TURBULENCE—A PHYSICAL THEORY OF ASTROPHYSICAL INTEREST*

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May I say at the outset that I have found myself deeply inadequate for the task of giving this third Henry Norris Russell Lecture. I am afraid that I have not discovered or paved a Royal Road that I can describe to you in the manner of Dr. Russell; neither have I the excellence of the material which Dr. Adams presented in his second Henry Norris Russell Lecture. And I am aware that no general interest attaches to matters in which I may claim some degree of competence. I have therefore chosen, after considerable hesitation, to describe to you the recent advances in our understanding of the phenomenon of turbulence, in the belief that these advances are relevant to the progress of astrophysics. Perhaps it is premature to take an occasion like this to describe a physical theory which has yet to establish its relations to astronomical developments. But the history of astronomy and astrophysics shows that major advances in our understanding of astrophysical phenomena have coincided with and depended upon advances in fundamental physical theory. While many examples illustrating this can be given, there is none more conspicuous or notable in recent history than that provided by the work of Henry Norris Russell; thus, during the great period in which the foundations both of atomic spectra and of stellar spectroscopy were laid, Russell was a great exponent of both subjects. As is well known, the main features of the theory of complex spectra emerged for the first time from the pioneering investigations of Drs. Russell and Saunders on the alkaline earths. The main conclusion of these investigations, stated by the authors in the words "both valency electrons may jump at once from outer to inner orbits, while the net energy lost is radiated as a single quantum," has since been incorporated into the analysis of stellar spectra as the "Russell-Saunders" coupling and is one of the keystones of atomic theory. In these early papers of Dr. Russell all the steps preliminary to the formulation of the exclusion principle were taken, and I do not believe that it is a misstatement of history to say that the honor of the discovery of the exclusion principle would have gone to Russell had his concern with the applications of the principles of atomic spectra to astrophyiscal problems been a little less. However that may be, as astronomers we may count ourselves fortunate that Russell's concern with astrophysical problems was as earnest then as it has always been, for otherwise, we should not have had so immediate or so complete an integration of physical and astrophysical theories as was, in fact, achieved when Russell's great work on the quantitative analysis of the solar spectrum and the first determination of the composition of the sun appeared in 1929. I have referred to this example of Russell's work to emphasize the interdependence of physical and astrophysical thought. And, as I have stated, it seems to me probable that the recent advances in the physics of turbulence, due in large measure to G. I. Taylor, von Karman, Kolmogoroff, and Heisenberg, may play an important part in the future developments of astrophysics. But, before I describe the nature of these advances in physical theory, I may perhaps indicate briefly the astrophysical contexts in which they may find their most fruitful applications.

The first person clearly to draw attention to the importance for astrophysics of tur-

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bulence with its correct hydrodynamical meaning was Rosseland. In a paper published in 1928, Rosseland[1] pointed out that if differential motions—i.e., motions of one part relative to another—occur in cosmical gas masses, then the motions should be turbulent in the sense that we should not expect to describe them in terms of the classical equations of motion of Stokes and Navier. In drawing this inference, Rosseland was guided by the experience in meteorology and oceanography and by the following reasoning.

We are all familiar with the fact that a linear flow of water in a tube can be obtained only for velocities below a certain critical limit and that, when the velocity exceeds this limit, laminar flow ceases and a complex, irregular, and fluctuating motion sets in. More generally than in this context of flow through a tube, it is known that motions governed by the equations of Stokes and Navier change into turbulent motion when a certain nondimensional constant called the "Reynolds number" exceeds a certain value of the order of 1000. This Reynolds number depends upon the linear dimension, L, of the system, the coefficient of viscosity μ, the density ρ, and the velocity v in the following manner:

$$R = \frac{\rho vL}{\mu}. \tag{1}$$

Since R depends directly on the linear dimension of the system, Rosseland argued that motions in the oceans, in terrestrial and planetary atmospheres, and still more in stellar atmospheres, once they occur, must become turbulent in this sense. Rosseland further pointed out that, if turbulence develops, the coefficients of viscosity and heat conduction may be expected to increase a million fold. And the importance of this enhanced efficiency of heat and momentum transport in a turbulent medium cannot be exaggerated.

Stimulated by Rosseland's ideas, McCrea[2] suggested in the same year that the solar chromosphere must be in a state of turbulence and that this turbulence may, in part, contribute to its support against gravity.

About a year later Harold Jeffreys[3] drew attention to a fact which had been ignored until then, namely, that if the generation of energy inside stars is confined to a small region at the center, then the radiation will not be able to dispose of it at a gradient under the adiabatic and that, if a superadiabatic gradient comes into being, vertical currents will be generated which will effectively restore the adiabatic gradient, leaving, however, a slight superadiabatic gradient to make possible the transport of heat. The condition for the occurrence of such convective transport of heat can be written down, and it follows from this condition that even a relatively mild concentration of the energy sources toward the center will lead to its occurrence near the center. Indeed, with the clarification of the source of stellar energy as due to nuclear transformations, it is now generally recognized that all stars must have convective cores in which turbulence prevails. And, as was shown, particularly by Cowling,[4] the existence of turbulence is of primary importance in all considerations relating to the stability of stars.

Returning to the role of turbulence in the atmospheres of the stars, we next observe that the investigations of Struve and Elvey[5] established the occurrence of large-scale motions in the atmospheres of stars like 17 Leporis, ϵ Aurigae, and α Persei. In investigations, which, it may be noted, were also the first to apply the then new method of the curve of growth to the analysis of stellar atmospheres—the method had already been applied to the solar atmosphere by Minnaert—Struve and Elvey showed that the linear portion of the curve of growth, as well as the line profiles themselves, cannot be explained in terms of the Doppler effect due to thermal motions alone and that large-scale

[1] *M.N.*, **89**, 49, 1929.

[2] *M.N.*, **89**, 718, 1929.

[3] *Nature*, **127**, 162, 1931; also *M.N.*, **91**, 121, 1931.

[4] *M.N.*, **94**, 768, 1934; **96**, 42, 1935.

[5] *Ap. J.*, **79**, 409, 1934; see also O. Struve, *Proc. Nat. Acad. Sci*, **18**, 585, 1932.

motions of a turbulent nature must be postulated. This conclusion has since been confirmed and extended by various other investigators.

That turbulence must play a part also in the solar atmosphere became clear after Unsöld[6] had shown that in the deeper layers of the solar photosphere, where hydrogen begins to get ionized, the radiative gradient must become unstable. Since that time the view first advanced by Siedentopf[7] and Biermann,[8] that the solar granulation must, in some way, be related to this hydrogen convection zone, has been steadily gaining ground.

Again the investigations of Struve and his associates during the past few years have shown that the shells surrounding early-type stars and the gaseous envelopes in which spectroscopic binaries are frequently imbedded must also be turbulent, the turbulence in these contexts arising, in the first instance, from the different parts of the shell or medium rotating with different angular velocities.

And, finally, it would appear that the interstellar clouds must also be in a state of turbulence; for, assuming that a typical cloud is 10 parsecs in diameter and that relative motions to the extent of 10 km/sec occur, we find that the Reynolds number must be of the order of 10^5; and the motions inside the cloud must therefore be turbulent. The even larger question now occurs whether we may not indeed regard the clouds of various dimensions in interstellar space as eddies in a medium occupying the whole of galactic space.

From this brief survey of the various problems in which turbulence may play a role, it would almost appear that, if we are in the mood for it, we may encounter turbulence no matter where we turn. But what is the picture of turbulence in terms of which we wish to interpret such a wide diversity of phenomena? It is that in a turbulent medium there are eddies which spontaneously form and disintegrate; that this process goes on continuously; that each eddy travels a certain average distance with a certain average speed before it loses its identity—a specific enough picture but not one derived from, or justified by, a physical theory. Thus, while the basic concepts of "mean free path" and "root-mean-square velocity" which underlie the picture are plausible enough, it was not known how these quantities were to be related with the physical conditions of the problem. Indeed, from the point of view of a rational physical theory, the situation has been so unsatisfactory that, in a recent conversation, Dr. Russell recalled that E. W. Brown, referring to the frequency with which appeals were being made to the action of a resisting medium to account for this or that anomaly in the motions of celestial bodies, once remarked: "What fifty years ago used to be attributed to the direct intervention of the Deity are now being attributed to a resisting medium." Dr. Russell added that he sometimes felt the same way about the frequency with which turbulence is currently being invoked to account for astrophysical phenomena. Nevertheless, it would seem that the application of the newer developments in the theory of turbulence may help to remove this element of the miraculous in astrophysics.

As I indicated at the outset, the study of turbulence in hydrodynamics started with investigations on the stability of laminar flow. In general, these investigations began with simple patterns of flow, like axial flow through a tube or plane-parallel flow over an infinite plate, and examined the stability of these flows to perturbations of particular types with a view to determining the critical value of the Reynolds number at which laminar flow becomes unstable. The mathematical analysis required for the investigation of stability along these lines is of a very treacherous kind, and, in spite of the enormous effort which has been expended on this problem by various authors, including Heisenberg, Tollmien, Lin, and Pekeris, no positive or general conclusions seem to have been reached. However, as Heisenberg has recently emphasized,[9] investigations of sta-

[6] *Zs. f. Ap.*, **1**, 138, and **2**, 209, 1931.

[7] *A.N.*, **247**, 297, 1933; **249**, 53, 1933; **255**, 157, 1935.

[8] *Zs. f. Ap.*, **22**, 244, 1943.

[9] *Zs. f. Naturforsch.*, **3**, 434, 1948.

bility along these lines, even if successful, cannot, in principle, lead to an understanding of the phenomenon of turbulence itself; for the basic problem of turbulence is of an entirely different character. That this is the case becomes apparent when we ask ourselves the very elementary question, "What is the reason that a phenomenon like turbulence can occur at all?" The answer must be that an ideal fluid is a mechanical system with a very large number of degrees of freedom and that, in consequence, it is theoretically capable of a very large number of different types of motions. Laminar motion is only one of the many possible motions that the system is capable of, and to expect that it will always be realized is as futile as to expect that in a gas we shall find all the molecules moving with the same velocity parallel to one another. It is far more likely that all the possible motions will be simultaneously present. The fundamental problem of turbulence would therefore appear to be a statistical one of specifying the probability with which the various types of motion may occur and are present. Stated in this way, it is clear that the problem of turbulence has an analogy with the problem of analyzing a continuous spectrum of radiation. In the latter case, the greatest interest is generally attached to the distribution of intensity in the spectrum and only secondarily to the phase relationships. Similarly, when we consider the motions in a turbulent fluid, we may make a harmonic analysis of the instantaneous velocity field $\boldsymbol{v}(\boldsymbol{r}, t)$ in the form

$$\boldsymbol{v}(\boldsymbol{r}, t) = \sum_{\boldsymbol{k}} \boldsymbol{v}_{\boldsymbol{k}}(t)\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \tag{2}$$

and ask for the average energy stored in the various wave lengths. We can visualize this formal procedure in the following manner.

Considering the state of motion at a given instant, we may analyze the fluctuating velocity field as the result of superposition of periodic variations with all possible wave lengths. We may picture the component with a wave length λ as corresponding to an eddy of size λ, and, since many wave lengths are needed to represent a general velocity field, we may speak of a "hierarchy of eddies." This hierarchy of eddies will be limited on the side of long wave lengths by the fact that no eddy of size larger than the dimension of the medium in which we analyze the turbulence can occur.

Instead of the wave length λ, it is often more convenient to speak of a wave number $k = 2\pi/\lambda$.

Analyzing the motion into eddies in this manner, we can ask: What is the energy per unit volume stored in eddies with wave numbers between k and $k + dk$? If $\rho F(k)dk$ denotes this energy, $F(k)$ is said to define the *spectrum* of turbulence.

It can be shown that most of the interesting features of turbulent motion can be deduced from its spectrum. For example, the correlations $\overline{u_i u_j'}$ between the instantaneous velocity components u_i and u_j' at two different points of the medium can be expressed simply in terms of the spectrum. Such correlations were first introduced by G. I. Taylor[10] as a basis for a phenomenological theory of turbulence, and they have since been studied extensively, both theoretically and experimentally. It is therefore natural that the question of the spectrum of turbulence should be in the forefront in all recent discussions of turbulence, since most of the available experimental data on the subject are capable of being interpreted in terms of the spectrum.

Now, returning to the optical analogy I referred to earlier, we know that under conditions of equilibrium the distribution of energy in the continuous spectrum will be that given by Planck's law. We may ask whether a similar equilibrium spectrum exists for turbulence. In answering this question, we must keep in mind one important distinction between the optical analogue and turbulence. In the optical case the equilibrium Planck

[10] *Proc. R. Soc. London, A*, **164**, 476, 1938. For a general account of these investigations see H. L. Dryden, *Quar. Appl. Math.*, 1, 7, 1943.

spectrum will be reached, no matter what the initial distribution is. In contrast, turbulence can be maintained only by an external agency, like continuous stirring, the energy available from thermal instability, or rotation in a differentially rotating atmosphere. In other words, energy is required for the maintenance of turbulence; in the absence of such an agency, turbulence will *decay*, and the spectrum will be a function of time. In discussing the spectrum of turbulence, we must therefore distinguish between two cases: the case in which the agency maintaining turbulence is communicating energy to the medium at a constant rate and a stationary condition prevails and the case in which there is no external agency maintaining turbulence and the turbulence, in consequence, is decaying.

In the stationary case it is clear that energy must be dissipated in the form of thermal energy at the same rate at which energy is being supplied.

According to the laws of hydrodynamics, the rate of dissipation of energy by viscosity is given by

$$\epsilon = \rho\nu \overline{|\operatorname{curl} \boldsymbol{v}|^{2}} \tag{3}$$

per unit volume. In equation (3) ν is the *kinematic viscosity* and is μ/ρ. In terms of the spectrum this expression for ϵ becomes

$$\epsilon = 2\rho\nu \int_0^\infty F(k)\, k^2 dk\,. \tag{4}$$

Under stationary conditions this must be the rate at which energy is being communicated to the medium by the external agency.

Considering, now, the problem of determining the spectrum of turbulence under stationary conditions, we may first remark that the presence of an external agency maintaining turbulence requires us to distinguish between the region of the spectrum in which the eddy sizes are comparable to the linear dimension —l_0, say—of the system and the region of the spectrum in which the eddy sizes are small compared to the linear dimension of the system. In the first region the nature of the spectrum must clearly depend on the external agency. We should not, therefore, expect to give a theory of the spectrum in this region which will be universally valid. Each situation will have to be analyzed separately. On the other hand, it does not seem unreasonable to suppose that the distribution of energy among the eddies which are small compared to the dimension of the system will be largely independent of the particular mechanism maintaining the turbulence and will depend only on the rate ϵ at which energy is being supplied. In terms of wave numbers we may express this in the following way.

Let $k_0 \sim 1/l_0$ denote the wave number of the largest eddies present. Then for $k \gg k_0$ we may expect the spectrum to approach a universal one, depending only on ϵ and ν. We may further expect that, as the Reynolds number tends to infinity, more and more of the spectrum will follow a universal law. When this is the case, we say that we have the equilibrium spectrum for a fully developed turbulence. In astronomical contexts turbulence, when it occurs, may be expected to be fully developed in this sense.

Turning, now, to the specification of the spectrum, we may suppose that the energy supplied by the external agency is communicated principally to the largest eddies, i.e., for $k \sim k_0$. Let ϵ denote this rate. As we have seen, energy is being dissipated by viscosity at this same rate; it is evident that this dissipation into thermal energy will be effected principally by the smallest eddies, in which the motions may be expected to be laminar. Consequently, energy at this same rate must flow through the entire hierarchy of eddies, and the equilibrium spectrum will be determined by this condition of constant flow of energy through the hierarchy. To translate this condition of constant flow of energy through the hierarchy into a quantitative expression, we consider the rate ϵ_k at which energy flows from eddies of all wave numbers less than a particular k to eddies of all wave numbers greater than this k. In a general way it is clear that we must distinguish between two different types of contributions to ϵ_k: First, there is the dissipation directly into thermal energy:

$$\epsilon_k \,(\text{thermal}) = 2\,\rho\nu \int_0^k F(k)\, k^2 dk\,. \tag{5}$$

Then there is the energy communicated to the eddies of smaller sizes in the form of kinetic energy of motion. We shall give an expression for this later, but we may note meantime that, for any given k, the relative importance of the two contributions will depend on k and the Reynolds number of the entire motion. If k_s denotes the wave number of the eddies in which the motion begins to be laminar, then we should expect that, for $k \gg k_s$, the transfer of energy into the kinetic energy of motion will be negligible. On the other hand, if the Reynolds number is sufficiently large, then in a significant portion of the spectrum the inequality $k_0 \ll k \ll k_s$ will be valid; this inequality means that there exists a range of sizes which is small compared to the largest eddies present but large compared to the eddies in which the dissipation by kinetic viscosity occurs; and this will certainly happen if we let the Reynolds number tend to infinity. Now let the Reynolds number be sufficiently large for the inequality $k_0 \ll k \ll k_s$ to be valid over a portion of the spectrum. In this portion of the spectrum, in contrast to the portion of the spectrum where $k \gg k_s$, the thermal contribution to ϵ_k must be negligible; when this is the case, the spectrum may be expected to become independent of the viscosity as well and depend only on ϵ. These ideas, which underlie the recent developments in the theory of turbulence, were first clearly recognized by L. F. Richardson, to whom the following rhyme is attributed:

Big whirls have little whirls,
That feed on their velocity;
And little whirls have lesser whirls,
And so on to viscosity.

However, mathematical expression was first given to these ideas by Kolmogoroff,[11] in the form of two principles. In our present context we may state the principles of Kolmogoroff in the following form:

1. The spectrum of turbulence for all k much greater than a certain k_0 must be determined uniquely by $e = \epsilon/\rho$ and the kinematic viscosity $\nu = \mu/\rho$.

2. For infinite Reynolds number the spectrum must, in addition, become independent of ν and depend only on e.

We shall see that in these forms the principles are valid also for the problem of decay.

We shall now show how the two principles of Kolmogoroff enable us to determine the form of the spectrum in the region $k_0 \ll k \ll k_s$.

Now $F(k)$ is of dimension (velocity)^2 $\times$ length, while k itself is of dimension (length)$^{-1}$. Quantities of these dimensions which can be constructed out of e and ν are

$$(\text{Velocity})^2 \times \text{length} = [(\nu^5 e)^{1/4}]\,, \qquad \text{Length} = \left[\left(\frac{\nu^3}{e}\right)^{1/4}\right]. \tag{6}$$

Consequently, Kolmogoroff's first principle requires that

$$F(k) \equiv (\nu^5 e)^{1/4} f(k\nu^{3/4} e^{-1/4})\,, \tag{7}$$

where f is a universal function of the argument specified. According to the second principle, $F(k)$ should be independent of ν in the region $k_0 \ll k \ll k_s$. Accordingly, in this region $f(x)$ must be of the form

$$f(x) \doteq C x^{-5/3}\,, \tag{8}$$

[11] 1941, *C.R. Acad. Sci. U.S.S.R.*, **30**, 301, and **32**, 16, 1942; see also G. K. Batchelor, *Proc. Cambridge Phil. Soc.*, **43**, 533, 1947.

where C is a constant; for then

$$F(k) \doteq C(\nu^5 e)^{1/4}\left(\frac{\nu^3}{e}\right)^{-5/12} k^{-5/3} = C e^{2/3} k^{-5/3}; \tag{9}$$

and the requirement that $F(k)$ be independent of ν is satisfied. Hence, when the Reynolds number tends to infinity, the spectrum will follow more and more closely a $k^{-5/3}$-law—this is the *Kolmogoroff spectrum*. I should perhaps mention at this stage that the $k^{-5/3}$-law was discovered independently, also by Onsager[12] and von Weizsäcker,[13] but several years later.

It is sometimes convenient to think of all eddies with wave numbers exceeding a certain k (i.e., with wave lengths less than a certain λ) as having a certain mean velocity, v_k. For this purpose we may adopt as definition the equation

$$v_k^2 = \int_k^\infty F(k)\,dk. \tag{10}$$

When the equilibrium Kolmogoroff spectrum prevails, this equation gives the law

$$v_k \propto k^{-1/3} \propto \lambda^{1/3}. \tag{11}$$

While Kolmogoroff's method of determining the form of the equilibrium spectrum of fully developed turbulence is very elegant, it does not, one must admit, give any real insight into the physical nature of turbulence. Also, even under equilibrium conditions, it does not give the part of the spectrum in which the dissipation by viscosity begins to be an important factor. An elementary theory which visualizes clearly the phenomenon of turbulence and which gives, at the same time, the complete equilibrium spectrum is due to Heisenberg.[14] The ideas underlying Heisenberg's development can be explained very simply.

Considering the rate at which eddies with wave numbers between 0 and k transfer energy to eddies with wave numbers exceeding k, Heisenberg writes

$$\epsilon_k\,(\text{mechanical}) = 2\rho\nu_k \int_0^k F(k)\,k^2 dk, \tag{12}$$

in analogy with expression (4) for the thermal part of the energy transfer. In writing ϵ_k (mechanical) in the form (12), we are assuming that the process of energy transfer between the sets of eddies $(0, k)$ and (k, ∞) can be visualized in terms of a suitably defined coefficient of viscosity, ν_k. We are, of course, familiar with the concept of eddy viscosity derived from the picture of eddies describing a certain mean free path l_k with a certain root-mean-square velocity, v_k. On this picture

$$\nu_k \sim l_k v_k \sim \frac{v_k}{k}, \tag{13}$$

since we may expect l_k to be of the order of $1/k$. However, for our purposes this is not a suitable expression for ν_k; to be useful, it must be expressed in terms of the spectrum. As the simplest of possible expressions, Heisenberg assumes that

$$\nu_k = \kappa \int_k^\infty \sqrt{\frac{F(k)}{k^3}}\,dk, \tag{14}$$

[12] *Phys. Rev.*, **68**, 286, 1945.

[13] *Zs. f. Phys.*, **124**, 614, 1948.

[14] *Zs. f. Phys.*, **124**, 628, 1948, and *Proc. R. Soc. London, A*, **195**, 402, 1948.

where κ is a certain numerical constant. Apart from the fact that the expression on the right-hand side is of the correct dimension, the justification for writing it in this particular form is the following.

If $F(k)$ follows a simple power law of the form k^{-n}, then

$$v_k = \left[\int_k^\infty F(k)\, dk \right]^{1/2} \propto k^{-(n-1)/2},$$

and

$$\nu_k \propto \frac{v_k}{k} \propto k^{-(n+1)/2} \propto \int_k^\infty \frac{dk}{k^{(n+3)/2}} \propto \int_k^\infty \sqrt{\frac{F(k)}{k^3}}\, dk\,. \tag{15}$$

In other words, formula (14) is a valid form when $F(k)$ follows a power law; we assume that we may use the same expression even when this is not the case.

According to equations (12) and (14), we have

$$\epsilon_k\,(\text{mechanical}) = 2\rho \int_k^\infty \sqrt{\frac{F(k'')}{k''^3}}\, dk'' \int_0^k F(k')\, k'^2 dk'\,. \tag{16}$$

This expression for ϵ_k admits of a simple interpretation. In a unit volume of the medium, eddies with wave numbers between k' and $k' + dk'$ transfer energy to eddies with wave numbers between k'' and $k'' + dk''$ ($k'' \geq k'$) at the rate

$$\epsilon(k';\, k'')\, dk' dk'' = 2\rho F(k')\, k'^2 \sqrt{\frac{F(k'')}{k''^3}}\, dk' dk''\,. \tag{17}$$

We may think of $\epsilon(k';\, k'')$ as a transition probability governing the process of energy transfer between eddies. The possibility of defining such a transition probability is, of course, implicit in our concept of the existence of a hierarchy of eddies.

Now, combining the expressions for ϵ_k(thermal) and ϵ_k(mechanical), we have

$$\epsilon_k = 2\rho \left\{ \nu + \kappa \int_k^\infty \sqrt{\frac{F(k'')}{k''^3}}\, dk'' \right\} \int_0^k F(k')\, k'^2 dk'\,. \tag{18}$$

This is the fundamental equation of Heisenberg's theory. It combines in a single expression the ideas underlying the picture of turbulence in terms of eddies describing mean free paths and the principle expressed by Richardson's rhyme.

For stationary turbulence, ϵ_k must be a constant, independent of k, and this condition suffices to determine the spectrum. Indeed, the exact solution of equation (18) for the case ϵ_k = Constant can be given explicitly.[15] We have

$$\begin{aligned} F(k) &= \text{Constant} \left(\frac{k_0}{k} \right)^{5/3} \frac{1}{[1 + (k/k_s)^4]^{4/3}} \qquad && (k > k_0) \\ &= 0 && (k < k_0)\,, \end{aligned} \tag{19}$$

where

$$k_s \doteq 0.2211\, k_0 (R_0 \kappa)^{3/4}\,, \tag{20}$$

and R_0 denotes the Reynolds number

$$\begin{aligned} R_0 = \frac{1}{\nu}\, &(\text{Root-mean-square velocity of all eddies present}) \\ &\times \text{diameter } (= \pi/k_0) \text{ of the largest eddies}\,. \end{aligned} \tag{21}$$

[15] S. Chandrasekhar, *Phys. Rev.*, **75**, 896, 1949.

We observe that, according to these equations, when $R_0 \to \infty$, $k_s \to \infty$, and we recover the Kolmogoroff spectrum. More generally, when the Reynolds number is finite, but sufficiently large, there will be a region of the spectrum in which the inequality $k_0 < k \ll k_s$ will be valid; and in this region the spectrum will follow closely the $k^{-5/3}$-law. However, the solution of Heisenberg's equation also shows that, for any finite Reynolds number, no matter how large, we must get departures from the $k^{-5/3}$-law when k approaches k_s and that for $k \gg k_s$ the spectrum follows an inverse seventh-power law:

$$F(k) \propto k^{-7} \qquad (k \gg k_s) . \tag{22}$$

Evidently this is the region of the spectrum where the dissipation by viscosity into thermal energy is the dominant factor. Accordingly, we may take k_s as defining the wave number of the eddies at which the dissipation by kinetic viscosity becomes comparable to the kinetic energy transferred to smaller eddies by eddy viscosity.

In astronomical contexts we shall probably be mostly concerned with stationary turbulence. But it is an important aspect of Heisenberg's theory that it also enables the treatment of the problem of the decay of turbulence. I shall therefore spend a few moments on this aspect of the subject.

Now, if there is no external agency maintaining turbulence, then clearly

$$\epsilon_k = -\rho \frac{\partial}{\partial t} \int_0^k F(k, t)\, dk , \tag{23}$$

since, by definition, ϵ_k is the net energy dissipated by the eddies with wave number between 0 and k, either in the form of molecular motion and thermal energy or in the form of the motions of the smaller eddies and kinetic energy. The decay of turbulence will therefore be described by the equation

$$-\frac{\partial}{\partial t} \int_0^k F(k, t)\, dk = 2 \left\{ \nu + \kappa \int_k^\infty \sqrt{\frac{F(k'', t)}{k''^3}}\, dk'' \right\} \times \int_0^k F(k', t)\, k'^2 dk' . \tag{24}$$

A case of some importance in this connection is the following: Suppose that we have initially an equilibrium spectrum and that, at a certain instant, the agency maintaining the turbulence is cut off. Then, in the decay of turbulence which will ensue, we may distinguish three stages—an early stage, during which the larger eddies ($k \sim k_0$) adjust themselves to the fact that no energy is being communicated to them; an intermediate stage, during which there is a sufficient store of energy among the larger eddies to maintain an equilibrium distribution among the lower members of the hierarchy and the Reynolds number remains constant; and, finally, a last stage, during which the store of energy among the larger eddies is getting exhausted and the Reynolds number decreases to zero.

While a unified discussion of all three stages of decay is a difficult problem, it appears that on the basis of Heisenberg's equation we can follow the second stage quite completely and in an explicit fashion; for, from the constancy of the Reynolds number which we expect during this stage, we conclude that the spectrum must be "self-preserving" in the sense that it keeps the same form, though the scale may change with time. From equation (24) the condition of self-preservation is seen to be equivalent to seeking solutions of this equation of the form

$$F(k, t) \equiv \frac{1}{\sqrt{t}} f(k\sqrt{t}) , \tag{25}$$

where f is a function of the argument $k\sqrt{t}$. The physical meaning of a solution of this form is that during the decay the eddies grow in size like $\sqrt{t}$ and that the total energy stored in turbulence decays like $1/t$:

$$\epsilon = \int_0^\infty F(k, t)\, dk = \frac{1}{t} \int_0^\infty f(x)\, dx . \tag{26}$$

However, the *form* of the spectrum remains unchanged.

With $F(k, t)$ given by equation (25), the equation determining $f(x)$ can be reduced to a second-order nonlinear differential equation which can be studied by standard methods.[16] And the discussion shows that, for any finite Reynolds number,

$$F(k, t) \propto k \qquad (k \to 0),$$

and

$$F(k, t) \propto k^{-7} \qquad (k \to \infty). \tag{27}$$

Moreover, when the Reynolds number is sufficiently large, there is a part of the spectrum which approximately follows the $k^{-5/3}$-law. And when the Reynolds number actually becomes infinite (or equivalently $\nu = 0$), the spectrum approaches the $k^{-5/3}$-law exactly as $k \to \infty$.

As I stated earlier, the decay spectrum predicted by these curves is valid only during the second stage, when there is a sufficient store of energy among the larger eddies to maintain an equilibrium distribution for $k \to \infty$. When this ceases to be the case, the decay will proceed much more rapidly and, as Batchelor and Townsend[17] have shown, the $1/t$-law is then replaced by $1/t^{5/2}$-law, and the Reynolds number, instead of remaining constant, starts decreasing to zero.

I think that that about describes the present state of the theory of turbulence. Having spent so much time on the physical theory, I should like to conclude by a brief reference to an application which von Weizsäcker has made of these ideas on turbulence.[18]

As is probably generally known, von Weizsäcker has outlined a general cosmogony, the essential feature of which is the prominent role which he ascribes to the interplay between turbulence and rotation.

It is the usual fate of cosmogonical theories not to survive. I do not suppose that von Weizsäcker's theory will prove the exception to this rule. However, I have been personally attracted by his writings for two reasons, first, because he expresses himself with a restraint and a modesty which is unusual among writers in this field and, second, because I think that we may accede to the importance he ascribes to turbulence without, at the same time, subscribing to his detailed picture of the manner in which he expects turbulence to operate. From one point of view he may be said to have scored already; for it was his emphasis on the role of turbulence in cosmogony that led Heisenberg to examine the basic physical theory, with the result that we have today the beginnings of a foundation on which we may build.

As I have said, von Weizsäcker's cosmogony rests on the effects which he expects from the interplay of rotation and turbulence. More particularly, the effects which he expects can be described in the following terms.

Consider, for example, a sheet of gas at very low density in the equatorial plane of a central mass, which we may identify with a star or with the nucleus of a galaxy. If we ignore, in the first instance, the effects of pressure and viscosity, each element of gas will describe a Keplerian orbit in the field of the central mass. If the system is assumed to have an axial symmetry, the orbits must be circular, and the angular velocity will vary with distance, s, from the axis, according to the law

$$\omega^2 = \frac{GM}{s^3}. \tag{28}$$

The successive rings of gas in the medium will therefore have motions relative to one another, and turbulence will ensue. As a result of this turbulence, viscous stresses will come into play and will perturb the motions, both in the radial and in the transverse direc-

[16] *Ibid.*, **76**, 1454, 1949, and *Proc. R. Soc. London, A* (in press).

[17] *Proc. R. Soc. London, A*, **193**, 539; **194**, 527, 1948.

[18] *Zs. f. Ap.*, **22**, 319, 1944; **24**, 181, 1947, and *Zs. f. Naturforsch.*, **3A**, 524, 1948.

tions. Examining the sense of these perturbations, von Weizsäcker concludes that all matter interior to a certain critical radius will fall toward the center, while the matter outside this radius will tend to move outward and dissipate into space. He invokes this mechanism in a variety of different contexts: for the dissipation of the gaseous envelope which, he imagines, once surrounded the sun and in which he presumes the solar system was formed; for interpreting the ring structure in extragalactic nebula. And, going further back to the beginning of things, he believes the linear dimensions of the present galaxies must have been determined by the condition that at these distances the forces which led to the expansion of the universe and the forces which result from turbulence compete on about equal terms; he infers the existence of such a distance from the fact that, while the mean velocities increase linearly with the distance in an expanding universe, the mean velocities increase only as the one-third power of the distance in a turbulent medium. I should emphasize again that it is not necessary to subscribe to all these speculations of von Weizsäcker to grant the importance of turbulence for the purpose of cosmogony. It is, indeed, entirely possible that the theory of turbulence which I have described may bear its first fruits in a much less spectacular way in the solution of more specific problems. Thus Martin Schwarzschild has already extended Heisenberg's theory to include the agency maintaining turbulence for the case when turbulence results from thermal instability. It is clear that the extension of Heisenberg's theory to the case of turbulence induced by thermal instability must have important applications to the interpretation of the solar granules.

While we shall have to wait for these and similar developments before we can finally pass on the importance of the recent developments in our understanding of turbulence for astrophysics, I think we may be sure of at least one thing.

"We cannot make bricks without straw"; that is a common enough saying. It is equally true that we cannot construct a rational astrophysical theory without an adequate base of physical knowledge. It would therefore seem to me that we cannot expect to incorporate the concept of turbulence in astrophysical theories in any essential manner without a basic physical theory of the phenomenon of turbulence itself. It appears that the first outlines of such a physical theory are just emerging.