THE THEORY OF THE FLUCTUATIONS IN BRIGHTNESS OF THE MILKY WAY. II

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ABSTRACT

In this paper the integral equation governing the fluctuations in brightness of the Milky Way is considered for the case in which the system extends to infinity in the direction of the line of sight. The equation is explicitly solved for the two cases in which all the clouds are equally transparent and when the frequency of occurrence of clouds with a transparency factor q is $(n+1)q^n$. The derived distributions of brightness are illustrated.

1. *Introduction.*—In the preceding paper[1] we derived a general integral equation governing the fluctuations in brightness of the Milky Way resulting from the varying number of absorbing clouds in the line of sight; and we showed how the moments of all orders of the distribution of brightness can be expressed explicitly in terms of the average number of clouds to be expected in the line of sight and the moments of the function giving the frequency of occurrence of clouds with a transparency factor q. While the illustrative example given in Paper I shows that the analysis of the observational material can be carried quite far with the aid of the moment relations only, it is clear that it will be useful to have explicit solutions of the basic integral equation (I, eq. [18]) at least for certain special forms of the frequency function $\psi(q)$ of q. As a preliminary to a general attack on this problem, we shall consider in this paper methods of solving the integral equation for the simpler case in which the system extends to infinity in the direction of the line of sight. In this latter case the equation to be considered is (cf. I, eq. [19])

$$g(u)+\frac{dg}{du}=\int_0^1 \frac{dq}{q}\, g\left(\frac{u}{q}\right)\psi(q). \tag{1}$$

Regarding the solution of this equation, we know that its moments are given by

$$\mu_n=\frac{n!}{\prod_{j=1}^{n}(1-q_j)}, \qquad q_j=\int_0^1 q^j\psi(q)\,dq. \tag{2}$$

This expression for μ_n is obtained by letting $\xi\to\infty$ in the solution given by Paper I, equation (36).

In this paper we shall restrict ourselves to two special forms of $\psi(q)$: the case in which all the clouds are equally transparent and the case in which $\psi(q)=(n+1)q^n$. It will appear that in both these cases equation (1) can be solved explicitly.

2. *The solution for the case in which all the clouds are equally transparent.*—Let q be the constant factor by which a cloud reduces the intensity of the light of the stars immediately behind it. In this case the equation governing $g(u)$ is

$$g(u)+\frac{dg}{du}=\frac{1}{q}\, g\left(\frac{u}{q}\right); \tag{3}$$

[1] See p. 380 of this issue; hereafter this paper will be referred to as "Paper I."

and its moments are given by

$$\mu_n = \frac{n!}{\prod_{j=1}^{n} (1 - q^j)} . \tag{4}$$

The form of equation (3) suggests that we seek a solution as a Dirichlet series in the form

$$g(u) = \sum_{k=0}^{\infty} A_k e^{-u/q^k} , \tag{5}$$

where the A_k's are certain constants unspecified for the present. Substituting the foregoing form for g in equation (3) and equating the coefficients of the terms having the same exponential factor, we find

$$A_k \left(1 - \frac{1}{q^k}\right) = \frac{A_{k-1}}{q} \qquad (k = 1, \ldots) . \tag{6}$$

By repeated applications of the recurrence relation (6), we can express all the coefficients $A_k (k \geqslant 1)$ in terms of A_0. Thus

$$A_k = A_0 \frac{(-1)^k}{q^k} \prod_{j=1}^{k} \frac{q^j}{1 - q^j} \qquad (k = 1, \ldots) . \tag{7}$$

The constant A_0 itself is left arbitrary.

With the constants A_k given by equation (7), the solution for $g(u)$ takes the form

$$g(u) = A_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{q^k} \prod_{j=1}^{k} \frac{q^j}{1 - q^j} e^{-u/q^k} . \tag{8}$$

The constant A_0 in solution (8) can be determined by evaluating the moments of the solution and comparing them with equation (4). For $g(u)$ given by equation (8),

$$\mu_n = A_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{q^k} \prod_{j=1}^{k} \frac{q^j}{1 - q^j} \int_0^{\infty} e^{-u/q^k} u^n du , \tag{9}$$

or

$$\mu_n = n! A_0 \sum_{k=0}^{\infty} (-1)^k \prod_{j=1}^{k} \frac{q^{j+n}}{1 - q^j} . \tag{10}$$

The infinite sum which occurs on the right-hand side of equation (10) can be expressed as an infinite product by using the identity

$$\prod_{j=1}^{\infty} (1 - x q^{n+j}) = \sum_{k=0}^{\infty} (-1)^k x^k \prod_{j=1}^{k} \frac{q^{j+n}}{1 - q^j} . \tag{11}$$ [2]

This identity can be established by considering the function

$$\phi(x) = \prod_{j=1}^{\infty} (1 - x q^{n+j}) \qquad (q < 1) , \tag{12}$$

[2] Identities similar to this one will be found in G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers* (Oxford: Clarendon Press, 1945), p. 275.

which satisfies the functional equation,

$$\phi(x) = (1 - xq^{n+1})\,\phi(xq)\,. \tag{13}$$

Thus, expanding both sides of equation (13) by Taylor's series, we have

$$\sum_{k=0}^{\infty} \phi^{(k)}(0)\frac{x^k}{k!} = (1 - xq^{n+1}) \sum_{k=0}^{\infty} \phi^{(k)}(0)\frac{(xq)^k}{k!}\,, \tag{14}$$

where $\phi^{(k)}(0)$ denotes the value of the kth derivative of $\phi(x)$ at $x=0$. Comparing the coefficients of x^k on either side of equation (14), we obtain the recurrence relation,

$$(1 - q^k)\,\phi^{(k)}(0) = -kq^{k+n}\phi^{(k-1)}(0) \qquad (k = 1, \ldots)\,. \tag{15}$$

By repeated applications of this relation and remembering that $\phi(0) = 1$, we find that

$$\phi^{(k)}(0) = (-1)^k k! \prod_{j=1}^{k} \frac{q^{j+n}}{1 - q^j} \qquad (k = 1, \ldots)\,. \tag{16}$$

Hence

$$\begin{aligned} \phi(x) &= \sum_{k=0}^{\infty} \phi^{(k)}(0)\frac{x^k}{k!} \\ &= \sum_{k=0}^{\infty} (-1)^k x^k \prod_{j=1}^{k} \frac{q^{j+n}}{1 - q^j}\,. \end{aligned} \tag{17}$$

This is the required identity.

Setting $x = 1$ in the identity we have just established, we have

$$\prod_{j=1}^{\infty} (1 - q^{n+j}) = \sum_{k=0}^{\infty} (-1)^k \prod_{j=1}^{k} \frac{q^{j+n}}{1 - q^j}\,. \tag{18}$$

The quantity on the right-hand side of this equation is the same as that which occurs in equation (10). Hence

$$\mu_n = n!A_0 \prod_{j=1}^{\infty} (1 - q^{j+n}) = n!A_0 \prod_{j=n+1}^{\infty} (1 - q^j)\,. \tag{19}$$

Comparing this expression for μ_n with that given by equation (4), we observe that, for agreement, we must choose

$$A_0 = \frac{1}{\prod_{j=1}^{\infty} (1 - q^j)}\,. \tag{20}$$

The fact that the value of A_0 determined in this fashion is independent of the order of the moment μ_n which we have chosen to compare provides the verification that the assumed form of the solution leads to the correct one.

With A_0 given by equation (20), the complete solution for $g(u)$ takes the form

$$g(u) = \frac{1}{\prod_{j=1}^{\infty} (1 - q^j)} \sum_{k=0}^{\infty} (-1)^k \frac{e^{-u/q^k}}{q^k} \prod_{j=1}^{k} \frac{q^j}{1 - q^j}\,. \tag{21}$$

The solution represented by series (21) is fairly rapidly convergent; though, for q approaching unity, larger and larger numbers of terms in the series must be included in an accurate numerical evaluation of $g(u)$.

Solutions for $g(u)$ for various values of q computed in accordance with equation (21) are given in Table 1; they are further illustrated in Figure 1.

TABLE 1

THE FUNCTION $g(u)$ FOR VARIOUS VALUES OF q

u	$g(u)$			u	$g(u)$		
	$q=0.5$	$q=0.6$	$q=0.7$		$q=0.75$	$q=0.80$	$q=0.85$
0.00	0	0	0	0.0	0	0	0
0.25	0.0220	0.0008	....	0.5	0.0002	....	....
0.50	.1526	.0260	0.0005	1.0	.0038	0.0001	....
0.75	.3163	.1072	.0098	1.5	.0437	.0030	....
1.00	.4207	.2168	.0323	2.0	.1323	.0238	....
1.25	.4547	.3124	.0838	2.5	.2308	.0767	0.0059
1.50	.4395	.3718	.1469	3.0	.2891	.1511	.0202
1.75	.3944	.3934	.2126	3.5	.2955	.2186	.0444
2.00	.3416	.3857	.2711	4.0	.2640	.2573	.0888
2.25	.2887	.3570	.3009	4.5	.2250	.2617	.1403
2.50	.2378	.3184	.3289	5.0	.1638	.2395	.1864
2.75	.1930	.2760	.3296	5.5	.1206	.2025	.2163
3.00	.1552	.2341	.3168	6.0	.0831	.1611	.2258
3.50	.0982	.1608	.2654	6.5	.0565	.1222	.2166
4.00	.0611	.1060	.2031	7.0	.0376	.0893	.1942
4.50	.0376	.0679	.1463	7.5	.0246	.0633	.1648
5.00	.0230	.0429	.1009	8.0	.0159	.0438	.1335
5.50	.0137	.0267	.0675	8.5	.0101	.0297	.1042
6.00	.0085	.0165	.0442	9.0	.0064	.0198	.0787
6.50	.0050	.0102	.0284	9.5	.0040	.0130	.0580
7.00	.0031	.0062	.0180	10.0	.0025	.0085	.0416
7.50	.0019	.0039	.0113	10.5	.0016	.0054	.0293
8.00	.0012	.0024	.0071	11.0	.0010	.0035	.0203
8.50	.0007	.0014	.0044	11.5	.0006	.0022	.0138
9.00	0.0004	0.0008	0.0027	12.0	0.0004	0.0014	0.0093

3. *The solution for the case $\psi(q) = (n+1)q^n$.*—Considering equation (1) quite generally and applying to it a Laplace transformation, we obtain

$$(s+1)G(s) = \int_0^1 dq\psi(q) \int_0^\infty \frac{du}{q}\, g\left(\frac{u}{q}\right) e^{-su}, \tag{22}$$

where

$$G(s) = \int_0^\infty g(u)\, e^{-su} du \tag{23}$$

denotes the Laplace transform of $g(u)$. The quantity on the right-hand side of equation (22) can also be expressed in terms of $G(s)$; for

$$\int_0^\infty \frac{du}{q}\, g\left(\frac{u}{q}\right) e^{-su} = \int_0^\infty dx\, g(x)\, e^{-sqx} = G(sq). \tag{24}$$

Equation (22) can therefore be written in the form

$$(s+1)G(s)=\int_0^1 dq\psi(q)G(sq). \tag{25}$$ [3]

Solutions of this equation must be sought which satisfy the boundary condition,

$$G(0)=\int_0^\infty g(u)\,du=1. \tag{26}$$

For the case

$$\psi(q)=(n+1)\,q^n, \tag{27}$$

equation (25) becomes

$$(s+1)G(s)=(n+1)\int_0^1 G(sq)\,q^n dq. \tag{28}$$

Letting $x=sq$ as the variable of integration on the right-hand side, we have

$$(s+1)G(s)=\frac{n+1}{s^{n+1}}\int_0^s G(x)\,x^n dx. \tag{29}$$

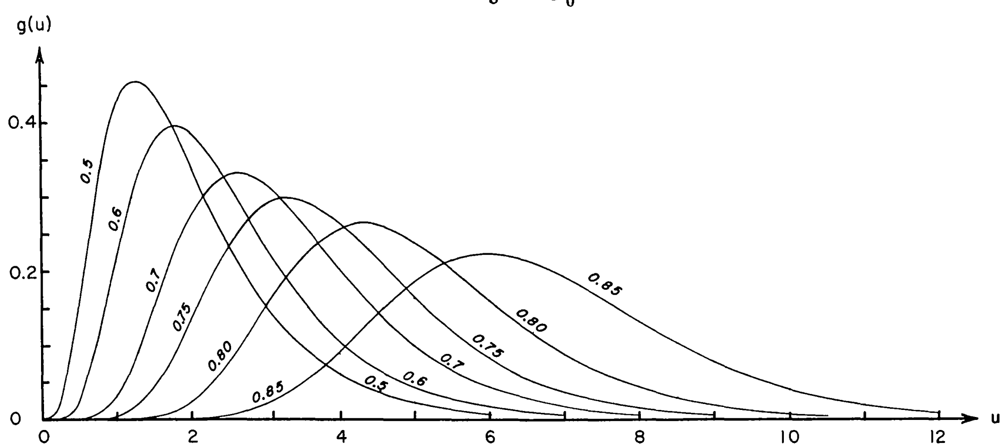

FIG. 1.—The frequency functions $g(u)$ for the case in which all the clouds are equally transparent. The different curves are labeled by the values of q to which they refer.

From this equation we derive

$$\frac{d}{ds}[s^{n+1}(s+1)G(s)]=(n+1)\,s^n G(s). \tag{30}$$

[3] For the case in which q takes only one discrete value, eq. (25) reduces to

$$(s+1)G(s)=G(sq).$$

The solution of this equation satisfying the boundary condition (26) is clearly

$$G(s)=\frac{1}{\prod_{n=0}^{\infty}(1+sq^n)}.$$

The inverse Laplace transform of $G(s)$ given by the foregoing equation can be found in accordance with the standard formula,

$$g(u)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{isu}\prod_{n=0}^{\infty}(1+sq^n)^{-1}.$$

The complex integral on the right-hand side can be evaluated by the method of residues and leads to the same solution as the one we derived by more elementary methods in § 2.

On further reduction, equation (30) becomes

$$(n+2)G(s)+(s+1)\frac{dG}{ds}=0. \quad (31)$$

The solution of this equation satisfying the boundary condition (26) is

$$G(s)=\frac{1}{(s+1)^{n+2}}. \quad (32)$$

The inverse Laplace transform of this elementary function is known, and we have

$$g(u)=\frac{1}{\Gamma(n+2)}e^{-u}u^{n+1}. \quad (33)$$

This function is properly normalized, and it can be verified that the moments of equation (33) are in agreement with those predicted by equation (2).

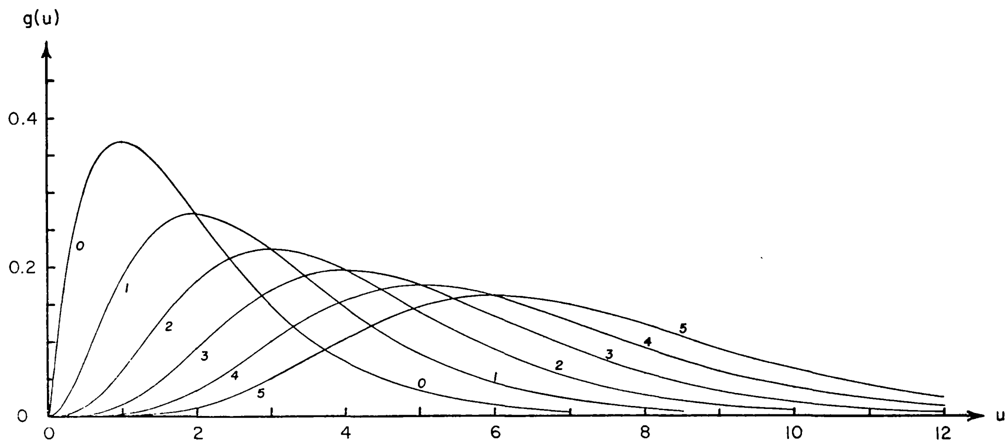

FIG. 2.—The frequency functions $g(u)$ for the case in which the occurrence of clouds with the transparency factor q is governed by $(n+1)\,q^n$. The different curves are labeled by the values of n to which they refer.

The family of frequency functions represented by equation (33) is illustrated in Figure 2. We notice the general similarity of the functions illustrated in Figures 1 and 2. However, in comparing the two sets of curves, it should be noted that, for the frequency function (27),

$$q_1=\bar{q}=\frac{n+1}{n+2} \quad \text{and} \quad q_2=\overline{q^2}=\frac{n+1}{n+3}. \quad (34)$$

Accordingly, the functions for the larger values of n should be compared with those for q approaching unity when q occurs with only one given value. And, when we make such a comparison, we observe that there are certain differences between the two sets of curves, which we may trace to the occurrence of clouds with varying absorptive power. Thus, comparing curves which correspond to the same mean value of q (for example, the curve for $q = 0.85$ in Fig. 1 and the curve for $n = 5$, $q_1 = 0.857$, in Fig. 2), we may conclude that the effect of a dispersion in q is to make the frequency function broader with excess both for large and for small values of u.