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THE EVOLUTION OF THE JACOBI ELLIPSOID BY GRAVITATIONAL RADIATION

S. CHANDRASEKHAR
University of Chicago
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ABSTRACT

The quasi-static evolution of the Jacobi ellipsoid by gravitational radiation is determined by integrating the equation that gives the rate of dissipation of the angular momentum with the constraint that the ellipsoid remains Jacobian at all times. It is found that the evolution is in the direction of increasing angular velocity toward a non-radiating state at the point of bifurcation with the Maclaurin sequence.

I. INTRODUCTION

In current discussions of pulsars (cf. Ostriker and Gunn 1969; Ferrari and Ruffini 1969) a problem which has arisen concerns the evolution of a uniformly rotating object that is radiating gravitationally. It has often been supposed that the evolution is in the direction of *decreasing* angular velocity on the presumption that the dissipation of energy by gravitational radiation is at the expense of the rotational kinetic energy. This need not necessarily be the case: the energy that is dissipated can equally be at the expense of the potential and/or the internal energy. Since configurations that are symmetric about the axis of rotation cannot radiate gravitationally, we must first know the origin of the non-axisymmetry of the configuration before we can be certain of its evolution. In the case of the classical ellipsoid of Jacobi, the origin of its triaxial nature is fully understood. Consequently, in this case, its evolution by gravitational radiation can be uniquely determined. It will appear that the evolution is actually in the direction of *increasing* angular velocity (see Chandrasekhar 1970*a*, where a preliminary account of this result is given).

II. THE RATES OF DISSIPATION OF ENERGY AND ANGULAR MOMENTUM BY A ROTATING OBJECT

The rates of dissipation of the energy E and the angular momentum L of an object by gravitational radiation are given by (cf. Chandrasekhar and Esposito 1970, eqs. [115] and [125])

$$\frac{dE}{dt} = -\frac{G}{45c^5}\left\langle\frac{d^3D_{\alpha\beta}}{dt^3}\frac{d^3D_{\alpha\beta}}{dt^3}\right\rangle \tag{1}$$

and

$$\frac{dL_\gamma}{dt} = \frac{4G}{5c^5}\left\langle\frac{d^3I_{\alpha\mu}}{dt^3}\frac{d^2I_{\beta\mu}}{dt^2}\right\rangle, \tag{2}$$

where

$$D_{\alpha\beta} = 3I_{\alpha\beta} - \delta_{\alpha\beta}I_{\mu\mu} \tag{3}$$

denotes the quadrupole moment and $I_{\alpha\beta}$ the moment-of-inertia tensor of the object. Also, in equation (2) it has been supposed that the indices $\alpha \neq \beta \neq \gamma$ are in cyclical order.

If the object is quasi-static in a frame of reference rotating with an angular velocity Ω about one of the principal axes of the moment-of-inertia tensor (say, the 3-axis), then (see Chandrasekhar 1970*b*, eqs. [17] and [18])

$$\left(\frac{d^2I}{dt^2}\right)_{\text{inertial frame}} = -2\Omega^2(I_{11} - I_{22})\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} \tag{4}$$

and

$$\left(\frac{d^3I}{dt^3}\right)_{\text{inertial frame}} = -4\Omega^3(I_{11} - I_{22})\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \tag{5}$$

where I_{11} and I_{22} are the components of the moment-of-inertia tensor (in the rotating frame) along its two principal axes in the equatorial plane.

Inserting the relations (5) and (4) in equations (1) and (2), we obtain

$$\frac{dE}{dt} = -\frac{32G\Omega^6}{5c^5}(I_{11} - I_{22})^2 \tag{6}$$

and

$$\frac{dL}{dt} = -\frac{32G\Omega^5}{5c^5}(I_{11} - I_{22})^2, \tag{7}$$

where L now denotes the angular momentum of the object about its axis of rotation. An immediate consequence of the relations (6) and (7) is

$$\frac{dL}{dt} = \frac{1}{\Omega}\frac{dE}{dt} \tag{8}$$

—a known result (cf. Ostriker and Gunn 1969, eq. [B19]).

It should be noted that in deriving equations (6) and (7) the assumption has been made that the rates of fractional decrease in the energy and in the angular momentum of the object are slow compared to Ω; but this assumption is already implicit in the premises which underlie the original equations (1) and (2).

III. THE EVOLUTION OF THE JACOBI ELLIPSOID BY GRAVITATIONAL RADIATION

The structure of the Jacobi ellipsoid is governed by two equations (cf. Chandrasekhar 1969, § 39; this book will be referred to hereafter as E.F.E.): the equation,

$$a_1^2a_2^2A_{12} = a_3^2A_3, \tag{9}$$

which determines the geometry (i.e., the ratio of the axes $a_1:a_2:a_3$) of the ellipsoid; and the equation,

$$\Omega^2 = 2B_{12}\,\pi G\rho, \tag{10}$$

which determines the angular velocity Ω that is to be associated with each Jacobian figure. In equations (9) and (10), $A_{ij\ldots}$ and $B_{ij\ldots}$ are the "index symbols" as usually defined in this theory (E.F.E., § 21).

The evolution of the Jacobi ellipsoid as it radiates gravitationally can be determined quite simply without an explicit appeal to the full equations of motion. Thus, equation (7) in the context of the Jacobi ellipsoid gives

$$\frac{d}{dt}[(a_1^2 + a_2^2)\Omega] = -\tfrac{32}{25}\frac{GM}{c^5}(a_1^2 - a_2^2)^2\Omega^5, \tag{11}$$

where M denotes the mass of the ellipsoid.

As we have already noted, the use of equation (11) presupposes that the emission of gravitational radiation alters the figure and the angular velocity of the ellipsoid at rates that are slow compared to its instantaneous angular velocity Ω. In other words, during the evolution of the Jacobi ellipsoid, equations (9) and (10) will continue to specify the figure and the angular velocity at each instant; and $a_1a_2a_3$ must also remain constant in view of the assumed homogeneity of the configuration.

Letting

$$\lambda_2 = a_2/a_1 \quad \text{and} \quad \lambda_3 = a_3/a_1 \,, \tag{12}$$

and inserting for Ω its value given in equation (10), we find that equation (11) can be rewritten in the form

$$\frac{d}{dt}\left[\frac{1+\lambda_2^2}{(\lambda_2\lambda_3)^{2/3}}\sqrt{B_{12}}\right] = -\tfrac{1}{2}\frac{(1-\lambda_2^2)^2}{(\lambda_2\lambda_3)^{4/3}}(B_{12})^{5/2}\,, \tag{13}$$

where t is measured in the unit

$$T = \tfrac{25}{144}\frac{\bar{a}^4c^5}{G^3M^3} = \tfrac{25}{18}\left(\frac{\bar{a}}{R_S}\right)^3\frac{\bar{a}}{c}\,, \tag{14}$$

$\bar{a} = (a_1a_2a_3)^{1/3}$ is the mean radius of the ellipsoid, and $R_S = 2GM/c^2$ is the Schwarzschild radius. Also, it might be noted that

$$B_{12} = \lambda_2\lambda_3\int_0^\infty \frac{udu}{(1+u)^{3/2}(\lambda_2^2+u)^{3/2}(\lambda_3^2+u)^{1/2}} \tag{15}$$

is a function of λ_2 and λ_3 only.

Expanding the left-hand side of equation (13) and making use of the relations

$$\frac{\partial B_{12}}{\partial\lambda_2} = \frac{1}{\lambda_2}(B_{12} - 3a_2^2B_{122}) \quad \text{and} \quad \frac{\partial B_{12}}{\partial\lambda_3} = \frac{1}{\lambda_3}(B_{12} - a_3^2B_{123}) \tag{16}$$

(which can be readily derived from eq. [15]), we obtain

$$\lambda_3[\tfrac{1}{3}(11\lambda_2^2 - 1)B_{12} - 3(1+\lambda_2^2)a_2^2B_{122}]\frac{d\lambda_2}{dt} - \lambda_2(1+\lambda_2^2)(\tfrac{1}{3}B_{12} + a_3^2B_{123})\frac{d\lambda_3}{dt}$$
$$= -(\lambda_2\lambda_3)^{1/3}(1-\lambda_2^2)^2(B_{12})^3\,. \tag{17}$$
[1]

Equation (17) must be considered together with the constraint provided by equation (9), namely,

$$F(\lambda_2, \lambda_3) = \lambda_2^2(a_1^2A_{12}) - \lambda_3^2A_3 = 0\,. \tag{18}$$

In differential form this constraint equation is equivalent to

$$\lambda_2\left(3a_1^2B_{122} - \frac{\lambda_3^2}{\lambda_2^2}B_{23}\right)\frac{d\lambda_2}{dt} = \lambda_3\left(3B_{33} - \frac{\lambda_2^2}{\lambda_3^2}a_1^2B_{123}\right)\frac{d\lambda_3}{dt}\,. \tag{19}$$

Equations (17) and (19) provide a pair of ordinary differential equations for λ_2 and λ_3 which can be integrated without difficulty. The results of the integration are given in Table 1 and further exhibited in Figure 1. For the sake of definiteness it has been supposed that the Jacobi ellipsoid at time $t = 0$ is the most elongated that is compatible with stability, namely, $a_2/a_1 = 0.43223$ and $a_3/a_1 = 0.34506$, where $\Omega^2/\pi G\rho = 0.28403$ (E.F.E., p. 110).

It will be observed that the angular velocity increases during the evolution of the Jacobi ellipsoid. This result is indeed to be expected since along the Jacobian sequence the angular velocity decreases while the angular momentum increases.

It is manifest from Figure 1 that the Jacobi ellipsoid approaches, asymptotically, the non-radiating Maclaurin spheroid (with $\lambda_2 = 1$ and $\lambda_3 = 0.58272$) at the point of bifurcation of the Maclaurin and the Jacobian sequences. We shall show in § IV below this approach toward the point of bifurcation is exponential.

[1] It should be noted that the combinations of the index symbols (such as $a_2^2B_{122}$ and $a_3^2B_{123}$, besides B_{ij}) which occur in this and similar equations below are functions of λ_2 and λ_3 only.

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TABLE 1

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a_2/a_1	e	Ω	t	a_2/a_1	e	Ω	t	a_2/a_1	e	Ω	t
0.432232	0.90176	0.53294	0	0.63....	0.77660	0.58627	11.5356	0.82....	0.57236	0.60693	25.7434
0.44....	0.89800	0.53597	0.4690	0.64....	0.76837	0.58793	12.1464	0.83....	0.55776	0.60750	26.7715
0.45....	0.89303	0.53973	1.0666	0.65....	0.75993	0.58952	12.7653	0.84....	0.54259	0.60803	27.8530
0.46....	0.88792	0.54334	1.6582	0.66....	0.75127	0.59103	13.3930	0.85....	0.52678	0.60851	28.9949
0.47....	0.88267	0.54680	2.2447	0.67....	0.74236	0.59247	14.0307	0.86....	0.51029	0.60896	30.2061
0.48....	0.87727	0.55013	2.8269	0.68....	0.73321	0.59384	14.6793	0.87....	0.49305	0.60937	31.4971
0.49....	0.87172	0.55332	3.4055	0.69....	0.72381	0.59515	15.3398	0.88....	0.47497	0.60974	32.8811
0.50....	0.86603	0.55638	3.9814	0.70....	0.71414	0.59639	16.0134	0.89....	0.45596	0.61008	34.3746
0.51....	0.86017	0.55931	4.5553	0.71....	0.70420	0.59757	16.7015	0.90....	0.43589	0.61038	35.9992
0.52....	0.85417	0.56213	5.1278	0.72....	0.69397	0.59868	17.4054	0.91....	0.41461	0.61065	37.7827
0.53....	0.84800	0.56483	5.6998	0.73....	0.68345	0.59974	18.1265	0.92....	0.39192	0.61089	39.7634
0.54....	0.84166	0.56742	6.2718	0.74....	0.67261	0.60075	18.8665	0.93....	0.36756	0.61110	41.9945
0.55....	0.83516	0.56989	6.8446	0.75....	0.66144	0.60169	19.6274	0.94....	0.34117	0.61128	44.5542
0.56....	0.82849	0.57227	7.4188	0.76....	0.64992	0.60259	20.4110	0.95....	0.31225	0.61142	47.5637
0.57....	0.82164	0.57454	7.9950	0.77....	0.63804	0.60343	21.2197	0.96....	0.28000	0.61155	51.2259
0.58....	0.81462	0.57672	8.5741	0.78....	0.62578	0.60423	22.0561	0.97....	0.24310	0.61163	55.9212
0.59....	0.80740	0.57881	9.1566	0.79....	0.61311	0.60497	22.9231	0.98....	0.19900	0.61169	62.5035
0.60....	0.80000	0.58080	9.7433	0.80....	0.60000	0.60567	23.8240	0.99....	0.14107	0.61173	73.6979
0.61....	0.79240	0.58271	10.3349	0.81....	0.58643	0.60632	24.7626	1.00....	0	0.61174	...
0.62....	0.78460	0.58453	10.9321								

* The column headed e is the eccentricity of the equatorial section, Ω is the angular velocity in the unit $(\pi G\rho)^{1/2}$, and t is the time in the unit T (defined in eq. [14]).

IV. THE APPROACH TO THE NON-RADIATING STATE

As we have seen in § III, the Jacobi ellipsoid, as it continues to radiate gravitationally, approaches asymptotically a non-radiating state characterized by

$$\lambda_2 = 1 , \quad \lambda_3 = (\lambda_3)_1 = 0.58272 , \quad \text{and} \quad \Omega^2/\pi G\rho = 0.37423 . \tag{20}$$

Therefore, for ascertaining the nature of the asymptotic approach, we shall let

$$\lambda_2 = 1 - \delta\lambda_2 \tag{21}$$

and determine the limiting form which equation (13) takes when $\delta\lambda_2 \to 0$. In this limit, the right-hand side of this equation becomes

$$-\frac{2(B_{11})^{5/2}}{(\lambda_3)_1^{4/3}}\,\delta\lambda_2^2 . \tag{22}$$

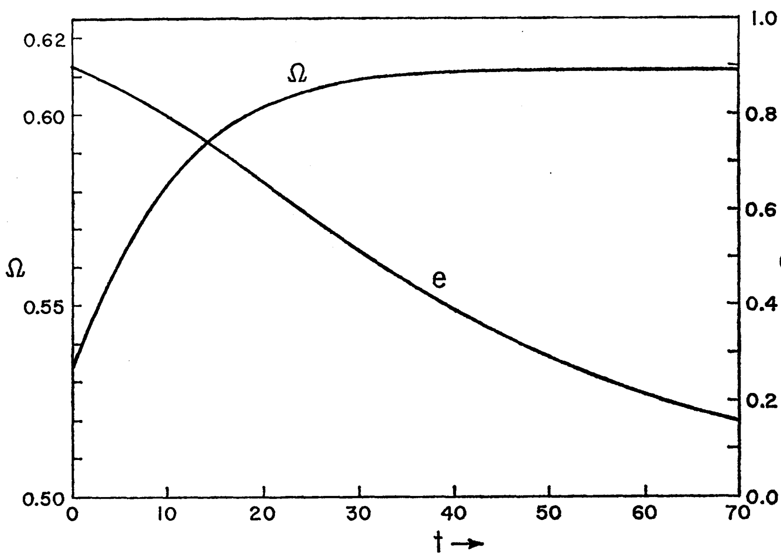

FIG. 1.—Illustrating the evolution of the Jacobi ellipsoid by gravitational radiation. The abscissa measures the time in the unit T (defined in eq. [14]), and the ordinates are the eccentricity e of the equatorial section (scale on the right-hand side) and the angular velocity Ω measured in the unit $(\pi G\rho)^{1/2}$ (scale on the left-hand side).

Therefore, the left-hand side of the equation must also be $O(\delta\lambda_2^2)$ as $\delta\lambda_2 \to 0$. We shall presently show that, in fact, both

$$G(\lambda_2, \lambda_3) = \frac{1 + \lambda_2^2}{(\lambda_2\lambda_3)^{2/3}} \quad \text{and} \quad B_{12} \tag{23}$$

are of $O(\delta\lambda_2^2)$ as $\delta\lambda_2 \to 0$.

First, we observe that it follows from quite elementary considerations that

$$\delta\lambda_3 = \tfrac{1}{2}(\lambda_3)_1\delta\lambda_2 + O(\delta\lambda_2^2) \quad \text{as} \quad \delta\lambda_2 \to 0 . \tag{24}$$[2]

[2] This relation follows from the fact that the Lagrangian displacement, which deforms the Maclaurin spheroid (with semiaxes a_1, a_1, and a_3) at the point of bifurcation into an adjacent Jacobi ellipsoid, results in an ellipsoid having the semiaxes $a_1 + \delta a_1$, $a_1 - \delta a_1$, and a_3. Therefore, for a Jacobi ellipsoid, near the point of bifurcation, $\lambda_2 = 1 - 2\delta a_1/a_1$ and $\lambda_3 = a_3(1 - \delta a_1/a_1)/a_1$. In other words, for $\lambda_2 \to 1$, $\delta\lambda_2 = -2\delta a_1/a_1$, and $\delta\lambda_3 = -(\lambda_3)_1\delta a_1/a_1$; and the result stated follows.

Also, it can be shown that δB_{12} and δG vanish to the first order for changes in λ_2 and λ_3 (in the neighborhood of the point of bifurcation) that are in the ratio given in equation (24). The relation between $\delta\lambda_3$ and $\delta\lambda_2$ to the second order in $\delta\lambda_2$ is, therefore, needed; and this relation can be determined by making use of the Taylor expansion of the relation (18), to the second order in $\delta\lambda_2$ and $\delta\lambda_3$, in the neighborhood of $\lambda_2 = 1$ and $\lambda_3 = (\lambda_3)_1$. Thus, we consider

$$\left(\frac{\partial F}{\partial\lambda_2}\right)_1 \delta\lambda_2 + \left(\frac{\partial F}{\partial\lambda_3}\right)_1 \delta\lambda_3 + \tfrac{1}{2}\left(\frac{\partial^2 F}{\partial\lambda_2^2}\right)_1 \delta\lambda_2^2 + \left(\frac{\partial^2 F}{\partial\lambda_2\partial\lambda_3}\right)_1 \delta\lambda_2\delta\lambda_3 + \tfrac{1}{2}\left(\frac{\partial^2 F}{\partial\lambda_3^2}\right)_1 \delta\lambda_3^2 = 0\,, \quad (25)$$

where the subscript 1 distinguishes that the quantity in parentheses is to be evaluated for $\lambda_2 = 1$ and $\lambda_3 = (\lambda_3)_1$.

Straightforward but somewhat lengthy reductions are required to establish the following formulae:

$$\begin{aligned}
\left(\frac{\partial F}{\partial\lambda_2}\right)_1 &= 3a_1^2 B_{111} - \frac{a_3^2}{a_1^2} B_{13} = +0.159732\,, \\
\left(\frac{\partial F}{\partial\lambda_3}\right)_1 &= \frac{a_3}{a_1}\left(\frac{a_1^4}{a_3^2} B_{113} - 3B_{33}\right) = -0.548228\,, \\
\left(\frac{\partial^2 F}{\partial\lambda_2^2}\right)_1 &= 6a_1^2 B_{111} - 15a_1^4 B_{1111} + 3a_3^2 B_{113} = -0.0813466\,, \qquad (26)\\
\left(\frac{\partial^2 F}{\partial\lambda_2\partial\lambda_3}\right)_1 &= 3\frac{a_1}{a_3}(a_1^2 B_{111} - a_1^2 a_3^2 B_{1113}) - 3\frac{a_3}{a_1}(B_{13} - a_3^2 B_{133}) = -0.0601069\,, \\
\left(\frac{\partial^2 F}{\partial\lambda_3^2}\right)_1 &= -6B_{33} - 3a_1^4 B_{1133} + 15a_3^2 B_{333} = -0.734506\,,
\end{aligned}$$

where all the index symbols are to be evaluated for the critical Maclaurin spheroid at the point of bifurcation. Inserting the foregoing results in equation (10), we find that to the second order in $\delta\lambda_2$

$$\delta\lambda_3 = \tfrac{1}{2}(\lambda_3)_1\delta\lambda_2 + B\delta\lambda_2^2 \quad \text{where} \quad B = -0.163003\,. \qquad (27)$$

We next determine the behavior of B_{12} near $\lambda_2 = 1$ and $\lambda_3 = (\lambda_3)_1$ from the Taylor expansion

$$\begin{aligned}
\delta B_{12} = \left(\frac{\partial B_{12}}{\partial\lambda_2}\right)_1 \delta\lambda_2 + \left(\frac{\partial B_{12}}{\partial\lambda_3}\right)_1 \delta\lambda_3 + \tfrac{1}{2}\left(\frac{\partial^2 B_{12}}{\partial\lambda_2^2}\right)_1 \delta\lambda_2^2 + \left(\frac{\partial^2 B_{12}}{\partial\lambda_2\partial\lambda_3}\right)_1 \delta\lambda_2\delta\lambda_3 \\
+ \tfrac{1}{2}\left(\frac{\partial^2 B_{12}}{\partial\lambda_3^2}\right)_1 \delta\lambda_3^2\,. \qquad (28)
\end{aligned}$$

We find

$$\begin{aligned}
\left(\frac{\partial B_{12}}{\partial\lambda_2}\right)_1 &= B_{11} - 3a_1^2 B_{111} = -0.0688244\,, \\
\left(\frac{\partial B_{12}}{\partial\lambda_3}\right)_1 &= \frac{a_1}{a_3}(B_{11} - a_3^2 B_{113}) = +0.236217\,, \\
\left(\frac{\partial^2 B_{12}}{\partial\lambda_2^2}\right)_1 &= -3(3a_1^2 B_{111} - 5a_1^4 B_{1111}) = -0.026196\,, \qquad (29)\\
\left(\frac{\partial^2 B_{12}}{\partial\lambda_2\partial\lambda_3}\right)_1 &= \frac{a_1}{a_3}(B_{11} - a_3^2 B_{113} - 3a_1^2 B_{111} + 3a_1^2 a_3^2 B_{1113}) = -0.0432234\,, \\
\left(\frac{\partial^2 B_{12}}{\partial\lambda_3^2}\right)_1 &= -3a_1^2(B_{113} - a_3^2 B_{1133}) = -0.257016\,.
\end{aligned}$$

To the first order in $\delta\lambda_2$, δB_{12} vanishes for $\delta\lambda_3 = \frac{1}{2}(\lambda_3)_1\delta\lambda_2$; and to the second order in $\delta\lambda_2$ we have

$$\delta B_{12} = \delta\lambda_2^2\left[B\left(\frac{\partial B_{12}}{\partial\lambda_3}\right)_1 + \frac{1}{2}\left(\frac{\partial^2 B_{12}}{\partial\lambda_2^2}\right)_1 + \frac{1}{2}(\lambda_3)_1\left(\frac{\partial^2 B_{12}}{\partial\lambda_2\partial\lambda_3}\right)_1 + \frac{1}{8}(\lambda_3)_1^2\left(\frac{\partial^2 B_{12}}{\partial\lambda_3^2}\right)_1\right]. \tag{30}$$

Inserting for the various coefficients in equation (30) their values given in equations (27) and (29), we obtain

$$\delta B_{12} = Q\delta\lambda_2^2 \quad \text{where} \quad Q = -0.0751050\,. \tag{31}$$

Finally, considering the function $G(\lambda_2, \lambda_3)$ defined in equation (23), we find

$$\begin{gathered}(G)_1 = \frac{2}{(\lambda_3)_1^{2/3}}\,, \quad \left(\frac{\partial G}{\partial\lambda_2}\right)_1 = \frac{2}{3}\frac{1}{(\lambda_3)_1^{2/3}}\,, \quad \left(\frac{\partial G}{\partial\lambda_3}\right)_1 = -\frac{4}{3}\frac{1}{(\lambda_3)_1^{5/3}}\,, \\ \left(\frac{\partial^2 G}{\partial\lambda_2^2}\right)_1 = \frac{14}{9}\frac{1}{(\lambda_3)_1^{2/3}}\,, \quad \left(\frac{\partial^2 G}{\partial\lambda_2\partial\lambda_3}\right)_1 = -\frac{4}{9}\frac{1}{(\lambda_3)_1^{5/3}}\,, \quad \left(\frac{\partial^2 G}{\partial\lambda_3^2}\right)_1 = \frac{20}{9}\frac{1}{(\lambda_3)_1^{8/3}}\,.\end{gathered} \tag{32}$$

The Taylor expansion for $G(\lambda_2, \lambda_3)$ at $\lambda_2 = 1$ and $\lambda_3 = (\lambda_3)_1$, together with equation (27), now gives

$$G(\lambda_2, \lambda_3) = \frac{2}{(\lambda_3)_1^{2/3}} + P\delta\lambda_2^2 \qquad (\delta\lambda_2 \to 0)\,, \tag{33}$$

where

$$P = -\frac{4B}{3(\lambda_3)_1^{5/3}} + \frac{5}{6(\lambda_3)_1^{2/3}} = 1.729075\,. \tag{34}$$

Returning to equation (13), we first rewrite it in the form

$$B_{12}\frac{dG}{dt} + \frac{1}{2}G\frac{dB_{12}}{dt} = -\frac{1}{2}\frac{(1-\lambda_2^2)^2}{(\lambda_2\lambda_3)^{4/3}}(B_{12})^3\,. \tag{35}$$

Next, making use of equations (31) and (32), we find that as $\lambda_2 \to 1$, the equation takes the limiting form

$$\left[B_{11}P + \frac{Q}{(\lambda_3)_1^{2/3}}\right]\frac{d}{dt}\delta\lambda_2 = -\frac{(B_{11})^3}{(\lambda_3)_1^{4/3}}\delta\lambda_2\,. \tag{36}$$

Hence,

$$\delta\lambda_2 \to \text{constant } e^{-t/\tau}\,, \tag{37}$$

where

$$\tau = \frac{(\lambda_3)_1^{4/3}}{(B_{11})^3}\left[B_{11}P + \frac{Q}{(\lambda_3)_1^{2/3}}\right] = 16.039\,. \tag{38}$$

Since $\Omega^2/\pi G\rho = 2B_{12}$, it follows that

$$\Omega_{\mathrm{Mc}}\delta\Omega/\pi G\rho = \delta B_{12} = Q\delta\lambda_2^2 \qquad (\delta\lambda_2 \to 0)\,, \tag{39}$$

where Ω_{Mc} denotes the angular velocity of the Maclaurin spheroid at the point of bifurcation. Combining equations (37) and (39), we can write

$$\delta\Omega = (\Omega_{\mathrm{Mc}} - \Omega) \to \text{constant } e^{-2t/\tau}\,. \tag{40}$$

Equations (37) and (40) specify the manner in which the Jacobi ellipsoid, radiating gravitationally, approaches the non-radiating state.

It may be noted here that for $\bar{a} = 20$ km and $M = 1$ solar mass, the unit of time T specified in equation (14) is 2.90×10^{-2} s. The time constant in equation (40), namely, 8.02 T, is 0.232 s whereas the period of rotation at the point of bifurcation is 2.92×10^{-3} s. The basic assumption that the Jacobi ellipsoid evolves adiabatically is thus amply justified even under these somewhat extreme conditions.

V. CONCLUDING REMARKS

The fact that the Jacobi ellipsoid evolves by gravitational radiation in the direction of increasing angular velocity toward the point of bifurcation with the Maclaurin sequence, together with the result established in the preceding paper (Chandrasekhar 1970*b*), gives rise to a curious dilemma: the dissipation of energy by gravitational radiation induces secular instability in the very spheroid toward which it is assisting the Jacobi ellipsoid to evolve.[3] It is as if radiation reaction is promoting evolution; but only toward a catastrophe! (See, however, the Note added in proof.) One may speculate whether this behavior has a moral in the larger context of gravitational collapse.

I am greatly indebted to Miss Donna D. Elbert for carrying out the integration of equations (17) and (19) which is included in Table 1 and exhibited in Figure 1.

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Note added in proof.—An alternative possibility suggested to me by Dr. J. P. Ostriker is the following:

A homogeneous uniformly rotating nonviscous mass may be expected to evolve up the Maclaurin sequence, in a time-scale τ_E (say), as its density increases. When the mass passes beyond the point of bifurcation, the secular instability due to gravitational radiation will result in the dissipation of *both* its energy and its angular momentum on the time-scale τ (defined in eq. [61] of the preceding paper). And if $\tau \ll \tau_E$, we may expect the configuration to hover about the point of bifurcation. As Dr. Ostriker points out, on these arguments (if correct), the angular velocity Ω_s at the point of bifurcation will represent the equilibrium angular velocity of a neutron star and the total energy output of a pulsar will just equal $\frac{1}{2}I_s\Omega_s^2$.

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[3] It should, however, be noted that the mode which is made secularly unstable is *not* the mode by which the Jacobi ellipsoid quasi-statically adjusts itself as it approaches the point of bifurcation.