian configurations have, for a given density* ρ, *the same baryon number and the same angular momentum.* In this paper, we shall carry out this normalization of the solution obtained in Paper III. The method is the same as in the preceding paper (Chandrasekhar 1971; this paper will be referred to hereafter as Paper V) where a similar normalization of the solution for the

[1] The following misprints in Paper III may be noted here. In equation (8) *replace* $1/c^2$ *by* ρ/c^2; and in the beginning of the third line of equation (12) insert a *plus sign.*

post-Newtonian Maclaurin spheroids (obtained in Chandrasekhar 1967*a*) is carried out: indeed, the analysis for the Jacobi ellipsoids is as direct and simple as for the Maclaurin spheroids.

II. THE ADJUSTMENT FOR EQUAL ANGULAR MOMENTUM AT CONSTANT PROPER VOLUME

Since the Jacobi ellipsoid is not stationary in an inertial frame, the expression for the linear momentum used in Paper V (eq. [3]) is not appropriate; we must use, instead, the general formula (Chandrasekhar and Nutku 1969, eq. [74])

$$\pi_\alpha = \rho v_\alpha + \frac{1}{c^2}\rho\left[v_\alpha\left(v^2 + 6U + \Pi + \frac{p}{\rho}\right) - 4U_\alpha + \frac{1}{2}\frac{\partial^2\chi}{\partial t\partial x_\alpha}\right]. \tag{3}$$

However, for the present problem (as for the problem considered in Paper V) we can set $\Pi = 0$.

Expressions for all the quantities that occur in equation (3), with the exception of $\partial^2\chi/\partial t\partial x_\alpha$, are given in Paper III (eqs. [18], [30], and [31]); and to obtain an expression for $\partial^2\chi/\partial t\partial x_\alpha$, we make use of the equation

$$\nabla^2\frac{\partial\chi}{\partial t} = -2\frac{\partial U}{\partial t} = 2v_\mu\frac{\partial U}{\partial x_\mu} \tag{4}$$

satisfied by $\partial\chi/\partial t$. Inserting for v_μ and U their values

$$v_1 = -\Omega x_2\,, \quad v_2 = +\Omega x_1\,, \quad v_3 = 0\,,$$

and

$$U = \pi G\rho\left(I - \sum_{\mu=1}^{3} A_\mu x_\mu^2\right) \qquad \left(I = \sum_{\mu=1}^{3} A_\mu a_\mu^2\right), \tag{5}$$

we obtain the equation

$$\nabla^2\frac{\partial\chi}{\partial t} = 4\pi G\rho\Omega(A_1 - A_2)x_1x_2\,. \tag{6}$$

Accordingly,

$$\frac{\partial\chi}{\partial t} = -\Omega(A_1 - A_2)\mathfrak{D}_{12}\,, \tag{7}$$

where $\mathfrak{D}_{ij}$ is the Newtonian potential for a distribution $\rho x_i x_j$ in the ellipsoid. Making use of the expression for $\mathfrak{D}_{ij}$ given in Chandrasekhar (1969, p. 57, eq. [132]), we obtain

$$\frac{\partial\chi}{\partial t} = -\pi G\rho\Omega(A_1 - A_2)a_1^2a_2^2\left(A_{12} - \sum_{\mu=1}^{3} A_{12\mu}x_\mu^2\right)x_1x_2\,; \tag{8}$$

and the required expressions for $\partial^2\chi/\partial t\partial x_\alpha$ follow from this equation.

We also have (cf. Chandrasekhar 1969, p. 56, eq. [125])

$$U_1 = -\Omega\mathfrak{D}_2 = -\pi G\rho\Omega a_2^2\left(A_2 - \sum_{\mu=1}^{3} A_{2\mu}x_\mu^2\right)x_2$$

and

$$U_2 = +\Omega\mathfrak{D}_1 = +\pi G\rho\Omega a_1^2\left(A_1 - \sum_{\mu=1}^{3} A_{1\mu}x_\mu^2\right)x_1\,. \tag{9}$$

Making use of the foregoing relations, we find that the angular momentum

$$L_3 = x_1\pi_2 - x_2\pi_1 \tag{10}$$

about the z-axis is given by

$$
\begin{aligned}
L_3 = {} & \rho\Omega(x_1^2 + x_2^2) \\
& + \frac{\pi G\rho}{c^2}\rho\Omega\Big\{6(a_1^2 A_1 + a_2^2 A_2 + a_3^2 A_3)(x_1^2 + x_2^2) + a_3^2 A_3(x_1^2 + x_2^2) \\
& \qquad - 4a_1^2 A_1 x_1^2 - 4a_2^2 A_2 x_2^2 - \tfrac{1}{2}a_1^2 a_2^2 (A_1 - A_2) A_{12}(x_1^2 - x_2^2) \\
& \qquad + \Big[2B_{12} - 6A_1 - \frac{a_3^2}{a_1^2}A_3 + 4a_1^2 A_{11} + \tfrac{1}{2}(A_1 - A_2)a_1^2 a_2^2 A_{112}\Big]x_1^4 \\
& \qquad + \Big[2B_{12} - 6A_2 - \frac{a_3^2}{a_2^2}A_3 + 4a_2^2 A_{22} - \tfrac{1}{2}(A_1 - A_2)a_1^2 a_2^2 A_{122}\Big]x_2^4 \\
& \qquad + \Big[4B_{12} - 6(A_1 + A_2) - \left(\frac{a_3^2}{a_1^2} + \frac{a_3^2}{a_2^2}\right)A_3 + 4a_1^2 A_{12} \\
& \qquad\qquad + 4a_2^2 A_{12} + \tfrac{3}{2}a_1^2 a_2^2 (A_1 - A_2)(A_{122} - A_{112})\Big]x_1^2 x_2^2 \\
& \qquad + [-7A_3 + 4a_1^2 A_{13} + \tfrac{1}{2}a_1^2 a_2^2 (A_1 - A_2) A_{123}]x_1^2 x_3^2 \\
& \qquad + [-7A_3 + 4a_2^2 A_{23} - \tfrac{1}{2}a_1^2 a_2^2 (A_1 - A_2) A_{123}]x_2^2 x_3^2\Big\} \\
= {} & \rho\Omega(x_1^2 + x_2^2) + \frac{1}{c^2}\mathfrak{L}_3 \quad \text{(say)} . \qquad (11)
\end{aligned}
$$

The angular momentum of the entire mass is obtained by integrating over the volume occupied by the fluid. Thus

$$\mathfrak{M} = \int_{\text{post N}} \rho\Omega(x_1^2 + x_2^2)d\boldsymbol{x} + \frac{1}{c^2}\int_{\text{Jacobi}} \mathfrak{L}_3 d\boldsymbol{x} , \qquad (12)$$

where the first integral must be evaluated, correctly to $O(c^{-2})$, over the deformed figure of the post-Newtonian configuration, while it will suffice to evaluate the second integral over the volume of the undeformed Jacobi ellipsoid.

The contribution to $\mathfrak{M}$ by the first integral on the right-hand side of equation (12) can be reduced in the manner

$$\int_{\text{post N}} \rho\Omega(x_1^2 + x_2^2)dx = [\Omega(I_{11} + I_{22})]_{\text{Jacobi}} + \delta\Omega(I_{11} + I_{22}) + \Omega(V_{11} + V_{22}) , \qquad (13)$$

where the first term represents the angular momentum of the Newtonian Jacobi ellipsoid having the same density and coordinate volume as the post-Newtonian configuration, while the remaining two terms represent the contribution arising from the differing angular velocities and moments of inertia of the Newtonian and the post-Newtonian configurations. We may accordingly write

$$\mathfrak{M} = [\Omega(I_{11} + I_{22})]_{\text{Jacobi}} + (\delta\mathfrak{M})_{\text{coord. vol.}} , \qquad (14)$$

where

$$(\delta\mathfrak{M})_{\text{coord. vol.}} = \delta\Omega(I_{11} + I_{22}) + \Omega(V_{11} + V_{22}) + \frac{1}{c^2}\int_{\text{Jacobi}} \mathfrak{L}_3 d\boldsymbol{x} . \qquad (15)$$

In equations (14) and (15) we have distinguished ($\delta\mathfrak{M}$) by a subscript "coord. vol." to emphasize that this is the difference in the angular momenta of the post-Newtonian and the Newtonian configurations at constant coordinate-volume. The adjustment to constant proper volume (or, equivalently, constant baryon number) is easily made by subjecting the post-Newtonian configuration to a uniform expansion of the form

$$\xi^{(0)} = \frac{\pi G \rho a_1^2}{c^2} S_0 x , \tag{16}$$

and determining S_0 by the condition (cf. Paper V, eq. [13])

$$\text{div}\, \xi^{(0)} = \frac{\pi G \rho a_1^2}{c^2}(3S_0) = -\frac{1}{(\frac{4}{3}\pi a_1 a_2 a_3)c^2} \int_{\text{Jacobi}} (\tfrac{1}{2}\Omega^2 \varpi^2 + 3U) dx . \tag{17}$$

Equation (17) gives

$$S_0 = -\frac{1}{3a_1^2}[\tfrac{1}{5}B_{12}(a_1^2 + a_2^2) + \tfrac{12}{5}I] . \tag{18}$$

Therefore, for equal proper volumes

$$\mathfrak{M} = [\Omega(I_{11} + I_{22})]_{\text{Jacobi}} + (\delta\mathfrak{M})_{\text{proper vol.}} , \tag{19}$$

where

$$\begin{aligned}(\delta\mathfrak{M})_{\text{proper vol.}} &= (\delta\mathfrak{M})_{\text{coord. vol.}} - \frac{1}{c^2}(\pi G \rho)\tfrac{5}{3}\Omega(I_{11} + I_{22})[\tfrac{1}{5}B_{12}(a_1^2 + a_2^2) + \tfrac{12}{5}I] \\ &= \delta\Omega(I_{11} + I_{22}) + \Omega(V_{11} + V_{22}) + \frac{1}{c^2}\int_{\text{Jacobi}} \mathfrak{L}_3 dx \\ &\quad - \frac{1}{c^2}(\pi G \rho)\tfrac{1}{3}\Omega(I_{11} + I_{22})[B_{12}(a_1^2 + a_2^2) + 12I] .\end{aligned} \tag{20}$$

Expressions for $\delta\Omega$, V_{11}, and V_{22} are given in Paper III (eqs. [64], [78], [88], and [89]); and the integration of $\mathfrak{L}_3$ over the volume of the ellipsoid is readily effected. In this manner, we obtain

$$\begin{aligned}(\delta\mathfrak{M})_{\text{proper vol.}} &= \frac{1}{c^2}\Omega(\pi G \rho)(\tfrac{4}{15}\pi a_1 a_2 a_3 \rho) \\ \times \Big\{ &(6a_1^2 A_1 + 6a_2^2 A_2 + 7a_3^2 A_3)(a_1^2 + a_2^2) - 4a_1^4 A_1 - 4a_2^4 A_2 + \tfrac{1}{2}a_1^2 a_2^2 (A_1 - A_2)^2 \\ &+ \tfrac{3}{7}a_1^4\Big[2B_{12} - 6A_1 - \frac{a_3^2}{a_1^2}A_3 + 4a_1^2 A_{11} + \tfrac{1}{2}a_1^2 a_2^2 (A_1 - A_2)A_{112}\Big] \\ &+ \tfrac{3}{7}a_2^4\Big[2B_{12} - 6A_2 - \frac{a_3^2}{a_2^2}A_3 + 4a_2^2 A_{22} - \tfrac{1}{2}a_1^2 a_2^2 (A_1 - A_2)A_{122}\Big] \\ &+ \tfrac{1}{7}a_1^2 a_2^2\Big[4B_{12} - 6(A_1 + A_2) - \Big(\frac{a_3^2}{a_1^2} + \frac{a_3^2}{a_2^2}\Big)A_3 + 4(a_1^2 + a_2^2)A_{12} \\ &\qquad + \tfrac{3}{2}a_1^2 a_2^2 (A_1 - A_2)(A_{122} - A_{112})\Big] \\ &+ \tfrac{1}{7}a_1^2 a_3^2[-7A_3 + 4a_1^2 A_{13} + \tfrac{1}{2}a_1^2 a_2^2 (A_1 - A_2)A_{123}] \\ &+ \tfrac{1}{7}a_2^2 a_3^2[-7A_3 + 4a_2^2 A_{23} - \tfrac{1}{2}a_1^2 a_2^2 (A_1 - A_2)A_{123}] \\ &+ \tfrac{2}{7}(7a_1^4 S_1 + a_1^4 S_3 - a_1^2 a_3^2 S_5) + \tfrac{2}{7}(7a_1^2 a_2^2 S_2 + a_2^4 S_2 - a_1^2 a_2^2 S_3) \\ &+ \frac{a_1^2(a_1^2 + a_2^2)}{4B_{12}}\frac{\delta\omega^2}{a_1^2} - \tfrac{5}{3}(a_1^2 + a_2^2)[\tfrac{1}{5}(a_1^2 + a_2^2)B_{12} + \tfrac{12}{5}I]\Big\} .\end{aligned} \tag{21}$$

In accordance with our stipulation, that the Newtonian and the post-Newtonian configurations have equal angular momenta for the same proper volume, we now require that

$$(\delta\mathfrak{M})_{\text{proper vol.}} = 0\,. \tag{22}$$

This requirement, together with the relation that follows from equating $\delta\omega^2$ given by equations (88) and (89) of Paper III, determine S_1 and S_2 and free the solution of any indeterminacy.

Tables 1 and 2 present the results of the calculations based on equations (21) and (22) and the formulae derived in Paper III; these tables replace Tables 1, 2, and 3 of Paper III. As already noted in Paper III the solutions exhibit a singularity for $a_2/a_1 = 0.29719$.

III. THE BINDING ENERGY

The binding energy E of the post-Newtonian Jacobi ellipsoid can be obtained by integrating the "conserved energy" (cf. Paper V, eqs. [17] and [18])

$$\mathfrak{E} = \tfrac{1}{2}\rho(v^2 - U) + \frac{1}{8c^2}\rho\left(5v^4 + 20v^2U - 20U^2 + 8v^2\frac{p}{\rho} - 16v_\mu P_\mu + 2v_\mu\frac{\partial^2\chi}{\partial t\partial x_\mu}\right) \tag{23}$$

over the volume of the fluid; and inserting for the various quantities their known values, we find

$$\begin{aligned}
\mathfrak{E} = {} & \tfrac{1}{2}\rho(v^2 - U) \\
& + \frac{1}{8c^2}\rho(\pi G\rho)^2\Big\{\left(5\Omega^4 - 20\Omega^2A_1 - 20A_1^2 - 8\frac{a_3^2}{a_1^2}A_3\Omega^2\right)x_1^4 \\
& \quad + \left(5\Omega^4 - 20\Omega^2A_2 - 20A_2^2 - 8\frac{a_3^2}{a_2^2}A_3\Omega^2\right)x_2^4 - 20A_3^2x_3^4 \\
& \quad + \left[10\Omega^4 - 20\Omega^2(A_1 + A_2) - 40A_1A_2 - 8\left(\frac{a_3^2}{a_1^2} + \frac{a_3^2}{a_2^2}\right)A_3\Omega^2\right]x_1^2x_2^2 \\
& \quad - (20\Omega^2A_3 + 40A_2A_3 + 8A_3\Omega^2)x_2^2x_3^2 \\
& \quad - (20\Omega^2A_3 + 40A_1A_3 + 8A_3\Omega^2)x_3^2x_1^2 \\
& \quad + (20\Omega^2I + 40IA_1 + 8a_3^2A_3\Omega^2)x_1^2 \\
& \quad + (20\Omega^2I + 40IA_2 + 8a_3^2A_3\Omega^2)x_2^2 - 40IA_3x_3^2 - 20I^2 \\
& \quad + 16\Omega^2[a_1^2A_{11}x_1^4 + a_2^2A_{22}x_2^4 + (a_1^2 + a_2^2)A_{12}x_1^2x_2^2 \\
& \qquad + a_1^2A_{13}x_1^2x_3^2 + a_2^2A_{23}x_2^2x_3^2 - a_1^2A_1x_1^2 - a_2^2A_2x_2^2] \\
& \quad + 2\Omega^2a_1^2a_2^2(A_1 - A_2)[A_{112}x_1^4 - A_{122}x_2^4 + 3(A_{122} - A_{112})x_1^2x_2^2 \\
& \qquad + A_{123}x_3^2(x_1^2 - x_2^2) - A_{12}(x_1^2 - x_2^2)]\Big\} \\
= {} & \tfrac{1}{2}\rho[\Omega^2(x_1^2 + x_2^2) - U] + \frac{1}{c^2}\mathfrak{E}_2 \quad \text{(say)}\,.
\end{aligned} \tag{24}$$

The required binding energy can therefore be expressed in the form

$$E = \tfrac{1}{2}\int_{\text{post N}}\rho[\Omega^2(x_1^2 + x_2^2) - U]d\boldsymbol{x} + \frac{1}{c^2}\int_{\text{Jacobi}}\mathfrak{E}_2 d\boldsymbol{x}\,, \tag{25}$$

TABLE 1

THE VALUES OF THE COEFFICIENTS S_i ($i = 1, \ldots, 5$)

a_2/a_1	S_1	S_2	S_3	S_4	S_5
1.00	$-\infty$	$+\infty$	−0.01743	+ 0.06971	+ 0.2908
0.97	−9.526	+9.621	−0.01362	+ 0.06275	+ 0.3112
0.96	−6.7196	+6.8081	−0.01211	+ 0.06037	+ 0.3184
0.95	−5.1939	+5.2800	−0.01038	+ 0.05797	+ 0.3257
0.90	−2.3221	+2.4063	+0.00131	+ 0.04557	+ 0.3649
0.85	−1.3750	+1.4584	+0.01793	+ 0.03223	+ 0.4105
0.80	−0.90086	+0.98356	+0.03965	+ 0.01749	+ 0.4651
0.75	−0.61655	+0.69868	+0.06678	+ 0.00070	+ 0.5329
0.70	−0.42821	+0.51004	+0.09986	− 0.01924	+ 0.6203
0.65	−0.29569	+0.37767	+0.13980	− 0.04411	+ 0.7376
0.60	−0.19905	+0.28190	+0.18823	− 0.07703	+ 0.9027
0.55	−0.12740	+0.21231	+0.24834	− 0.12389	+ 1.149
0.50	−0.07450	+0.16351	+0.32673	− 0.19688	+ 1.543
0.45	−0.03707	+0.13417	+0.43939	− 0.32507	+ 2.243
0.40	−0.01491	+0.12973	+0.63459	− 0.59616	+ 3.721
0.35	−0.01823	+0.18738	+1.146	− 1.4357	+ 8.253
0.32	−0.07628	+0.39549	+2.475	− 3.7936	+20.88
0.31	−0.1658	+0.6923	+4.286	− 7.068	+38.37
0.29	+0.4214	−1.2118	−7.162	+13.802	−72.96
0.28	+0.2054	−0.5060	−2.891	+ 6.052	−31.59
0.25	+0.10869	−0.18613	−0.9279	+ 2.550	−12.820
0.20	+0.07813	−0.09195	−0.3384	+ 1.604	− 7.606
0.15	+0.06053	−0.05693	−0.14667	+ 1.407	− 6.313
0.10	+0.04087	−0.03284	−0.05683	+ 1.443	− 6.157

TABLE 2

THE COEFFICIENTS P_μ AND $P_{\mu\nu}$ IN THE EXPRESSION FOR THE PRESSURE

a_2/a_1	P_1	P_2	P_3	P_{11}	P_{22}	P_{33}	P_{12}	P_{23}	P_{31}
1.00	$-\infty$	$+\infty$	−4.1697	+0.39826	+0.39826	+1.7778	+0.79652	+1.5328	+1.5328
0.97	−7.863	+4.672	−4.042	+0.38349	+0.41352	+1.7778	+0.79563	+1.5715	+1.4942
0.96	−5.992	+2.825	−3.990	+0.37856	+0.41881	+1.7778	+0.79509	+1.5846	+1.4811
0.95	−4.960	+1.824	−3.945	+0.37363	+0.42420	+1.7778	+0.79435	+1.5980	+1.4679
0.90	−2.931	−0.039	−3.728	+0.34912	+0.45293	+1.7779	+0.78765	+1.6668	+1.3999
0.85	−2.170	−0.6306	−3.5080	+0.32485	+0.48477	+1.7782	+0.77557	+1.7397	+1.3286
0.80	−1.7230	−0.9055	−3.2834	+0.30099	+0.52011	+1.7785	+0.75732	+1.8169	+1.2538
0.75	−1.4059	−1.0490	−3.0543	+0.27770	+0.55939	+1.7790	+0.73204	+1.8988	+1.1750
0.70	−1.1574	−1.1222	−2.8208	+0.25523	+0.60309	+1.7796	+0.69874	+1.9856	+1.0920
0.65	−0.9522	−1.1504	−2.5834	+0.23388	+0.65178	+1.7656	+0.65625	+2.0779	+1.0042
0.60	−0.7782	−1.1458	−2.3425	+0.21405	+0.70611	+1.7818	+0.60320	+2.1762	+0.9107
0.55	−0.6294	−1.1148	−2.0985	+0.19639	+0.76684	+1.7835	+0.53766	+2.2809	+0.8101
0.50	−0.5032	−1.0605	−1.8523	+0.18200	+0.83496	+1.7861	+0.45653	+2.3930	+0.6998
0.45	−0.3994	−0.9842	−1.6041	+0.17323	+0.91195	+1.7902	+0.35318	+2.5142	+0.5737
0.40	−0.3221	−0.8848	−1.3534	+0.17663	+1.00091	+1.7982	+0.20824	+2.6493	+0.4135
0.35	−0.2948	−0.7513	−1.0906	+0.2230	+1.1145	+1.8207	−0.07278	+2.8208	+0.1319
0.32	−0.3918	−0.6091	−0.8852	+0.3849	+1.2396	+1.8811	−0.661	+3.026	−0.420
0.31	−0.5664	−0.4873	−0.7497	+0.6174	+1.3529	+1.9641	−1.416	+3.228	−1.115
0.29	+0.6353	−1.0489	−1.2261	−0.8820	+0.7996	+1.4373	+3.230	+2.161	+3.1242
0.28	+0.2052	−0.79394	−0.9699	−0.3284	+1.0414	+1.6335	+1.469	+2.605	+1.5092
0.25	+0.02931	−0.60754	−0.74177	−0.08194	+1.2063	+1.7229	+0.6195	+2.875	+0.7186
0.20	+0.00085	−0.43501	−0.50729	−0.01871	+1.3478	+1.7484	+0.3027	+3.0645	+0.4029
0.15	+0.00058	−0.28372	−0.31455	−0.004586	+1.4732	+1.7550	+0.1566	+3.2132	+0.2384
0.10	...	−0.14928	−0.15785	−0.000819	+1.5921	+1.7558	+0.0666	+3.3429	+0.1202

where the first integral on the right-hand side must be evaluated, correctly to $O(c^{-2})$, over the deformed figure of the post-Newtonian configuration while it will suffice to evaluate the second integral over the undeformed Jacobi ellipsoid.

The contribution to E by the first integral on the right-hand side of equation (25) can be reduced in the manner

$$\tfrac{1}{2}\int_{\text{post N}}\rho[\Omega^2(x_1^2+x_2^2)-U]dx=[\tfrac{1}{2}\Omega^2(I_{11}+I_{22})+\mathfrak{W}]_{\text{Jacobi}}$$
$$+\tfrac{1}{2}\delta\Omega^2(I_{11}+I_{22})+\tfrac{1}{2}\Omega^2(V_{11}+V_{22})+\delta\mathfrak{W}$$
$$=[\tfrac{1}{2}\Omega^2(I_{11}+I_{22})+\mathfrak{W}]_{\text{Jacobi}}+\tfrac{1}{2}\delta\Omega^2(I_{11}+I_{22})+\Omega^2(V_{11}+V_{22})\,. \quad (26)$$

The first term on the right-hand side of equation (26) represents the binding energy of the Newtonian Jacobi ellipsoid having the same density and coordinate volume as the post-Newtonian configuration; and the remaining terms represent the contribution arising from the fact that the angular velocity, the moments of inertia, and the potential energy of the post-Newtonian configuration differ from those of the Newtonian configuration by the amounts $\delta\Omega$, δI_{11} $(=V_{11})$, δI_{22} $(=V_{22})$, and $\delta\mathfrak{W}=\frac{1}{2}\Omega^2(V_{11}+V_{22})$ (as may be readily verified). We may, accordingly, write

$$E=E_0+(\Delta E)_{\text{coord. vol.}}\,, \quad (27)$$

where

$$E_0=[\tfrac{1}{2}\Omega^2(I_{11}+I_{22})+\mathfrak{W}]_{\text{Jacobi}}$$
$$=\pi G\rho(\tfrac{4}{15}\pi a_1a_2a_3\rho)[B_{12}(a_1^2+a_2^2)-2I] \quad (28)$$

and

$$(\Delta E)_{\text{coord. vol.}}=\tfrac{1}{2}\delta\Omega^2(I_{11}+I_{22})+\Omega^2(V_{11}+V_{22})+\frac{1}{c^2}\int_{\text{Jacobi}}\mathfrak{E}_2dx\,. \quad (29)$$

In equations (27) and (29) we have distinguished (ΔE) by the subscript "coord. vol." to emphasize that this is the difference in the binding energies of the Newtonian and the post-Newtonian configurations at constant coordinate-volume. The adjustment to constant proper volume is readily made: the considerations of § II lead to the result

$$(\Delta E)_{\text{proper vol.}}=(\Delta E)_{\text{coord. vol.}}$$
$$-\frac{1}{c^2}(\pi G\rho)^2(\tfrac{4}{15}\pi a_1a_2a_3\rho)\tfrac{1}{3}[B_{12}(a_1^2+a_2^2)-2I][B_{12}(a_1^2+a_2^2)+12I]\,. \quad (30)$$

Expressions for $\delta\Omega$, V_{11}, and V_{22} are given in Paper III; and the integration of $\mathfrak{E}_2$ over the volume of the ellipsoid is readily effected. In this manner we obtain

$$(\Delta E)_{\text{proper vol.}}=\frac{1}{c^2}(\pi G\rho)^2(\tfrac{4}{15}\pi a_1a_2a_3\rho)$$
$$\times\Big[\!\Big[\tfrac{1}{56}\Big\{5\Omega^4[3(a_1^4+a_2^4)+2a_1^2a_2^2]-20\Omega^2[3(a_1^4A_1+a_2^4A_2)+a_1^2a_2^2(A_1+A_2)]$$
$$-20[3(a_1^4A_1^2+a_2^4A_2^2)+2a_1^2a_2^2A_1A_2]-60a_3^4A_3^2-420I^2$$
$$-4\Omega^2a_3^2(a_1^2+a_2^2)A_3-40(a_1^2A_1+a_2^2A_2)a_3^2A_3+140\Omega^2I(a_1^2+a_2^2)$$
$$-2a_1^2a_2^2(a_1^2-a_2^2)(A_1-A_2)\Omega^2(7A_{12}-3a_1^2A_{112}-3a_2^2A_{122}-A_{123}a_3^2)$$
$$+16\Omega^2[3(a_1^6A_{11}+a_2^6A_{22})+a_1^2a_2^2(a_1^2+a_2^2)A_{12}+a_3^2(a_1^4A_{13}+a_2^4A_{23})$$
$$-7(a_1^4A_1+a_2^4A_2)]$$
$$+\Omega^2\Big[\frac{\delta\omega^2}{4B_{12}}(a_1^2+a_2^2)+\tfrac{2}{7}(7a_1^4S_1+a_1^4S_3-a_1^2a_3^2S_5)$$
$$+\tfrac{2}{7}(7a_1^2a_2^2S_2+a_2^4S_4-a_1^2a_2^2S_3)\Big]$$
$$-\tfrac{5}{3}[B_{12}(a_1^2+a_2^2)-2I][\tfrac{1}{5}B_{12}(a_1^2+a_2^2)+\tfrac{12}{5}I]\Big\}\Big]\!\Big]\,. \quad (31)$$

In Table 3 we list the values of $(\Delta E)_{\text{proper vol.}}/E_0$ obtained with the aid of the foregoing formula.

IV. CONCLUDING REMARKS

The singularity in the solution for the constants S_i that occur for $a_2/a_1 = 0.29719$ and $a_3/a_1 = 0.25746$ along the Jacobian sequence has been noted in Paper III; and its origin, as stated in that paper, must be traced to the Newtonian instability of the Jacobi ellipsoid at this point to a cubic deformation of the form (1) and the excitation of this instability by the post-Newtonian effects of general relativity. However, we observe that in the present normalization of the solutions, there is no singularity either in the angular velocity or in the binding energy at this point (see the entries in the corresponding line in Table 3).

While there is no singularity, in the chosen normalization, in $\delta\Omega^2$ and ΔE at the point where all the S_i's diverge, a new situation has arisen at the point where the Jacobian sequence joins the Maclaurin sequence and $a_2 \to a_1$ it appears that $S_1 \to -\infty$ and $S_2 \to +\infty$, while $S_1 + S_2$ tends to a finite limit (namely, 0.20170). On the formal side, the origin of this singularity is the following.

In Paper III, in which the indeterminacy in the solutions for S_1, S_2, and $\delta\omega^2 = c^2\delta\Omega^2/(\pi G\rho)^2$ was not eliminated by our present requirement of $(\delta\mathfrak{M})_{\text{proper vol.}} = 0$, it was shown that as we approach the point of bifurcation from the Jacobian side, S_3, S_4, and S_5 tend to well-defined limits (namely, those given in Paper III, eqs. [103] and [111]). At the same time, the pair of equations (Paper III, eqs. [88] and [89]), which at a general point along the Jacobian sequence leads to a determinate linear relation between S_1 and

TABLE 3

THE BINDING ENERGIES AND THE ANGULAR VELOCITIES OF THE POST-NEWTONIAN JACOBI ELLIPSOIDS

a_2/a_1	S_0	$(c^2\delta\Omega)/(\pi G\rho a_1^2\Omega)$	E_0	$(c^2\Delta E)/(E_0\pi G\rho a_1^2)$
1.00	−1.11339	$+\infty$	−2.34688	...
0.97	−1.07980	+0.811	−2.27595	...
0.96	−1.06853	+0.7849	−2.25212	+0.024
0.95	−1.05721	+0.7703	−2.22815	+0.0231
0.90	−1.00006	+0.7239	−2.10677	+0.0210
0.85	−0.94191	+0.6791	−1.98270	+0.0184
0.80	−0.88279	+0.6328	−1.85604	+0.0151
0.75	−0.82275	+0.5850	−1.72689	+0.0114
0.70	−0.76183	+0.5357	−1.59543	+0.00725
0.65	−0.70011	+0.4851	−1.46189	+0.00264
0.60	−0.63768	+0.4333	−1.32659	−0.00230
0.55	−0.57469	+0.3806	−1.18996	−0.00747
0.50	−0.51132	+0.3272	−1.05257	−0.01272
0.45	−0.44781	+0.2738	−0.91518	−0.01785
0.40	−0.38449	+0.2208	−0.77875	−0.02263
0.35	−0.32182	+0.1692	−0.64457	−0.02673
0.32	−0.28475	+0.1393	−0.56581	−0.02868
0.31	−0.27252	+0.1295	−0.53994	−0.0289
$(a_2/a_1)^*$	−0.25696	+0.1173	−0.50711	−0.02985
0.29	−0.24829	+0.1106	−0.48888	−0.03015
0.28	−0.23631	+0.1014	−0.46373	−0.03052
0.25	−0.20096	+0.07504	−0.39005	−0.03114
0.20	−0.14472	+0.03631	−0.27468	−0.0303
0.15	−0.09324	+0.00672	−0.17187	−0.0266
0.10	−0.04888	−0.01007	−0.08657	−0.0194
0.05	−0.01536	...	−0.02561	...

NOTE.—$(a_2/a_1)^* = 0.29719$.

S_2 and leaves $\delta\omega^2$ correspondingly indeterminate, become a pair of equations (Paper III, eqs. [112] and [113]) for $\delta\omega^2$ and $S_1 + S_2$; these equations determine $\delta\omega^2$ and $S_1 + S_2$ without any ambiguity but leave $S_1 - S_2$ arbitrary. This last fact was interpreted (correctly, it would appear in that context) by the statement, "*The continuous range of post-Newtonian configurations which occur at the point of bifurcation is a general-relativistic manifestation of the Newtonian instability which sets in at this point if some dissipative mechanism is present.*"

When we now supplement equations (112) and (113) of Paper III by equation (22), we obtain, in the limit $a_2 \to a_1$, a further equation between $\delta\omega^2$ and $S_1 + S_2$, as is apparent from the expression (21) for $(\delta\mathfrak{M})_{\text{proper vol.}}$; and these three equations for $\delta\omega^2$ and $S_1 + S_2$ are simply not consistent. The origin of this inconsistency must be traced to the circumstance that at the Newtonian point of bifurcation the Newtonian angular momentum remains constant to the first order in the displacement as we pass to the Jacobian sequence. Consequently an analysis valid only to the first order in the displacements will require an infinite displacement to make the Newtonian and the post-Newtonian configurations have equal angular momenta. It is on this account that $S_1 \to -\infty$ and $S_2 \to +\infty$ as we approach the Newtonian point of bifurcation. Actually, the Newtonian point of bifurcation is not strictly relevant to the post-Newtonian theory: as shown in Paper III (eqs. [128] and [129]), on the relativistic theory, bifurcation occurs at a slightly displaced point.

In judging the foregoing results on the post-Newtonian Jacobi ellipsoids, we must bear in mind that there can be no analogue to the Jacobi ellipsoids in the exact framework of general relativity since such objects will radiate gravitational waves and will tend to evolve toward a nonradiating axisymmetric state and presumable secular instability (cf. Chandrasekhar 1970*a*, *b*, where these problems are solved explicitly in the context of the Jacobian sequence by including radiation-reaction in a suitable approximation). It is indeed possible that in the exact framework of general relativity there are no stable stationary solutions representing rotating configurations beyond points that correspond to points of bifurcation.

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