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THE INSTABILITY OF THE CONGRUENT DARWIN ELLIPSOIDS. II

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ABSTRACT

The resonant oscillations of one of two congruent Darwin ellipsoids, forced by the natural oscillations of the other, are considered; and the instability of the ellipsoids to synchronous coupled oscillations is traced to this resonance.

I. INTRODUCTION

In the first of two papers on the Darwin ellipsoids (Chandrasekhar 1964, 1969; these papers will be referred to hereafter as Papers I and II, respectively) the natural modes of oscillation of either component by itself (with the other remaining static) were considered; and it was shown that "*all the physically realizable* congruent *ellipsoids are stable with respect to their own natural oscillations.*" In the second paper, a class of synchronous coupled oscillations was considered; and it was shown that "*two of the five modes of oscillation belonging to this class excite instabilities along the entire Darwin sequence.*" This last result was characterized as "unexpected" since it is at variance with Darwin's surmises with respect to the "limiting" and the "partial" stability of his figures.

The question to which this paper is addressed is: Should the instability of the congruent Darwin ellipsoids be considered as really "unexpected"? Actually, on second thoughts, it would rather appear that the instability is indeed natural under the circumstances! Thus, suppose that one of the two congruent figures is set into a mode of natural oscillation (considered in Paper I). The tidal potential of this component over the other will then vary periodically. As a result, the other component will be set into *forced* oscillations. Since the natural frequencies of oscillation of the two components (being congruent) are the same, it is clear that we shall have a case of *resonant* forced oscillations. The amplitude of the forced oscillations should, therefore, be expected to grow linearly with time. The energy in the oscillation of the first component will thus begin to be drained to the second component; and this transfer of energy must result in the excitation of the coupled modes of oscillation (considered in Paper II) and the eventual instability.

II. THE RESONANT OSCILLATIONS OF ONE OF THE COMPONENTS FORCED BY THE NATURAL OSCILLATIONS OF THE OTHER

As shown in Paper I (eqs. [39] and [47]–[49]; see also Paper II, eqs. [58]–[61]), the equations governing the even modes of natural oscillation of one of the components (while the other remains static) are

$$\left(\tfrac{1}{2}\frac{d^2}{dt^2} + 3B_{11} - B_{12} - \beta_{11}\right) V_{11} + \left(-\tfrac{1}{2}\frac{d^2}{dt^2} - 3B_{22} + B_{12} + \beta_{22}\right) V_{22} + (B_{13} - B_{23}) V_{33} - 2\Omega \frac{dV_{12}}{dt} = 0\,, \tag{1}$$

$$\frac{d}{dt}\left(\frac{1}{2}\frac{d^2}{dt^2}+3B_{11}+B_{12}-2B_{13}+2\Omega^2-\beta_{11}\right)V_{11}$$

$$+\frac{d}{dt}\left(\frac{1}{2}\frac{d^2}{dt^2}+3B_{22}+B_{12}-2B_{23}+2\Omega^2-\beta_{22}\right)V_{22} \tag{2}$$

$$+\frac{d}{dt}\left(-\frac{d^2}{dt^2}-6B_{33}+B_{13}+B_{23}+2\beta_{33}\right)V_{33}+2(\beta_{11}-\beta_{22})\Omega V_{12}=0\,,$$

$$\Omega\frac{dV_{11}}{dt}-\Omega\frac{dV_{22}}{dt}+\left(\frac{d^2}{dt^2}+4B_{12}-\beta_{11}-\beta_{22}\right)V_{12}=0\,, \tag{3}$$

and

$$\frac{1}{a_1^2}V_{11}+\frac{1}{a_2^2}V_{22}+\frac{1}{a_3^2}V_{33}=0\,, \tag{4}$$

where the various symbols have the same meanings as in Paper II.

Letting

$$(V_{11},\ V_{22},\ V_{33},\ V_{12})=(X_1,\ X_2,\ X_3,\ X_4)\,, \tag{5}$$

we can write equations (1)–(4), in matrix notation, in the form

$$L_{ij}(d/dt)X_j(t)=0\,, \tag{6}$$

where the coefficients L_{ij}, as indicated, include terms in d/dt. Seeking solutions having a time-dependence of the form

$$X_j(t)=X_je^{\lambda t}\,, \tag{7}$$

where X_j's now denote the amplitudes of the oscillation and λ is a characteristic-value parameter to be determined, we obtain the characteristic equation

$$L_{ij}(\lambda)X_j=0\,. \tag{8}$$

It has been shown in Paper I that equation (8) allows three characteristic values for λ^2 (which are all negative corresponding to the stability of these modes). Let λ now denote a characteristic root and X_j a suitably normalized characteristic vector of the matrix $\boldsymbol{L}$, belonging to it. Also, let $X^\dagger_i$ denote a characteristic vector of the transposed matrix $\boldsymbol{L}^\dagger$, belonging to the same λ, so that

$$L_{ij}(\lambda)X^\dagger_i=0\,. \tag{9}$$

During the oscillation, with an amplitude appropriate to the solution X_j (say), the orientation of the ellipsoid in the equatorial plane, as well as its semiaxes, will vary periodically with amplitudes that can be deduced from the formulae given in Paper II, § III (eqs. [27] and [28]). The resulting variation of the tidal potential over the other component can be similarly written with the aid of the formulae given in Paper II, § IV (eqs. [33]–[37]) and § V (eqs. [62], [66], and [67]): the required expressions for $\delta\Omega$ and $\delta\beta_{ij}$ are formally the same; but they should now be evaluated in terms of V_{11}, V_{22}, V_{33}, and V_{12} that follow from the adopted solution of equation (8). (Since the centers of mass of the two ellipsoids are at relative rest during the natural modes of oscillation, we should set $V_1=V_2=0$.)

Under the influence of the varying tidal potential, the equations governing the forced oscillations of the second component are, again, formally the same as in Paper II, equations (58)–(61); but the terms in $\delta\Omega$ and $\delta\beta_{ij}$ must now be interpreted as arising from the natural oscillations of the other component. Thus, if

$$V_{11},\ V_{22},\ V_{33},\ V_{12})=(Y_1,\ Y_2,\ Y_3,\ Y_4) \tag{10}$$

denote the virials of the component executing the forced oscillations, the equations governing Y_j are represented by

$$L_{ij}(d/dt)Y_j(t) = F_i e^{\lambda t} , \tag{11}$$

where L_{ij} denotes the same matrix defined by equations (1)–(4) and the forcing term F_i is given by (cf. Paper II, eqs. [59]–[62], [66], and [67])

$$\begin{aligned} F_1 &= I_{11}\delta\beta_{11} - I_{22}\delta\beta_{22} , \quad F_3 = -\lambda(I_{11} - I_{22})\delta\Omega + (I_{11} + I_{22})\delta\beta_{12} , \quad F_4 = 0 , \\ F_2 &= -2\lambda\Omega(I_{11} + I_{22})\delta\Omega + 2\Omega(I_{11} - I_{22})\delta\beta_{12} + \lambda I_{11}\delta\beta_{11} + \lambda I_{22}\delta\beta_{22} - 2\lambda I_{33}\delta\beta_{33} , \end{aligned} \tag{12}$$

where

$$\delta\beta_{11} = 2\delta a_1 + \delta a_2 + \delta a_3 , \quad \delta\beta_{22} = 2\delta a_1 - \delta a_2 , \quad \delta\beta_{33} = -\delta a_3 , \quad \text{and} \quad \Omega\delta\Omega = \delta a_1 , \tag{13}$$

and

$$\begin{aligned} \delta a_1 &= \frac{5}{2M}(Q_{11}X_1 - a_{12}X_2 - a_{13}X_3) , \\ \delta a_2 &= \frac{5}{2M}(Q_{21}X_1 - 3a_{22}X_2 - a_{23}X_3) , \\ \delta a_3 &= \frac{5}{2M}(Q_{31}X_1 - a_{32}X_2 - 3a_{33}X_3) , \end{aligned} \tag{14}$$

and

$$\delta\beta_{12} = -\frac{5}{M}\frac{4Ra_1a_2a_3}{(R^2 + a_2^2 - a_1^2)^{3/2}(R^2 + a_3^2 - a_1^2)^{1/2}}\frac{X_4}{(a_1^2 - a_2^2)} .$$

(It should be noticed that in accordance with our earlier remarks, we have set $V_1 = 0$ in Paper II, eq. [66].)

The complementary solutions of equation (11) are the same as those of equation (8); and it remains only to determine a particular integral. We set

$$Y_j(t) = \kappa X_j e^{\lambda t} t + Z_j e^{\lambda t} , \tag{15}$$

where X_j denotes the (normalized) characteristic vector of $\boldsymbol{L}$ (belonging to λ) and κ and Z_j are constants, unspecified for the present. Inserting this form for the solution in equation (11), we obtain

$$\kappa L_{ij}X_j e^{\lambda t} t + \kappa\frac{\partial L_{ij}(\lambda)}{\partial\lambda}X_j e^{\lambda t} + L_{ij}Z_j e^{\lambda t} = F_i e^{\lambda t} . \tag{16}$$

Since X_j has been chosen as a characteristic vector of $\boldsymbol{L}$, belonging to λ, equation (16) reduces to

$$L_{ij}Z_j = F_i - \kappa\frac{\partial L_{ij}}{\partial\lambda}X_j . \tag{17}$$

Contracting this equation with $X^{\dagger}{}_i$, we obtain (cf. eq. [9])

$$\kappa X^{\dagger}{}_i\frac{\partial L_{ij}}{\partial\lambda}X_j = X^{\dagger}{}_i F_i . \tag{18}$$

This equation determines κ; and it will be noted that it is independent of the adopted normalizations of both X_i and $X^{\dagger}{}_i$.

With κ determined by equation (18), equation (17) will clearly allow nontrivial

TABLE 1

THE CHARACTERISTIC VECTOR OF L AND $L^{\dagger}$

ϕ_R	λ^2	V_{22}/V_{11}	V_{33}/V_{11}	iV_{12}/V_{11}	$V^{\dagger}_{22}/V^{\dagger}_{11}$	$V^{\dagger}_{33}/V^{\dagger}_{11}$	$iV^{\dagger}_{12}/V^{\dagger}_{11}$
Mode 1							
14°	-0.33060	-0.51087	-0.14221	-0.66469	0.13925	-0.93168	0.12075
16°	-0.27401	-0.45218	-0.14133	-0.60704	0.15561	-0.91124	0.13500
17°	-0.24749	-0.42378	-0.14008	-0.57678	0.16392	-0.89987	0.14126
17°.5	-0.23464	-0.40985	-0.13927	-0.56127	0.16811	-0.89391	0.14414
18°	-0.22205	-0.39612	-0.13834	-0.54551	0.17232	-0.88779	0.14686
18°.5	-0.20972	-0.38260	-0.13728	-0.52950	0.17654	-0.88153	0.14940
19°	-0.19762	-0.36929	-0.13612	-0.51322	0.18078	-0.87515	0.15176
19°.5	-0.18576	-0.35620	-0.13484	-0.49667	0.18502	-0.86868	0.15393
20°	-0.17413	-0.34334	-0.13345	-0.47983	0.18928	-0.86214	0.15591
20°.5	-0.16272	-0.33073	-0.13196	-0.46270	0.19353	-0.85560	0.15770
21°	-0.15152	-0.31836	-0.13036	-0.44526	0.19779	-0.84911	0.15928
21°.5	-0.14053	-0.30625	-0.12867	-0.42750	0.20204	-0.84276	0.16066
22°	-0.12975	-0.29439	-0.12689	-0.40938	0.20629	-0.83665	0.16183
22°.5	-0.11916	-0.28280	-0.12501	-0.39088	0.21053	-0.83091	0.16279
23°	-0.10877	-0.27149	-0.12305	-0.37197	0.21475	-0.82575	0.16353
24°	-0.08855	-0.24968	-0.11888	-0.33267	0.22313	-0.81821	0.16438
Mode 2							
14°	-1.18773	+0.20108	-0.75472	0.56381	1.84648	1.71814	0.32860
16°	-1.17374	+0.11517	-0.62244	0.60901	1.45730	1.64725	0.30727
17°	-1.16215	+0.08057	-0.56523	0.62831	1.31633	1.62537	0.29860
17°.5	-1.15523	+0.06525	-0.53870	0.63722	1.25574	1.61722	0.29461
18°	-1.14758	+0.05122	-0.51352	0.64567	1.20095	1.61079	0.29083
18°.5	-1.13920	+0.03840	-0.48959	0.65370	1.15109	1.60588	0.28719
19°	-1.13014	+0.02674	-0.46687	0.66134	1.10577	1.60244	0.28369
19°.5	-1.12038	+0.01616	-0.44527	0.66862	1.06439	1.60034	0.28027
20°	-1.10998	+0.00661	-0.42476	0.67558	1.02658	1.59950	0.27691
20°.5	-1.09893	-0.00199	-0.40525	0.68225	0.99190	1.59984	0.27359
21°	-1.08727	-0.00968	-0.38670	0.68866	0.96006	1.60130	0.27028
21°.5	-1.07502	-0.01655	-0.36904	0.69485	0.93074	1.60380	0.26697
22°	-1.06221	-0.02263	-0.35222	0.70084	0.90370	1.60731	0.26363
22°.5	-1.04885	-0.02801	-0.33619	0.70666	0.87866	1.61175	0.26024
23°	-1.03498	-0.03271	-0.32091	0.71235	0.85546	1.61712	0.25680
24°	-1.00576	-0.04035	-0.29238	0.72339	0.81380	1.63040	0.24969
Mode 3							
14°	-1.52515	-1.93737	1.08505	1.36032	-0.48256	0.88526	0.04504
16°	-1.55384	-2.29401	1.42058	1.48337	-0.57510	0.84785	0.03830
17°	-1.56891	-2.50708	1.61609	1.55486	-0.61902	0.82825	0.03436
17°.5	-1.57666	-2.62292	1.72134	1.59315	-0.64013	0.81833	0.03232
18°	-1.58450	-2.74485	1.83152	1.63300	-0.66057	0.80839	0.03028
18°.5	-1.59249	-2.87303	1.94680	1.67445	-0.68029	0.79846	0.02825
19°	-1.60057	-3.00762	2.06734	1.71745	-0.69928	0.78854	0.02624
19°.5	-1.60877	-3.14853	2.19312	1.76196	-0.71747	0.77869	0.02427
20°	-1.61705	-3.29596	2.32436	1.80794	-0.73489	0.76891	0.02236
20°.5	-1.62542	-3.45008	2.46124	1.85540	-0.75151	0.75921	0.02051
21°	-1.63386	-3.61091	2.60384	1.90428	-0.76734	0.74960	0.01873
21°.5	-1.64236	-3.77853	2.75229	1.95453	-0.78238	0.74010	0.01704
22°	-1.65091	-3.95314	2.90684	2.00612	-0.79666	0.73071	0.01543
22°.5	-1.65950	-4.13470	3.06753	2.05901	-0.81016	0.72145	0.01391
23°	-1.66812	-4.32359	3.23476	2.11318	-0.82293	0.71230	0.01249
24°	-1.68539	-4.72375	3.58950	2.22524	-0.84635	0.69436	0.00991

TABLE 2*

THE PARTICULAR INTEGRAL DESCRIBING FORCED RESONANT OSCILLATIONS OF THE DARWIN ELLIPSOID

ϕ_R	λ^2	κ	Z_2	Z_3	iZ_4	V_1	V_2
			Mode 1				
14°	-0.33060	-0.16841	0.04342	-0.03736	-0.75888	-6.34640	9.4414
16°	-0.27401	-0.14722	0.03624	-0.03073	-0.67653	-8.42128	12.1567
17°	-0.24749	-0.13531	0.03296	-0.02778	-0.64002	-9.80638	14.0685
17°.5	-0.23464	-0.12896	0.03139	-0.02639	-0.62272	-10.6063	15.2024
18°	-0.22205	-0.12232	0.02986	-0.02504	-0.60602	-11.4827	16.4683
18°.5	-0.20972	-0.11536	0.02838	-0.02375	-0.58987	-12.4342	17.8710
19°	-0.19762	-0.10804	0.02693	-0.02249	-0.57426	-13.4526	19.4066
19°.5	-0.18576	-0.10034	0.02553	-0.02129	-0.55914	-14.5173	21.0550
20°	-0.17413	-0.09221	0.02417	-0.02012	-0.54452	-15.5891	22.7708
20°.5	-0.16272	-0.08360	0.02285	-0.01900	-0.53037	-16.6036	24.4711
21°	-0.15152	-0.07444	0.02157	-0.01791	-0.51670	-17.4663	26.0264
21°.5	-0.14053	-0.06468	0.02033	-0.01687	-0.50351	-18.0519	27.2551
22°	-0.12975	-0.05421	0.01913	-0.01586	-0.49082	-18.2214	27.9430
22°.5	-0.11916	-0.04292	0.01797	-0.01489	-0.47868	-17.8556	27.8878
23°	-0.10877	-0.03069	0.01684	-0.01396	-0.46712	-16.8984	26.9639
24°	-0.08855	-0.00257	0.01471	-0.01219	-0.44616	-13.4859	22.7196
			Mode 2				
14°	-1.18773	-0.09479	0.99636	-0.85721	-0.01975	0.49852	-1.04339
16°	-1.17374	-0.11798	0.78373	-0.66462	-0.09278	0.51991	-1.04078
17°	-1.16215	-0.12855	0.68489	-0.57734	-0.12608	0.52823	-1.03709
17°.5	-1.15523	-0.13352	0.63851	-0.53682	-0.14157	0.53198	-1.03494
18°	-1.14758	-0.13825	0.59446	-0.49857	-0.15621	0.53548	-1.03268
18°.5	-1.13920	-0.14277	0.55277	-0.46258	-0.17003	0.53881	-1.03039
19°	-1.13014	-0.14706	0.51354	-0.42890	-0.18300	0.54199	-1.02812
19°.5	-1.12038	-0.15114	0.47674	-0.39746	-0.19518	0.54508	-1.02596
20°	-1.10998	-0.15502	0.44233	-0.36820	-0.20658	0.54809	-1.02395
20°.5	-1.09893	-0.15872	0.41024	-0.34104	-0.21727	0.55109	-1.02215
21°	-1.08727	-0.16224	0.38038	-0.31588	-0.22729	0.55408	-1.02062
21°.5	-1.07502	-0.16561	0.35264	-0.29259	-0.23670	0.55712	-1.01941
22°	-1.06221	-0.16884	0.32692	-0.27107	-0.24555	0.56023	-1.01854
22°.5	-1.04885	-0.17193	0.30307	-0.25119	-0.25390	0.56343	-1.01807
23°	-1.03498	-0.17492	0.28099	-0.23285	-0.26180	0.56676	-1.01803
24°	-1.00576	-0.18062	0.24164	-0.20029	-0.27650	0.57390	-1.01939
			Mode 3				
14°	-1.52515	-0.12233	2.69919	-2.32221	0.34446	0.85563	-1.96919
16°	-1.55384	-0.09455	3.35145	-2.84209	0.59014	0.86199	-1.91259
17°	-1.56891	-0.08208	3.68363	-3.10521	0.70513	0.86489	-1.89220
17°.5	-1.57666	-0.07627	3.85051	-3.23728	0.76054	0.86610	-1.88335
18°	-1.58450	-0.07076	4.01694	-3.36898	0.81432	0.86709	-1.87510
18°.5	-1.59249	-0.06557	4.18299	-3.50049	0.86649	0.86778	-1.86733
19°	-1.60057	-0.06067	4.34864	-3.63186	0.91709	0.86815	-1.85990
19°.5	-1.60877	-0.05608	4.51344	-3.76286	0.96599	0.86814	-1.85262
20°	-1.61705	-0.05177	4.67763	-3.89374	1.01329	0.86770	-1.84537
20°.5	-1.62542	-0.04774	4.84123	-4.02464	1.05900	0.86682	-1.83811
21°	-1.63386	-0.04397	5.00410	-4.15551	1.10312	0.86544	-1.83064
21°.5	-1.64236	-0.04046	5.16613	-4.28635	1.14559	0.86354	-1.82293
22°	-1.65091	-0.03719	5.32749	-4.41740	1.18650	0.86109	-1.81484
22°.5	-1.65950	-0.03414	5.48785	-4.54845	1.22573	0.85805	-1.80630
23°	-1.66812	-0.03131	5.64750	-4.67984	1.26338	0.85441	-1.79726
24°	-1.68539	-0.02623	5.96438	-4.94365	1.33387	0.84527	-1.77740

* The values of Z_j have been derived with the choice $Z_1(=V_{11}) = 1$.

solutions for Z_j even though the determinant of L vanishes. With the choice $Z_1 = 1$, for example, equation (17) can be solved uniquely. A particular integral of equation (11) can thus be obtained.

Associated with the forced oscillation of the figure, which can be determined with the aid of the foregoing solution, the center of mass of the ellipsoid will also execute oscillations in the equatorial plane. And the amplitudes of the displacements can be determined with the aid of the equations (cf. Paper II, eqs. [29], [72], and [73])

$$(\lambda^2 - \beta_{11}) V_1 - 2\lambda\Omega V_2 = 0 \tag{19}$$

and

$$(\lambda^2 - \beta_{22}) V_2 + 2\lambda\Omega V_1 = -\frac{5a_2 R}{a_1^2 - a_2^2} X_4 + \tfrac{5}{4}\lambda \frac{R}{\Omega} (Q_{11} X_1 - \alpha_{12} X_2 - \alpha_{13} X_3) ; \tag{20}$$

and this completes the solution.

III. NUMERICAL RESULTS

In Table 1, the characteristic vectors of L and $L^\dagger$ are given for a sequence of Darwin figures. And in Table 2 the solutions for the forced resonant oscillations corresponding to the three even modes of natural oscillation are given; the vector Z_j has been evaluated with the choice $Z_1 = 1$.

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