ON THE RADIATIVE EQUILIBRIUM OF A STELLAR ATMOSPHERE. XXIV

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ABSTRACT

In this paper the functional equations governing the X- and Y-functions (introduced in Paper XXII) are solved by an iteration process, starting with the "first approximation" $X^{(1)} = 1$ and $Y^{(1)} = e^{-\tau/\mu}$. The iteration of these functions leads to a "second approximation." The solutions in these approximations are then corrected by a semi-empirical method to satisfy, exactly, an integral property of the X- and Y-functions. It appears that, in this fashion, solutions for the X- and Y-functions, with an error probably not exceeding 1 per cent for $\tau < 0.5$, can be found. The construction of standard solutions in conservative cases is also considered. In an appendix (worked jointly with Mrs. Frances H. Breen) tables of the auxiliary functions, for use in conjunction with the exact solutions of Paper XXII, are provided.

1. *Introduction.*—In Paper XXII[1] we have shown how the various standard problems in the theory of radiative transfer in plane-parallel atmospheres of finite optical thicknesses can be reduced to the solution of one or more pairs of functional equations of the type

$$X(\mu) = 1 + \mu \int_0^1 \frac{\Psi(\mu')}{\mu + \mu'} [X(\mu) X(\mu') - Y(\mu) Y(\mu')] d\mu' \tag{1}$$

and

$$Y(\mu) = e^{-\tau/\mu} + \mu \int_0^1 \frac{\Psi(\mu')}{\mu - \mu'} [Y(\mu) X(\mu') - X(\mu) Y(\mu')] d\mu', \tag{2}$$

where the characteristic function $\Psi(\mu)$ is an even polynomial in μ, satisfying the condition

$$\int_0^1 \Psi(\mu) d\mu \leqslant \tfrac{1}{2}, \tag{3}$$

and τ is the optical thickness of the atmosphere.[2]

For general values of τ the solution of equations (1) and (2) can be found only by an iteration process of the kind which was adopted, for example, in the evaluation of the H-functions.[3] However, in certain important applications of the theory, interest is mainly attached to small values of the optical thickness (of the order of 0.1 or 0.2). It will therefore be convenient if simple analytical representations of the solutions of equations (1) and (2) can be found which will be accurate enough for these applications. The importance of having such analytical representations has been emphasized by van de Hulst,[4] who has recently reconsidered the problem of diffuse reflection by an isotropically scattering atmosphere with a view to such applications.

Now for an isotropically scattering atmosphere the characteristic function is a constant $(= \frac{1}{2}\varpi_0)$, and the solutions for the reflected and the transmitted intensities are expressed very simply in terms of the X- and Y-functions (cf. Paper XXII, Sec. II, §7,

[1] *Ap. J.*, **107**, 48, 188, 1948.

[2] In Papers XXI (*Ap. J.*, **106**, 152, 1947) and XXII the optical thickness of the atmosphere was denoted by τ_1. It is convenient to suppress the subscript 1 when, as in the present instance, there will be no occasion to refer to the radiation field in the interior.

[3] Cf. Papers XVI, XIX, and XXIII (*Ap. J.*, **105**, 435; **106**, 143, 1947; **107**, 216, 1948).

[4] *Ap. J.*, **107**, 220, 1948.

eqs. [178] and [179]). By analyzing the emergent radiations in terms of the light which has been scattered once, twice, etc., in the atmosphere and consistently arranging that, at each stage of the approximation, the solutions have the forms required by the exact theory as developed in Paper XXII, van de Hulst has shown that it is possible to write down relatively simple expressions which will adequately represent the solutions of equations (1) and (2) for the case

$$\Psi(\mu) = \text{Constant} = \tfrac{1}{2}\varpi_0 \tag{4}$$[5]

and valid for small values of τ. In the general case when no very simple meanings can be attached to the X- and Y-functions,[6] a more directly mathematical approach to the solution of equations (1) and (2) than was adopted by van de Hulst appears necessary. It is the object of this paper to develop such an approach and to present a method for solving equations (1) and (2) for suitably small values of τ.

2. *The first and the second approximation.*—Quite generally we may expect to solve equations (1) and (2) by an iteration process in which the nth iterates are obtained by evaluating the integrals on the right-hand sides of equations (1) and (2) in terms of the $(n-1)$th iterates. Thus

$$X^{(n+1)}(\mu) = 1 + \mu \int_0^1 \frac{\Psi(\mu')}{\mu+\mu'} [X^{(n)}(\mu)\, X^{(n)}(\mu') - Y^{(n)}(\mu)\, Y^{(n)}(\mu')]\, d\mu' \tag{5}$$

and

$$Y^{(n+1)}(\mu) = e^{-\tau/\mu} + \mu \int_0^1 \frac{\Psi(\mu')}{\mu-\mu'} [Y^{(n)}(\mu)\, X^{(n)}(\mu') - X^{(n)}(\mu)\, Y^{(n)}(\mu')]\, d\mu' . \tag{6}$$

The success of such an iteration scheme will largely depend on how well the first "trial" functions $X^{(1)}(\mu)$ and $Y^{(1)}(\mu)$ are chosen. The third or the fourth approximation in the method of solution of the earlier papers[7] of this series could, of course, be used; indeed, for the purposes of numerical iterations the approximate solutions (Paper XXII, eqs. [155]–[161]) in terms of characteristic roots and points of the Gaussian division are found to be quite satisfactory; for the purposes of analytical iterations, on the other hand, these solutions are not very satisfactory, as the resulting expressions are too inconvenient to handle. An alternative method which appears more suitable, particularly for small values of τ, is to start the iteration with the functions

$$X^{(1)}(\mu) = 1 \quad \text{and} \quad Y^{(1)}(\mu) = e^{-\tau/\mu} . \tag{7}$$

The appropriateness of these functions in our present context arises from the fact that with these expressions for X and Y we essentially recover the formulae for the reflected and the transmitted light which have suffered a single scattering process in the atmosphere.

Iterating the functions (7) in accordance with equations (5) and (6), we have

$$X^{(2)}(\mu) = 1 + \mu \int_0^1 \frac{\Psi(\mu')}{\mu+\mu'} \left[1 - \exp\left\{-\tau\left(\frac{1}{\mu}+\frac{1}{\mu'}\right)\right\}\right] d\mu' \tag{8}$$

[5] Van de Hulst uses a in place of our ϖ_0.

[6] It should be noted in this connection that different source functions can give rise to the same laws for the angular distributions of the emergent radiations. Indeed, it is on this account that the functional equations governing the angular distributions of the emergent radiations are simpler to handle than are the equations governing the source functions themselves.

[7] Specifically, Paper XXI in this context.

and

$$Y^{(2)}(\mu) = e^{-\tau/\mu} + \mu \int_0^1 \frac{\Psi(\mu')}{\mu - \mu'} [e^{-\tau/\mu} - e^{-\tau/\mu'}] d\mu' . \tag{9}$$

Let

$$\Psi(\mu) = \sum_j a_j \mu^j . \tag{10}$$

(In actual fact, only even powers of μ can occur on the right-hand side, since the characteristic function is an even polynomial in μ.)

For $\Psi(\mu)$ of the form (10) we can clearly express $X^{(2)}(\mu)$ and $Y^{(2)}(\mu)$ in the forms

$$X^{(2)}(\mu) = 1 + \sum_j a_j F_{j+1}(\tau, -\mu) \tag{11}$$

and

$$Y^{(2)}(\mu) = e^{-\tau/\mu}\Big[1 + \sum_j a_j F_{j+1}(\tau, \mu)\Big] , \tag{12}$$

where

$$F_{j+1}(\tau, \mu) = \mu \int_0^1 \frac{\mu'^j}{\mu - \mu'} \left[1 - \exp\left\{-\tau\left(\frac{1}{\mu'} - \frac{1}{\mu}\right)\right\}\right] d\mu' . \tag{13}$$

The functions $F_{j+1}(\tau, \mu)$ ($j = 0, 1, \ldots$), defined as in equation (13), are the same as the functions

$$F_{j+1}(\tau, \mu) = \int_0^\tau e^{t/\mu} E_{j+1}(t)\, dt \tag{14}$$

introduced by van de Hulst (*op. cit.*). To see the identity of the definitions (13) and (14), write

$$s = \frac{1}{\mu} \quad \text{and} \quad s' = \frac{1}{\mu'} \tag{15}$$

in equation (13). We obtain

$$F_{j+1}(\tau, s) = \int_1^\infty \frac{ds'}{s'^{j+1}(s' - s)} [1 - e^{-\tau(s'-s)}] . \tag{16}$$

Since

$$\int_0^\tau e^{-t(s'-s)} dt = \frac{1}{s' - s} [1 - e^{-\tau(s'-s)}] , \tag{17}$$

we can re-write equation (16) in the form

$$F_{j+1}(\tau, s) = \int_1^\infty \frac{ds'}{s'^{j+1}} \int_0^\tau dt\, e^{-t(s'-s)} , \tag{18}$$

or, inverting the order of the integrations, we have

$$F_{j+1}(\tau, s) = \int_0^\tau dt\, e^{ts} \int_1^\infty \frac{ds'}{s'^{j+1}} e^{-ts'} ; \tag{19}$$

and this is in agreement with van de Hulst's definition.

For the problems considered in Paper XXII, the characteristic functions $\Psi(\mu)$ are, at most, of degree 2 in μ^2; consequently, the solutions of the various problems in the second approximation (and for small values of τ) can all be written down in terms of the functions F_n of the odd orders 1, 3, and 5. These functions, for various values of τ and μ, are tabulated in the appendix.

The further iteration of $X^{(2)}(\mu)$ and $Y^{(2)}(\mu)$ to obtain a third approximation does not seem practicable, since a large number of other transcendental functions are introduced. However, in § 4 we shall show how the solutions $X^{(2)}(\mu)$ and $Y^{(2)}(\mu)$ can be "corrected" to satisfy certain exact integrals of the problem by a procedure essentially equivalent to taking into account the third and higher orders of scattering in a semi-empirical fashion.

3. *The moments of* $X^{(2)}(\mu)$ *and* $Y^{(2)}(\mu)$.—For several purposes the moments (cf. Paper XXII, eqs. [10] and [11])

$$\alpha_n=\int_0^1 X(\mu)\,\mu^n d\mu\,,\qquad \beta_n=\int_0^1 Y(\mu)\,\mu^n d\mu\,,$$
$$x_n=\int_0^1 X(\mu)\Psi(\mu)\,\mu^n d\mu\,,\qquad\text{and}\qquad y_n=\int_0^1 Y(\mu)\Psi(\mu)\,\mu^n d\mu \tag{20}$$

of the functions $X(\mu)$ and $Y(\mu)$ are needed. In the second approximation of § 2, these moments can all be expressed in terms of the functions

$$G_{n,m}(\tau)=\int_0^1 F_n(\tau,-\mu)\,\mu^m\frac{d\mu}{\mu^2}=\int_1^\infty F_n(\tau,-s)\,\frac{ds}{s^m}$$

and

$$G'_{n,m}(\tau)=\int_0^1 e^{-\tau/\mu}F_n(\tau,\mu)\,\mu^m\frac{d\mu}{\mu^2}=\int_1^\infty e^{-\tau s}F_n(\tau,s)\frac{ds}{s^m}, \tag{21}$$

also introduced by van de Hulst. We clearly have

$$\alpha_n^{(2)}=\frac{1}{n+1}+\sum_j a_j G_{j+1,\,n+2}(\tau)\,,$$
$$\beta_n^{(2)}=E_{n+2}(\tau)+\sum_j a_j G'_{j+1,\,n+2}(\tau)\,, \tag{22}$$
$$x_n^{(2)}=\sum_j\frac{a_j}{j+n+1}+\sum_j\sum_k a_j a_k G_{j+1,\,k+n+2}(\tau)\,,$$

and

$$y_n^{(2)}=\sum_j a_j E_{j+n+2}(\tau)+\sum_j\sum_k a_j a_k G'_{j+1,\,k+n+2}(\tau)\,.$$

For the various problems considered in Paper XXII the functions $G_{n,m}$ and $G'_{n,m}$ (symmetrical in the indices n and m) are required for $m \geqslant n$ and $n = 1, 2, 3, 4$, and 5. These functions are also tabulated in the appendix.

4. *The correction of the approximate solutions.*—In Paper XXII (Sec. I, § 4) a number of integral properties of the functions X and Y were established. The most important of these is the relation (Paper XXII, eq. [32])

$$x_0=1-\left[1-2\int_0^1\Psi(\mu)\,d\mu+y_0^2\right]^{1/2}. \tag{23}$$

An alternative form of this relation is

$$[1-(x_0+y_0)][1-(x_0-y_0)]=1-2\int_0^1\Psi(\mu)\,d\mu\,. \tag{24}$$

The approximate solutions of equations (1) and (2) obtained in the iteration scheme of § 2 will not, naturally, satisfy equation (24) exactly. We shall now show how, on this account, we can make use of equation (24) to "correct" the approximate solutions. For this purpose, we shall write

$$X(\mu) = X^{(n)}(\mu) + \Delta(\tau)\,\mu(1 - e^{-\tau/\mu})$$

and

$$Y(\mu) = Y^{(n)}(\mu) + \Delta(\tau)\,\mu(1 - e^{-\tau/\mu}), \tag{25}$$

and determine $\Delta(\tau)$ by requiring that x_0 and y_0, evaluated with the aid of the functions (25), satisfy the relation (24) exactly. This procedure of adding a correction term of the form $\Delta(\tau)\mu(1 - e^{-\tau/\mu})$ to both X and Y can, in a sense, be regarded as equivalent to assuming that the sources giving rise to the radiation which has been scattered more than n times in the atmosphere are uniformly distributed (cf. van de Hulst, *op. cit.*).[8] In conservative cases the procedure can also be described as a device by which we arrange for the flux integral to be satisfied exactly.

For $X(\mu)$ and $Y(\mu)$ defined as in equations (25), we have

$$x_0 = x_0^{(n)} + \Delta(\tau)\int_0^1 \Psi(\mu)\,\mu(1 - e^{-\tau/\mu})\,d\mu$$

and

$$y_0 = y_0^{(n)} + \Delta(\tau)\int_0^1 \Psi(\mu)\,\mu(1 - e^{-\tau/\mu})\,d\mu, \tag{26}$$

where $x_0^{(n)}$ and $y_0^{(n)}$ are the respective quantities evaluated with the functions $X^{(n)}$ and $Y^{(n)}$. Inserting equations (26) in equation (24), we obtain, after some minor rearranging of the terms,

$$\Delta(\tau) = \frac{1}{2\int_0^1 \Psi(\mu)\,\mu(1 - e^{-\tau/\mu})\,d\mu}\left\{1 - [x_0^{(n)} + y_0^{(n)}] - \frac{1 - 2\int_0^1 \Psi(\mu)\,d\mu}{1 - [x_0^{(n)} - y_0^{(n)}]}\right\}, \tag{27}$$

where it may be noted that, for $\Psi(\mu)$ given by equation (10),

$$\int_0^1 \Psi(\mu)\,\mu(1 - e^{-\tau/\mu})\,d\mu = \sum_j a_j\left\{\frac{1}{j+2} - E_{j+3}(\tau)\right\}. \tag{28}$$

In conservative cases equation (27) simplifies to

$$\Delta(\tau) = \frac{1 - [x_0^{(n)} + y_0^{(n)}]}{2\int_0^1 \Psi(\mu)\,\mu(1 - e^{-\tau/\mu})\,d\mu}. \tag{29}$$

As an illustration of the foregoing procedure of correcting the solutions $X^{(n)}$ and $Y^{(n)}$ $(n = 1$ and $2)$ consider the isotropic case (4) and the correction to the first approximation (7). In this case

$$x_0^{(1)} = \tfrac{1}{2}\varpi_0 \quad \text{and} \quad y_0^{(1)} = \tfrac{1}{2}\varpi_0 E_2(\tau), \tag{30}$$

[8] This remark should not be taken too literally, since, even in the simplest case of isotropic scattering, the solutions obtained by successive iterations of the equations for X and Y are not in strict 1:1 correspondence with the solutions obtained by including successively the higher orders of scattering. And, even apart from this, the method of correcting by the moment relation (24) is not exactly the same as distributing the sources for the higher orders of scattering uniformly through the atmosphere.

and

$$\Delta(\tau)=\frac{1}{\varpi_0[\frac{1}{2}-E_3(\tau)]}\left\{1-\tfrac{1}{2}\varpi_0[1+E_2(\tau)]-\frac{1-\varpi_0}{1-\frac{1}{2}\varpi_0[1-E_2(\tau)]}\right\}. \tag{31}$$

For the case $\varpi_0=1$ we have, in particular,

$$\Delta(\tau)=\frac{1-E_2(\tau)}{1-2E_3(\tau)}. \tag{32}$$

Actual numerical comparisons show that the first approximation, corrected according to equations (25) and (31), differs from the second approximation,

$$X^{(2)}(\mu)=1+\tfrac{1}{2}\varpi_0 F_1(\tau,-\mu)$$

and

$$Y^{(2)}(\mu)=e^{-\tau/\mu}[1+\tfrac{1}{2}\varpi_0 F_1(\tau,\mu)], \tag{33}$$

by less than 5 per cent for $\tau\leqslant 0.2$.[9] It will therefore appear that the second approximation, corrected in the manner we have indicated, will give entirely satisfactory representations for $\tau<0.5$.

For $X^{(2)}(\mu)$ and $Y^{(2)}(\mu)$, given by equations (11) and (12), the moments $x_0^{(2)}$ and $y_0^{(2)}$ are (cf. eqs. [22])

$$x_0^{(2)}=\sum_j\frac{a_j}{j+1}+\sum_j\sum_k a_j a_k G_{j+1,k+2}$$

and

$$y_0^{(2)}=\sum_j a_j E_{j+2}+\sum_j\sum_k a_j a_k G'_{j+1,k+2}. \tag{34}$$

The resulting expression for $\Delta(\tau)$ is

$$\Delta(\tau)=\left[1-\left\{\sum_j a_j\left(\frac{1}{j+1}+E_{j+2}\right)+\sum_j\sum_k a_j a_k\,(G_{j+1,k+2}+G'_{j+1,k+2})\right\}\right.$$
$$\left.-\frac{1-2\int_0^1\Psi(\mu)\,d\mu}{1-\left\{\sum_j a_j\left(\frac{1}{j+1}-E_{j+2}\right)+\sum_j\sum_k a_j a_k\,(G_{j+1,k+2}-G'_{j+1,k+2})\right\}}\right] \tag{35}$$
$$\div\,2\left[\sum_j a_j\left(\frac{1}{j+2}-E_{j+3}\right)\right].$$

In the conservative case the foregoing equation simplifies to

$$\Delta(\tau)=\frac{1-\left\{\sum_j a_j\left(\frac{1}{j+1}+E_{j+2}\right)+\sum_j\sum_k a_j a_k\,(G_{j+1,k+2}+G'_{j+1,k+2})\right\}}{2\sum_j a_j\left(\frac{1}{j+2}-E_{j+3}\right)}. \tag{36}$$

[9] Thus, for the case $\varpi_0=1$ and $\mu=1$, the corrected first approximation gives for X the values 1.16 and 1.26 for $\tau=0.1$ and $\tau=0.2$, respectively; on the uncorrected second approximation the corresponding values are 1.133 and 1.197. And van de Hulst estimates that the true values in these cases are probably 1.159 and 1.263.

As an illustration of the foregoing expressions, consider again the isotropic case. In this case the only nonvanishing a is

$$a_0 = \tfrac{1}{2}\varpi_0 ; \tag{37}$$

and equation (35) becomes

$$\Delta(\tau) = \left[1 - \tfrac{1}{2}\varpi_0\{1 + E_2 + \tfrac{1}{2}\varpi_0(G_{12} + G'_{12})\} - \frac{1-\varpi_0}{1-\tfrac{1}{2}\varpi_0\{1 - E_2 + \tfrac{1}{2}\varpi_0(G_{12} - G'_{12})\}}\right]\frac{1}{\varpi_0(\tfrac{1}{2} - E_3)} . \tag{38}$$

And for the case $\varpi_0 = 1$, we have

$$\Delta(\tau) = \frac{1}{4}\frac{1}{\tfrac{1}{2} - E_3}(2 - 2E_2 - G_{12} - G'_{12}) . \tag{39}$$

The foregoing expression for $\Delta(\tau)$ reduces to the correction factor $g(\tau)/\tau$ given by van de Hulst (*op. cit.*, eqs. [28] and [30]) if we replace $\tfrac{1}{2} - E_3(\tau)$ by τ as sufficiently accurate.

Finally, we may note that on the corrected second approximation the various moments are given by

$$\alpha_n - \alpha_n^{(2)} = \beta_n - \beta_n^{(2)} = \Delta(\tau)\left(\frac{1}{n+2} - E_{n+3}\right)$$

and

$$x_n - x_n^{(2)} = y_n - y_n^{(2)} = \Delta(\tau)\sum_j a_j\left(\frac{1}{j+n+2} - E_{j+n+3}\right), \tag{40}$$

where the quantities $\alpha_n^{(2)}$, etc., are given by equations (22).

5. *The standard solutions.*—In Paper XXII (Sec. I, theorem 5) we have shown that in conservative cases the solutions of equations (1) and (2) are not unique and that they form, instead, a one-parametric family. A member of this family which plays a particularly important role in the theory is the *standard solution,* defined by the property (Paper XXII, eqs. [64] and [65])

$$x_0^{(s)} = \int_0^1 X^{(s)}(\mu)\Psi(\mu)\,d\mu = 1 \quad \text{and} \quad y_0^{(s)} = \int_0^1 Y^{(s)}(\mu)\Psi(\mu)\,d\mu = 0 . \tag{41}$$ [10]

Now if $X(\mu)$ and $Y(\mu)$ are particular solutions of equations (1) and (2) (in the conservative case), then the standard solutions can be derived from them by writing

$$X^{(s)}(\mu) = X(\mu) + q\mu[X(\mu) + Y(\mu)]$$

and

$$Y^{(s)}(\mu) = Y(\mu) - q\mu[X(\mu) + Y(\mu)] \tag{42}$$

and determining the constant q by conditions (10).[11] In this manner we find that

$$q = \frac{y_0}{x_1 + y_1} . \tag{43}$$

Since the approximate solutions constructed as in § 4 satisfy the moment condition $x_0 + y_0 = 1$ exactly, standard solutions can be constructed from them according to

[10] We have used a superscript s to distinguish the standard solutions from the rest.

[11] Since $x_0 + y_0 = 1$ in conservative cases, the two conditions $x_0 = 1$ and $y_0 = 0$ are equivalent.

equations (42) and with q given by equation (43). In the corrected second approximation we therefore have

$$q=\Big[\sum_j a_j E_{j+2}+\sum_j\sum_k a_j a_k G'_{j+1,\,k+2}+\Delta(\tau)\sum_j a_j\Big(\frac{1}{j+2}-E_{j+3}\Big)\Big]$$
$$\div\Big[\sum_j a_j\Big(\frac{1}{j+2}+E_{j+3}\Big)+\sum_j\sum_k a_j a_k\,(G_{j+1,\,k+3}+G'_{j+1,\,k+3})$$
$$+\,2\Delta(\tau)\sum_j a_j\Big(\frac{1}{j+3}-E_{j+4}\Big)\Big]. \tag{44}$$

Substituting for $\Delta(\tau)$ according to equation (36) in the numerator of the foregoing equation, we obtain the more symmetrical expression

$$q=\frac{1}{2}\Big[1-\sum_j a_j\Big(\frac{1}{j+1}-E_{j+2}\Big)-\sum_j\sum_k a_j a_k\,(G_{j+1,\,k+2}-G'_{j+1,\,k+2})\Big]$$
$$\div\Big[\sum_j a_j\Big(\frac{1}{j+2}+E_{j+3}\Big)+\sum_j\sum_k a_j a_k\,(G_{j+1,\,k+3}+G'_{j+1,\,k+3})$$
$$+\,2\Delta(\tau)\sum_j a_j\Big(\frac{1}{j+3}-E_{j+4}\Big)\Big]. \tag{45}$$

The moments $\alpha_j^{(s)}$ and $\beta_j^{(s)}$ of the standard solutions are related to the moments α_j and β_j of the solutions from which they are derived by

$$\alpha_j^{(s)}=\alpha_j+q\,(\alpha_{j+1}+\beta_{j+1})$$
and
$$\beta_j^{(s)}=\beta_j-q\,(\alpha_{j+1}+\beta_{j+1}). \tag{46}$$

6. *The approximate solution for the conservative isotropic case.*—In Paper XXII we have shown that for the problem of diffuse reflection by an isotropically scattering atmosphere the solutions for X and Y appropriate to the problem are (Paper XXII, eqs. [180], [181], and [202])

$$X^*(\mu)=X^{(s)}(\mu)+Q\mu\,[X^{(s)}(\mu)+Y^{(s)}(\mu)]$$
and
$$Y^*(\mu)=Y^{(s)}(\mu)-Q\mu\,[X^{(s)}(\mu)+Y^{(s)}(\mu)], \tag{47}$$

where

$$Q=-\frac{\alpha_1^{(s)}-\beta_1^{(s)}}{[\alpha_1^{(s)}+\beta_1^{(s)}]\,\tau+2\,[\alpha_2^{(s)}+\beta_2^{(s)}]}. \tag{48}$$

When the standard solutions are themselves derived from another set, $X(\mu)$ and $Y(\mu)$, the required solutions can be expressed in terms of these latter solutions in the forms

$$X^*(\mu)=X(\mu)+(Q+q)\,\mu\,[X(\mu)+Y(\mu)]$$
and
$$Y^*(\mu)=Y(\mu)-(Q+q)\,\mu\,[X(\mu)+Y(\mu)], \tag{49}$$

where, according to equations (46) and (48),

$$Q=-\frac{\alpha_1-\beta_1+2q\,(\alpha_2+\beta_2)}{(\alpha_1+\beta_1)\,\tau+2\,(\alpha_2+\beta_2)}, \tag{50}$$

the moments now referring to $X(\mu)$ and $Y(\mu)$.

Since (in this case)

$$q=\frac{y_0}{x_1+y_1}=\frac{\beta_0}{\alpha_1+\beta_1}, \tag{51}$$

we obtain from equations (50) and (51)

$$Q+q=\frac{\beta_0\tau-(\alpha_1-\beta_1)}{(\alpha_1+\beta_1)\tau+2(\alpha_2+\beta_2)}. \tag{52}$$

The solutions $X^*(\mu)$ and $Y^*(\mu)$ appropriate to the problem of diffuse reflection are, therefore,

$$X^*(\mu)=X(\mu)+\frac{\beta_0\tau-(\alpha_1-\beta_1)}{(\alpha_1+\beta_1)\tau+2(\alpha_2+\beta_2)}\mu[X(\mu)+Y(\mu)]$$

and

$$Y^*(\mu)=Y(\mu)-\frac{\beta_0\tau-(\alpha_1-\beta_1)}{(\alpha_1+\beta_1)\tau+2(\alpha_2+\beta_2)}\mu[X(\mu)+Y(\mu)]. \tag{53}$$

If we now use in equations (53) the solutions in the corrected second approximation, we shall obtain solutions which will *exactly* satisfy both the integrals of the problem, namely, the flux and the K-integrals (cf. Paper XXII, eqs. [188] and [189]).[12]

I am indebted to Dr. van de Hulst for discussions on various aspects of scattering by planetary atmospheres.

APPENDIX

S. CHANDRASEKHAR AND FRANCES H. BREEN

The practical use of the developments given in this paper will largely depend on the availability of tables of the various auxiliary functions in terms of which the solutions are expressed. In this appendix we shall therefore provide the necessary tables which may prove adequate for use in conjunction with the exact solutions given in Paper XXII. For this latter purpose, the functions $F_n(\tau, -\mu)$ and $e^{-\tau/\mu}F_n(\tau, \mu)$ of orders 1, 3, and 5 and the functions $G_{n,m}(\tau)$ and $G'_{n,m}(\tau)$ for $m \geqslant n$ and $n = 1, 2, 3, 4$, and 5 are needed. These functions are given in Tables 1 and 2 for values of the arguments which are probably sufficiently close for practical applications. (The tables can, in any case, be supplemented by the series expansions given in van de Hulst's paper.)

As the solutions given in this paper are intended only for small values of $\tau(< 0.5)$, it did not seem necessary to carry the tabulations beyond $\tau = 1.0$.

The formulae which were used in the computation of the various functions are given in van de Hulst's paper. But for convenience we note here the principal formulae.

The functions $F_1(\tau, -\mu)$ and $e^{-\tau/\mu}F_1(\tau, \mu)$ are given by

$$F_1(\tau,-\mu)=\mu\left[\log\left(\frac{1}{\mu}+1\right)-e^{-\tau/\mu}E_1(\tau)+E_1\left(\frac{\tau}{\mu}+\tau\right)\right] \qquad (0\leqslant\mu\leqslant 1),$$

$$e^{-\tau/\mu}F_1(\tau,\mu)=\mu\left[E_1(\tau)+e^{-\tau/\mu}Ei\left(\frac{\tau}{\mu}-\tau\right)-e^{-\tau/\mu}\log\left(\frac{1}{\mu}-1\right)\right] \qquad (0\leqslant\mu<1),$$

and

$$e^{-\tau}F(\tau,1)=e^{-\tau}(\gamma+\log\tau)+E_1(\tau),$$

[12] In this connection it is of interest to note that, by writing the reflected and the transmitted intensities in the forms given in Paper XXII, eqs. (178) and (179) (with $\varpi_0 = 1$), we can readily show that the flux and the K-integrals are equivalent to the conditions

$$\alpha_0+\beta_0=2 \quad \text{and} \quad \tau=\frac{\alpha_1-\beta_1}{\beta_0}.$$

And it may be verified that, with the solutions for X and Y in the forms given by eqs. (53), the latter condition is satisfied in virtue of the former.

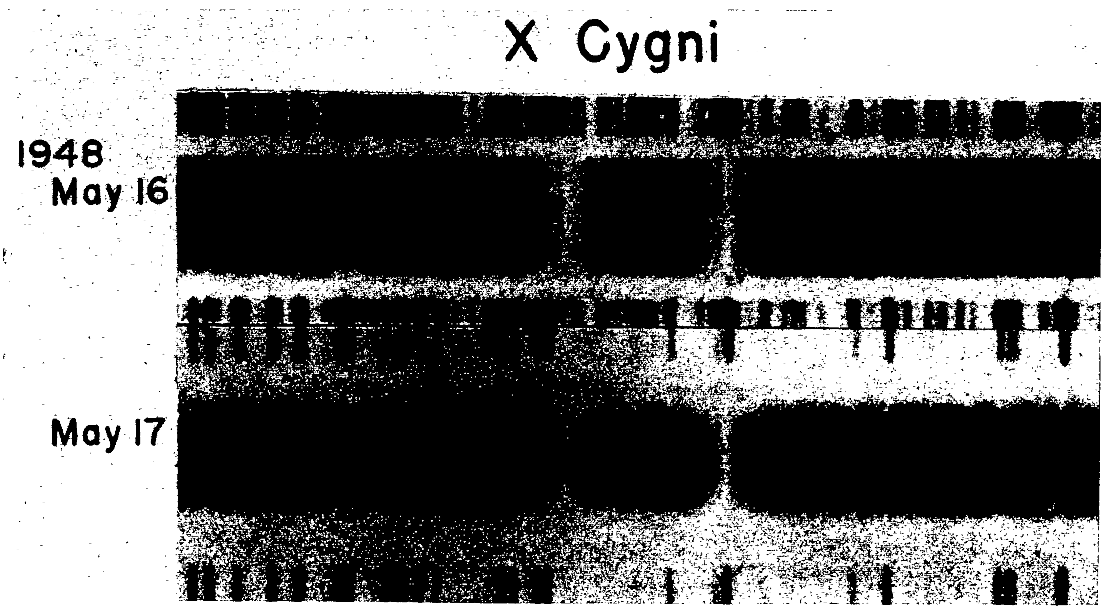

FIG. 1.—The *Ca* II lines of X Cygni

TABLE 1

τ	μ	$F_1(-\mu,\tau)$	$F_3(-\mu,\tau)$	$F_5(-\mu,\tau)$	$e^{-\tau/\mu}F_1(\mu,\tau)$	$e^{-\tau/\mu}F_3(\mu,\tau)$	$e^{-\tau/\mu}F_5(\mu,\tau)$
	0	0	0	0	0	0	0
	0.010	0.023024	0.001958	0.000981	0.022045	0.001957	0.000980
	0.025	0.026166	0.002255	0.001129	0.025714	0.002254	0.001129
	0.050	0.027353	0.002367	0.001186	0.027116	0.002367	0.001186
	0.075	0.027766	0.002407	0.001205	0.027605	0.002406	0.001205
	0.1	0.027975	0.002427	0.001215	0.027854	0.002426	0.001215
	0.2	0.028294	0.002457	0.001230	0.028232	0.002457	0.001230
	0.3	0.028401	0.002467	0.001236	0.028360	0.002467	0.001236
0.005	0.4	0.028455	0.002472	0.001238	0.028424	0.002472	0.001238
	0.5	0.028488	0.002475	0.001240	0.028463	0.002475	0.001240
	0.6	0.028509	0.002477	0.001241	0.028489	0.002477	0.001241
	0.7	0.028525	0.002479	0.001241	0.028508	0.002479	0.001242
	0.8	0.028537	0.002480	0.001242	0.028521	0.002480	0.001242
	0.9	0.028546	0.002481	0.001242	0.028532	0.002481	0.001242
	1.0	0.028553	0.002481	0.001243	0.028540	0.002481	0.001243
	0	0	0	0	0	0	0
	0.010	0.033454	0.003135	0.001572	0.030347	0.003125	0.001568
	0.025	0.042319	0.004083	0.002048	0.040680	0.004078	0.002046
	0.050	0.046072	0.004489	0.002251	0.045170	0.004486	0.002250
	0.075	0.047432	0.004636	0.002325	0.046810	0.004634	0.002325
	0.1	0.048133	0.004712	0.002364	0.047659	0.004711	0.002363
	0.2	0.049214	0.004830	0.002422	0.048971	0.004829	0.002422
0.01	0.3	0.049582	0.004870	0.002443	0.049418	0.004869	0.002442
	0.4	0.049767	0.004890	0.002453	0.049644	0.004889	0.002453
	0.5	0.049879	0.004902	0.002459	0.049780	0.004902	0.002459
	0.6	0.049954	0.004910	0.002463	0.049871	0.004910	0.002463
	0.7	0.050007	0.004916	0.002466	0.049936	0.004915	0.002466
	0.8	0.050047	0.004920	0.002468	0.049985	0.004920	0.002468
	0.9	0.050078	0.004924	0.002470	0.050023	0.004923	0.002470
	1.0	0.050103	0.004926	0.002471	0.050054	0.004926	0.002471
	0	0	0	0	0	0	0
	0.010	0.042087	0.004266	0.002142	0.033934	0.004214	0.002124
	0.025	0.062645	0.006768	0.003402	0.057223	0.006733	0.003390
	0.050	0.073290	0.008093	0.004070	0.070022	0.008072	0.004063
	0.075	0.077453	0.008616	0.004333	0.075168	0.008601	0.004328
	0.1	0.079667	0.008894	0.004474	0.077868	0.008883	0.004470
	0.2	0.083167	0.009336	0.004696	0.082222	0.009329	0.004694
0.02	0.3	0.084384	0.009489	0.004774	0.083743	0.009485	0.004772
	0.4	0.085002	0.009567	0.004813	0.084517	0.009564	0.004812
	0.5	0.085376	0.009615	0.004837	0.084986	0.009612	0.004836
	0.6	0.085626	0.009646	0.004853	0.085301	0.009644	0.004852
	0.7	0.085806	0.009669	0.004864	0.085526	0.009667	0.004864
	0.8	0.085941	0.009686	0.004873	0.085696	0.009685	0.004873
	0.9	0.086046	0.009699	0.004880	0.085828	0.009698	0.004879
	1.0	0.086131	0.009710	0.004885	0.085934	0.009709	0.004885
	0	0	0	0	0	0	0
	0.010	0.044804	0.004674	0.002349	0.032150	0.004557	0.002308
	0.025	0.074335	0.008533	0.004298	0.064157	0.008438	0.004265
	0.050	0.092426	0.010989	0.005540	0.085759	0.010926	0.005518
	0.075	0.100023	0.012033	0.006068	0.095138	0.011987	0.006052
	0.1	0.104183	0.012607	0.006358	0.100338	0.012571	0.006346
	0.2	0.110924	0.013541	0.006831	0.108856	0.013522	0.006824
0.03	0.3	0.113315	0.013873	0.006999	0.111901	0.013860	0.006994
	0.4	0.114538	0.014044	0.007085	0.113465	0.014033	0.007082
	0.5	0.115282	0.014147	0.007137	0.114417	0.014139	0.007135
	0.6	0.115782	0.014217	0.007173	0.115057	0.014210	0.007170
	0.7	0.116141	0.014267	0.007198	0.115517	0.014261	0.007196
	0.8	0.116411	0.014304	0.007217	0.115864	0.014299	0.007215
	0.9	0.116622	0.014334	0.007232	0.116135	0.014329	0.007230
	1.0	0.116791	0.014357	0.007244	0.116352	0.014353	0.007242

TABLE 1-Continued

τ	μ	$F_1(-\mu,\tau)$	$F_3(-\mu,\tau)$	$F_5(-\mu,\tau)$	$e^{-\tau/\mu}F_1(\mu,\tau)$	$e^{-\tau/\mu}F_3(\mu,\tau)$	$e^{-\tau/\mu}F_5(\mu,\tau)$
	0	0	0	0	0	0	0
	0.010	0.045696	0.004821	0.002424	0.029468	0.004628	0.002356
	0.025	0.081341	0.009695	0.004891	0.066117	0.009508	0.004825
	0.050	0.106443	0.013316	0.006727	0.095670	0.013183	0.006680
	0.075	0.117698	0.014968	0.007566	0.109572	0.014868	0.007530
	0.1	0.124032	0.015860	0.008040	0.117540	0.015824	0.008012
	0.2	0.134552	0.017467	0.008834	0.130977	0.017423	0.008819
0.04	0.3	0.138358	0.018035	0.009123	0.135895	0.018005	0.009112
	0.4	0.140320	0.018328	0.009272	0.138443	0.018305	0.009263
	0.5	0.141518	0.018507	0.009363	0.140001	0.018489	0.009356
	0.6	0.142325	0.018628	0.009424	0.141052	0.018612	0.009418
	0.7	0.142905	0.018715	0.009468	0.141809	0.018701	0.009463
	0.8	0.143342	0.018781	0.009501	0.142380	0.018769	0.009497
	0.9	0.143684	0.018832	0.009527	0.142826	0.018821	0.009523
	1.0	0.143958	0.018873	0.009548	0.143185	0.018863	0.009545
	0	0	0	0	0	0	0
	0.010	0.045996	0.004875	0.002452	0.026979	0.004600	0.002354
	0.025	0.085630	0.010459	0.005283	0.065447	0.010155	0.005174
	0.050	0.116925	0.015186	0.007686	0.101593	0.014953	0.007603
	0.075	0.131828	0.017490	0.008859	0.119937	0.017309	0.008794
	0.1	0.140440	0.018833	0.009543	0.130799	0.018686	0.009491
	0.2	0.155087	0.021134	0.010715	0.149653	0.021051	0.010686
0.05	0.3	0.160488	0.021987	0.011150	0.156714	0.021929	0.011129
	0.4	0.163294	0.022431	0.011376	0.160406	0.022387	0.011360
	0.5	0.165014	0.022704	0.011515	0.162674	0.022668	0.011502
	0.6	0.166175	0.022888	0.011609	0.164209	0.022858	0.011598
	0.7	0.167011	0.023020	0.011676	0.165316	0.022994	0.011667
	0.8	0.167643	0.023121	0.011727	0.166153	0.023098	0.011719
	0.9	0.168137	0.023199	0.011767	0.166808	0.023179	0.011760
	1.0	0.168533	0.023262	0.011799	0.167334	0.023244	0.011793
	0	0	0	0	0	0	0
	0.010	0.046098	0.004894	0.002462	0.024873	0.004538	0.002333
	0.025	0.088290	0.010963	0.005542	0.063430	0.010521	0.005383
	0.050	0.124867	0.016690	0.008461	0.104712	0.016327	0.008331
	0.075	0.143271	0.019657	0.009976	0.127218	0.019368	0.009872
	0.1	0.154179	0.021436	0.010885	0.140976	0.021198	0.010799
	0.2	0.173163	0.024559	0.012481	0.165551	0.024421	0.012431
0.06	0.3	0.180296	0.025740	0.013085	0.174968	0.025644	0.013050
	0.4	0.184030	0.026360	0.013402	0.179935	0.026287	0.013375
	0.5	0.186327	0.026742	0.013597	0.183002	0.026682	0.013575
	0.6	0.187882	0.027001	0.013729	0.185084	0.026951	0.013711
	0.7	0.189004	0.027188	0.013825	0.186589	0.027144	0.013809
	0.8	0.189853	0.027329	0.013897	0.187728	0.027291	0.013883
	0.9	0.190517	0.027440	0.013954	0.188620	0.027406	0.013941
	1.0	0.191050	0.027529	0.013999	0.189338	0.027498	0.013988
	0	0	0	0	0	0	0
	0.010	0.046133	0.004901	0.002465	0.023116	0.004464	0.002307
	0.025	0.089955	0.011294	0.005712	0.060785	0.010699	0.005498
	0.050	0.130936	0.017899	0.009087	0.105840	0.017379	0.008899
	0.075	0.152621	0.021520	0.010941	0.132117	0.021095	0.010787
	0.1	0.165785	0.023749	0.012083	0.148685	0.023393	0.011953
	0.2	0.189217	0.027759	0.014138	0.179136	0.027549	0.014062
0.07	0.3	0.198228	0.029314	0.014935	0.191111	0.029178	0.014886
	0.4	0.202913	0.030124	0.015351	0.197425	0.030010	0.015310
	0.5	0.205833	0.030630	0.015611	0.201366	0.030538	0.015577
	0.6	0.207814	0.030974	0.015787	0.204049	0.030896	0.015759
	0.7	0.209247	0.031223	0.015915	0.205994	0.031156	0.015890
	0.8	0.210332	0.031412	0.016012	0.207467	0.031352	0.015990
	0.9	0.211181	0.031559	0.016087	0.208623	0.031506	0.016068
	1.0	0.211865	0.031678	0.016148	0.209553	0.031630	0.016131

TABLE 1-Continued

τ	μ	$F_1(-\mu,\tau)$	$F_3(-\mu,\tau)$	$F_5(-\mu,\tau)$	$e^{-\tau/\mu}F_1(\mu,\tau)$	$e^{-\tau/\mu}F_3(\mu,\tau)$	$e^{-\tau/\mu}F_5(\mu,\tau)$
	0	0	0	0	0	0	0
	0.010	0.046145	0.004903	0.002466	0.021635	0.004388	0.002278
	0.025	0.091003	0.011512	0.005827	0.057916	0.010755	0.005550
	0.050	0.135606	0.018871	0.009593	0.105555	0.018170	0.009337
	0.075	0.160311	0.023122	0.011775	0.135154	0.022533	0.011559
	0.1	0.175655	0.025804	0.013152	0.154390	0.025306	0.012970
	0.2	0.203569	0.030749	0.015694	0.190755	0.030448	0.015584
0.08	0.3	0.214445	0.032694	0.016694	0.205336	0.032480	0.016616
	0.4	0.220221	0.033731	0.017228	0.213164	0.033565	0.017167
	0.5	0.223801	0.034374	0.017559	0.218044	0.034239	0.017510
	0.6	0.226237	0.034813	0.017785	0.221376	0.034699	0.017743
	0.7	0.228002	0.035131	0.017948	0.223796	0.035032	0.017912
	0.8	0.229339	0.035372	0.018073	0.225633	0.035285	0.018041
	0.9	0.230388	0.035561	0.018170	0.227075	0.035483	0.018141
	1.0	0.231232	0.035713	0.018248	0.228237	0.035643	0.018222
	0	0	0	0	0	0	0
	0.010	0.046149	0.004904	0.002467	0.020367	0.004312	0.002248
	0.025	0.091667	0.011656	0.005902	0.055043	0.010729	0.005561
	0.050	0.139218	0.019653	0.010002	0.104275	0.018749	0.009668
	0.075	0.166668	0.024499	0.012494	0.136731	0.023720	0.012207
	0.1	0.184089	0.027632	0.014108	0.158453	0.026964	0.013861
	0.2	0.216464	0.033543	0.017154	0.200679	0.033131	0.017002
0.09	0.3	0.229300	0.035913	0.018377	0.217993	0.035618	0.018268
	0.4	0.236165	0.037186	0.019034	0.227371	0.036956	0.018949
	0.5	0.240438	0.037980	0.019444	0.233245	0.037792	0.019374
	0.6	0.243351	0.038522	0.019724	0.237269	0.038363	0.019665
	0.7	0.245466	0.038916	0.019927	0.240197	0.038778	0.019876
	0.8	0.247070	0.039215	0.020081	0.242423	0.039094	0.020036
	0.9	0.248329	0.039450	0.020202	0.244173	0.039341	0.020162
	1.0	0.249343	0.039639	0.020300	0.245584	0.039541	0.020264
	0	0	0	0	0	0	0
	0.010	0.046150	0.004904	0.002467	0.019263	0.004237	0.002219
	0.025	0.092088	0.011750	0.005952	0.052282	0.010651	0.005544
	0.050	0.142022	0.020283	0.010333	0.102306	0.019155	0.009914
	0.075	0.171944	0.025684	0.013116	0.137160	0.024690	0.012747
	0.10	0.191327	0.029257	0.014961	0.161161	0.028393	0.014640
	0.15	0.214651	0.033617	0.017213	0.191256	0.032946	0.016964
0.10	0.2	0.228096	0.036155	0.018526	0.209127	0.035611	0.018323
	0.3	0.242926	0.038973	0.019983	0.229234	0.038580	0.019837
	0.4	0.250914	0.040498	0.020773	0.240222	0.040191	0.020659
	0.5	0.255903	0.041453	0.021267	0.247139	0.041201	0.021173
	0.6	0.259314	0.042107	0.021605	0.251890	0.041894	0.021526
	0.7	0.261793	0.042582	0.021852	0.255355	0.042398	0.021783
	0.8	0.263676	0.042944	0.022039	0.257994	0.042781	0.021978
	0.9	0.265156	0.043228	0.022186	0.260069	0.043082	0.022132
	1.0	0.266348	0.043458	0.022305	0.261745	0.043326	0.022256
	0	0	0	0	0	0	0
	0.010	0.046151	0.004905	0.002467	0.018289	0.004164	0.002190
	0.025	0.092357	0.011813	0.005985	0.049685	0.010539	0.005509
	0.050	0.144206	0.020789	0.010600	0.099875	0.019422	0.010088
	0.075	0.176337	0.026702	0.013653	0.136687	0.025471	0.013192
	0.10	0.197558	0.030702	0.015722	0.162750	0.029618	0.015317
	0.15	0.223490	0.035667	0.018294	0.196129	0.034813	0.017975
0.11	0.2	0.238624	0.038597	0.019813	0.216280	0.037899	0.019552
	0.3	0.255467	0.041882	0.021517	0.239216	0.041374	0.021327
	0.4	0.264600	0.043672	0.022446	0.251862	0.043275	0.022297
	0.5	0.270326	0.044798	0.023031	0.259859	0.044471	0.022908
	0.6	0.274252	0.045572	0.023432	0.265371	0.045295	0.023329
	0.7	0.277108	0.046135	0.023725	0.269398	0.045894	0.023634
	0.8	0.279281	0.046564	0.023947	0.272470	0.046351	0.023868
	0.9	0.280990	0.046901	0.024122	0.274889	0.046711	0.024051
	1.0	0.282368	0.047173	0.024264	0.276844	0.047001	0.024199

TABLE 1-Continued

τ	μ	$F_1(-\mu,\tau)$	$F_3(-\mu,\tau)$	$F_5(-\mu,\tau)$	$e^{-\tau/\mu}F_1(\mu,\tau)$	$e^{-\tau/\mu}F_3(\mu,\tau)$	$e^{-\tau/\mu}F_5(\mu,\tau)$
	0	0	0	0	0	0	0
	0.010	0.046151	0.004905	0.002467	0.017420	0.004092	0.002162
	0.025	0.092529	0.011854	0.006007	0.047150	0.010406	0.005462
	0.050	0.145912	0.021196	0.010815	0.097146	0.019577	0.010205
	0.075	0.180005	0.027579	0.014117	0.135511	0.026089	0.013556
	0.10	0.202938	0.031987	0.016403	0.163411	0.030660	0.015903
	0.15	0.231381	0.037552	0.019292	0.199899	0.036492	0.018893
0.12	0.2	0.248180	0.040880	0.021022	0.222288	0.040008	0.020693
	0.3	0.267040	0.044647	0.022981	0.248069	0.044007	0.022740
	0.4	0.277337	0.046716	0.024057	0.262408	0.046212	0.023868
	0.5	0.283816	0.048022	0.024737	0.271521	0.047607	0.024581
	0.6	0.288268	0.048920	0.025205	0.277815	0.048566	0.025072
	0.7	0.291514	0.049577	0.025547	0.282432	0.049271	0.025431
	0.8	0.293986	0.050077	0.025807	0.285955	0.049806	0.025705
	0.9	0.295931	0.050471	0.026012	0.288729	0.050224	0.025918
	1.0	0.297501	0.050789	0.026178	0.290981	0.050570	0.026095
	0	0	0	0	0	0	0
	0.010	0.046151	0.004905	0.002467	0.016635	0.004023	0.002134
	0.025	0.092640	0.011881	0.006021	0.045054	0.010260	0.005407
	0.050	0.147248	0.021524	0.010990	0.094242	0.019642	0.010276
	0.075	0.183076	0.028332	0.014518	0.124954	0.026515	0.013847
	0.10	0.207594	0.033130	0.017010	0.163307	0.031536	0.016406
	0.15	0.238440	0.039285	0.020213	0.202711	0.037996	0.019725
0.13	0.2	0.256874	0.043015	0.022156	0.227282	0.041946	0.021751
	0.3	0.277746	0.047276	0.024378	0.255904	0.046486	0.024079
	0.4	0.289217	0.049633	0.025608	0.271963	0.049010	0.025372
	0.5	0.296463	0.051127	0.026388	0.282219	0.050612	0.026193
	0.6	0.301452	0.052158	0.026926	0.289329	0.051720	0.026760
	0.7	0.305097	0.052913	0.027320	0.294547	0.052531	0.027175
	0.8	0.307875	0.053488	0.027620	0.298538	0.053151	0.027492
	0.9	0.310063	0.053942	0.027857	0.301685	0.053635	0.027739
	1.0	0.311831	0.054309	0.028049	0.304241	0.054034	0.027944
	0	0	0	0	0	0	0
	0.010	0.046151	0.004905	0.002467	0.015923	0.003955	0.002106
	0.025	0.092711	0.011899	0.006031	0.043008	0.010107	0.005348
	0.050	0.148296	0.021788	0.011131	0.091252	0.019634	0.010309
	0.075	0.185651	0.028981	0.014864	0.131650	0.026919	0.014077
	0.10	0.211631	0.034147	0.017553	0.162569	0.032266	0.016835
	0.15	0.244769	0.040880	0.021064	0.204687	0.039338	0.020476
0.14	0.2	0.264799	0.045011	0.023222	0.231373	0.043724	0.022731
	0.3	0.287669	0.049776	0.025712	0.262816	0.048818	0.025347
	0.4	0.300322	0.052431	0.027101	0.280615	0.051671	0.026811
	0.5	0.308343	0.054120	0.027985	0.292036	0.053492	0.027745
	0.6	0.313940	0.055311	0.028606	0.300030	0.054816	0.028433
	0.7	0.317929	0.056146	0.029045	0.305818	0.055679	0.028867
	0.8	0.321020	0.056800	0.029387	0.310293	0.056387	0.029230
	0.9	0.323457	0.057316	0.029658	0.313830	0.056945	0.029516
	1.0	0.325427	0.057734	0.029876	0.316697	0.057398	0.029748
	0	0	0	0	0	0	0
	0.010	0.046151	0.004905	0.002467	0.015270	0.003888	0.002079
	0.025	0.092756	0.011910	0.006037	0.041126	0.009951	0.005286
	0.050	0.149120	0.022001	0.011244	0.088242	0.019568	0.010311
	0.075	0.187815	0.029539	0.015163	0.129196	0.027167	0.014253
	0.10	0.215138	0.035052	0.018038	0.161310	0.032863	0.017198
	0.15	0.250454	0.042347	0.021850	0.205935	0.040530	0.021153
0.15	0.2	0.272036	0.046879	0.024222	0.234658	0.045352	0.023658
	0.3	0.296884	0.052154	0.026986	0.268892	0.051009	0.026547
	0.4	0.310720	0.055114	0.028538	0.288439	0.054203	0.028188
	0.5	0.319522	0.057005	0.029530	0.301043	0.056249	0.029240
	0.6	0.325611	0.058317	0.030218	0.309834	0.057671	0.029970
	0.7	0.330073	0.059280	0.030723	0.316313	0.058717	0.030507
	0.8	0.333483	0.060016	0.031110	0.321284	0.059517	0.030918
	0.9	0.336173	0.060598	0.031415	0.325218	0.060150	0.031243
	1.0	0.338350	0.061069	0.031662	0.328410	0.060662	0.031506

TABLE 1-Continued

τ	μ	$F_1(-\mu,\tau)$	$F_3(-\mu,\tau)$	$F_5(-\mu,\tau)$	$e^{-\tau/\mu}F_1(\mu,\tau)$	$e^{-\tau/\mu}F_3(\mu,\tau)$	$e^{-\tau/\mu}F_5(\mu,\tau)$
	0	0	0	0	0	0	0
	0.010	0.046151	0.004905	0.002467	0.014669	0.003824	0.002052
	0.025	0.092786	0.011918	0.006041	0.039392	0.009793	0.005222
	0.050	0.149769	0.022172	0.011336	0.085258	0.019456	0.010289
	0.075	0.189636	0.030020	0.015421	0.126512	0.027324	0.014382
	0.10	0.218189	0.035857	0.018470	0.159625	0.033343	0.017501
	0.15	0.255567	0.043696	0.022575	0.206547	0.041583	0.021760
0.16	0.2	0.278656	0.048626	0.025161	0.237223	0.046837	0.024471
	0.3	0.305454	0.054415	0.028201	0.274203	0.053065	0.027681
	0.4	0.320471	0.057687	0.029921	0.295503	0.056608	0.029505
	0.5	0.330059	0.059786	0.031025	0.309303	0.058889	0.030679
	0.6	0.336706	0.061245	0.031792	0.318958	0.060478	0.031496
	0.7	0.341586	0.062318	0.032357	0.326088	0.061648	0.032098
	0.8	0.345318	0.063140	0.032789	0.331568	0.062545	0.032560
	0.9	0.348266	0.063789	0.033131	0.335910	0.063255	0.032925
	1.0	0.350653	0.064316	0.033408	0.339435	0.063831	0.033221
	0	0	0	0	0	0	0
	0.010	0.046151	0.004905	0.002467	0.014113	0.003760	0.002025
	0.025	0.092804	0.011923	0.006044	0.037791	0.009636	0.005158
	0.050	0.150280	0.022310	0.011411	0.082332	0.019307	0.010247
	0.075	0.191170	0.030433	0.015644	0.123664	0.027403	0.014469
	0.10	0.220847	0.036573	0.018857	0.157593	0.033717	0.017749
	0.15	0.260173	0.044937	0.023245	0.206605	0.042508	0.022303
0.17	0.2	0.284719	0.050260	0.026043	0.239143	0.048189	0.025240
	0.3	0.313435	0.056567	0.029362	0.278816	0.054992	0.028752
	0.4	0.329628	0.060156	0.031253	0.301864	0.058892	0.030763
	0.5	0.340003	0.062466	0.032471	0.316872	0.061414	0.032063
	0.6	0.347213	0.064077	0.033320	0.327403	0.063175	0.032971
	0.7	0.352513	0.065263	0.033946	0.335196	0.064475	0.033640
	0.8	0.356573	0.066173	0.034426	0.341194	0.065473	0.034154
	0.9	0.359781	0.066893	0.034805	0.345953	0.066264	0.034561
	1.0	0.362381	0.067477	0.035114	0.349820	0.066905	0.034892
	0	0	0	0	0	0	0
	0.010	0.046151	0.004905	0.002467	0.013597	0.003698	0.001999
	0.025	0.092817	0.011927	0.006046	0.036310	0.009480	0.005093
	0.050	0.150684	0.022421	0.011471	0.079487	0.019130	0.010189
	0.075	0.192465	0.030789	0.015837	0.120706	0.027415	0.014522
	0.10	0.223166	0.037211	0.019202	0.155283	0.033998	0.017950
	0.15	0.264327	0.046079	0.023863	0.206178	0.043313	0.022785
0.18	0.2	0.290279	0.051789	0.026870	0.240485	0.049416	0.025945
	0.3	0.320878	0.058613	0.030470	0.282789	0.056796	0.029762
	0.4	0.338239	0.062523	0.032535	0.307582	0.061060	0.031965
	0.5	0.349401	0.065050	0.033870	0.323799	0.063828	0.033394
	0.6	0.357176	0.066816	0.034803	0.335215	0.065768	0.034395
	0.7	0.362899	0.068119	0.035492	0.343681	0.067201	0.035134
	0.8	0.367288	0.069120	0.036021	0.350208	0.068304	0.035703
	0.9	0.370761	0.069912	0.036440	0.355391	0.069178	0.036154
	1.0	0.373576	0.070555	0.036780	0.359607	0.069888	0.036520
	0	0	0	0	0	0	0
	0.010	0.046151	0.004905	0.002467	0.013115	0.003637	0.001973
	0.025	0.092825	0.011929	0.006047	0.034935	0.009326	0.005029
	0.050	0.151003	0.022510	0.011519	0.076737	0.018931	0.010118
	0.075	0.193558	0.031096	0.016003	0.117684	0.027369	0.014544
	0.10	0.225191	0.037779	0.019510	0.152750	0.029195	0.018056
	0.15	0.268078	0.047131	0.024434	0.205330	0.044009	0.023211
0.19	0.2	0.295384	0.053219	0.027648	0.241309	0.050544	0.026592
	0.3	0.327826	0.060560	0.031528	0.286176	0.058481	0.030714
	0.4	0.346345	0.064795	0.033769	0.312698	0.063115	0.033111
	0.5	0.358292	0.067542	0.035224	0.330130	0.066135	0.034673
	0.6	0.366632	0.069466	0.036243	0.342438	0.068258	0.035770
	0.7	0.372782	0.070888	0.036997	0.351585	0.069829	0.036582
	0.8	0.377502	0.071982	0.037577	0.358647	0.071040	0.037207
	0.9	0.381240	0.072849	0.038036	0.364263	0.072001	0.037704
	1.0	0.384273	0.073553	0.038409	0.368835	0.072782	0.038107

TABLE 1-Continued

τ	μ	$F_1(-\mu,\tau)$	$F_3(-\mu,\tau)$	$F_5(-\mu,\tau)$	$e^{-\tau/\mu}F_1(\mu,\tau)$	$e^{-\tau/\mu}F_3(\mu,\tau)$	$e^{-\tau/\mu}F_5(\mu,\tau)$
	0	0	0	0	0	0	0
	0.025	0.092830	0.011930	0.006048	0.033655	0.009174	0.004965
	0.050	0.151255	0.022582	0.011559	0.074091	0.018806	0.010037
	0.075	0.194483	0.031359	0.016147	0.114632	0.027274	0.014539
	0.10	0.226962	0.038284	0.019786	0.150045	0.034317	0.018224
	0.15	0.271469	0.048098	0.024962	0.204115	0.044603	0.023586
	0.2	0.300076	0.054558	0.028378	0.241668	0.051521	0.027182
0.20	0.3	0.334319	0.062413	0.032538	0.289023	0.060054	0.031609
	0.4	0.353983	0.066974	0.034958	0.317258	0.065060	0.034204
	0.5	0.366712	0.069945	0.036534	0.335904	0.068339	0.035902
	0.6	0.375618	0.072030	0.037641	0.349110	0.070648	0.037097
	0.7	0.382194	0.073574	0.038461	0.358945	0.072362	0.037984
	0.8	0.387248	0.074763	0.039093	0.366550	0.073684	0.038668
	0.9	0.391254	0.075706	0.039594	0.372605	0.074734	0.039211
	1.0	0.394506	0.076473	0.040001	0.377538	0.075588	0.039653
	0	0	0	0	0	0	0
	0.025	0.092838	0.011933	0.006049	0.028372	0.008458	0.004654
	0.050	0.151917	0.022779	0.011667	0.062494	0.017499	0.009544
	0.075	0.197372	0.032220	0.016621	0.099713	0.026286	0.014233
	0.10	0.233015	0.040090	0.020780	0.135156	0.034072	0.018357
	0.15	0.284183	0.051896	0.027052	0.194068	0.046303	0.024800
	0.2	0.318317	0.060061	0.031410	0.238013	0.055056	0.029392
0.25	0.3	0.361083	0.070416	0.036943	0.296555	0.066387	0.035321
	0.4	0.386232	0.076620	0.040267	0.332976	0.073293	0.038927
	0.5	0.402782	0.080735	0.042473	0.357607	0.077911	0.041336
	0.6	0.414483	0.083658	0.044042	0.375319	0.081209	0.043056
	0.7	0.423189	0.085840	0.045213	0.388650	0.083680	0.044343
	0.8	0.429916	0.087530	0.046121	0.399039	0.085599	0.045343
	0.9	0.435271	0.088878	0.046844	0.407359	0.087132	0.046141
	1.0	0.439633	0.089977	0.047435	0.414172	0.088385	0.046794
	0	0	0	0	0	0	0
	0.025	0.092839	0.011933	0.006049	0.024393	0.007809	0.004364
	0.050	0.152127	0.022846	0.011704	0.053363	0.016240	0.008997
	0.075	0.198647	0.032629	0.016849	0.086399	0.024838	0.013652
	0.10	0.236174	0.041101	0.021345	0.120007	0.032908	0.017981
	0.15	0.292022	0.054408	0.028455	0.180235	0.046421	0.025176
	0.2	0.330878	0.064032	0.033625	0.228251	0.056665	0.030599
0.30	0.3	0.380582	0.076671	0.040439	0.295719	0.070557	0.037927
	0.4	0.410730	0.084480	0.044660	0.339482	0.079340	0.042548
	0.5	0.430875	0.089750	0.047512	0.369784	0.085340	0.045700
	0.6	0.445263	0.093536	0.049563	0.391908	0.089683	0.047980
	0.7	0.456044	0.096384	0.051107	0.408737	0.092968	0.049703
	0.8	0.464421	0.098604	0.052310	0.421954	0.095537	0.051050
	0.9	0.471114	0.100382	0.053274	0.432603	0.097600	0.052132
	1.0	0.476585	0.101838	0.054064	0.441364	0.099293	0.053019
	0	0	0	0	0	0	0
	0.025	0.092839	0.011933	0.006049	0.021259	0.007222	0.004092
	0.050	0.152194	0.022869	0.011717	0.046147	0.015041	0.008454
	0.075	0.199218	0.032822	0.016959	0.075087	0.023242	0.012960
	0.10	0.237845	0.041669	0.021666	0.105975	0.031282	0.017328
	0.15	0.296920	0.056073	0.029399	0.165072	0.045511	0.024985
	0.20	0.339272	0.066887	0.035243	0.215178	0.056881	0.031060
0.35	0.25	0.370769	0.075132	0.039715	0.256040	0.065852	0.035836
	0.3	0.394977	0.081567	0.043214	0.289307	0.073006	0.039634
	0.4	0.429584	0.090894	0.048295	0.339397	0.083571	0.045234
	0.5	0.453045	0.097293	0.051787	0.374902	0.090946	0.049133
	0.6	0.469962	0.101940	0.054326	0.401220	0.096354	0.051990
	0.7	0.482725	0.105464	0.056253	0.421451	0.100483	0.054170
	0.8	0.492691	0.108225	0.057763	0.437466	0.103735	0.055886
	0.9	0.500686	0.110446	0.058979	0.450446	0.106361	0.057271
	1.0	0.507242	0.112271	0.059978	0.461175	0.108525	0.058412

TABLE 1-Continued

τ	μ	$F_1(-\mu,\tau)$	$F_3(-\mu,\tau)$	$F_5(-\mu,\tau)$	$e^{-\tau/\mu}F_1(\mu,\tau)$	$e^{-\tau/\mu}F_3(\mu,\tau)$	$e^{-\tau/\mu}F_5(\mu,\tau)$
	0	0	0	0	0	0	0
	0.050	0.152216	0.022877	0.011722	0.040346	0.013929	0.007937
	0.075	0.199476	0.032914	0.017011	0.065622	0.021648	0.012236
	0.10	0.238737	0.041987	0.021849	0.093515	0.029459	0.016534
	0.15	0.300010	0.057177	0.030033	0.149930	0.043953	0.024418
	0.20	0.345029	0.068945	0.036424	0.200539	0.056107	0.030972
	0.25	0.379133	0.078123	0.041432	0.243406	0.066014	0.036289
0.40	0.3	0.405707	0.085405	0.045417	0.279233	0.074097	0.040613
	0.4	0.444236	0.096136	0.051305	0.334577	0.086309	0.047130
	0.5	0.470711	0.103613	0.055416	0.374727	0.095002	0.051757
	0.6	0.489974	0.109100	0.058437	0.404941	0.101468	0.055194
	0.7	0.504601	0.113291	0.060747	0.428414	0.106443	0.057837
	0.8	0.516079	0.116594	0.062568	0.447140	0.110402	0.059938
	0.9	0.525343	0.119278	0.064054	0.462409	0.113610	0.061639
	1.0	0.532924	0.121461	0.065255	0.475091	0.116266	0.063047
	0	0	0	0	0	0	0
	0.050	0.152223	0.022879	0.011723	0.035591	0.012909	0.007449
	0.075	0.199594	0.032958	0.017037	0.057724	0.020121	0.011520
	0.10	0.239217	0.042167	0.021953	0.082673	0.027589	0.015681
	0.15	0.301975	0.057911	0.030459	0.135520	0.042010	0.023610
	0.20	0.349006	0.070431	0.037288	0.185430	0.054643	0.030485
	0.25	0.385208	0.080393	0.042751	0.229261	0.065265	0.036231
0.45	0.3	0.413766	0.088417	0.047167	0.266841	0.074124	0.041006
	0.4	0.455710	0.100425	0.053797	0.326377	0.087807	0.048357
	0.5	0.484894	0.108915	0.058497	0.370572	0.097747	0.053682
	0.6	0.506311	0.115208	0.061987	0.404333	0.105238	0.057688
	0.7	0.522675	0.120048	0.064674	0.430841	0.111066	0.060801
	0.8	0.535576	0.123883	0.066805	0.452154	0.115721	0.063286
	0.9	0.546002	0.126993	0.068534	0.469639	0.119521	0.065312
	1.0	0.554602	0.129566	0.069965	0.484229	0.122679	0.066996
	0	0	0	0	0	0	0
	0.050	0.152225	0.022880	0.011724	0.031625	0.011975	0.006991
	0.075	0.199648	0.032979	0.017049	0.051098	0.018690	0.010832
	0.10	0.239476	0.042267	0.022012	0.073318	0.025756	0.014817
	0.15	0.303231	0.058399	0.030746	0.122187	0.039861	0.022656
	0.20	0.351772	0.071505	0.037919	0.170530	0.052716	0.029715
	0.25	0.389649	0.082118	0.043765	0.214468	0.063845	0.035784
0.50	0.30	0.419856	0.090783	0.048558	0.253085	0.073327	0.040933
	0.35	0.444423	0.097943	0.052530	0.286655	0.081394	0.045300
	0.4	0.464750	0.103937	0.055861	0.315805	0.088292	0.049026
	0.5	0.496351	0.113367	0.061114	0.363436	0.099382	0.055003
	0.6	0.519730	0.120423	0.065052	0.400456	0.107884	0.059570
	0.7	0.537698	0.125887	0.068106	0.429674	0.114504	0.063132
	0.8	0.551926	0.130237	0.070540	0.453422	0.119854	0.066002
	0.9	0.563466	0.133781	0.072523	0.473023	0.124245	0.068356
	1.0	0.573011	0.136721	0.074171	0.489458	0.127910	0.070320
	0	0	0	0	0	0	0
	0.050	0.152226	0.022881	0.011724	0.025390	0.010331	0.006163
	0.075	0.199684	0.032994	0.017058	0.040715	0.016134	0.009561
	0.10	0.239695	0.042356	0.022065	0.058327	0.022351	0.013150
	0.15	0.304560	0.058940	0.031068	0.099140	0.035392	0.020553
	0.20	0.355060	0.072845	0.038718	0.142798	0.048099	0.027658
	0.25	0.395316	0.084429	0.045144	0.185064	0.059706	0.034088
0.60	0.30	0.428008	0.094108	0.050542	0.224007	0.070002	0.039759
	0.35	0.454994	0.102256	0.055103	0.259089	0.079036	0.044715
	0.4	0.477597	0.109179	0.058989	0.290403	0.086947	0.049042
	0.5	0.513234	0.120258	0.065226	0.343132	0.100007	0.056164
	0.6	0.539988	0.128692	0.069987	0.385250	0.110246	0.061732
	0.7	0.560774	0.135307	0.073728	0.419380	0.118439	0.066179
	0.8	0.577371	0.140626	0.076740	0.447474	0.125121	0.069801
	0.9	0.590920	0.144991	0.079214	0.470945	0.130664	0.072802
	1.0	0.602187	0.148635	0.081282	0.490815	0.135332	0.075328

TABLE 1-Continued

τ	μ	$F_1(-\mu,\tau)$	$F_3(-\mu,\tau)$	$F_5(-\mu,\tau)$	$e^{-\tau/\mu}F_1(\mu,\tau)$	$e^{-\tau/\mu}F_3(\mu,\tau)$	$e^{-\tau/\mu}F_5(\mu,\tau)$
	0	0	0	0	0	0	0
	0.050	0.152226	0.022881	0.011724	0.020724	0.008942	0.005437
	0.075	0.199692	0.032997	0.017060	0.033030	0.013958	0.008437
	0.10	0.239761	0.042385	0.022082	0.047108	0.019372	0.011632
	0.15	0.305118	0.059181	0.031215	0.080650	0.031094	0.018423
	0.20	0.356688	0.073548	0.039146	0.118800	0.043134	0.025276
	0.25	0.398420	0.085770	0.045959	0.157819	0.054661	0.031763
0.7	0.30	0.432780	0.096170	0.051796	0.195387	0.065275	0.037691
	0.35	0.461484	0.105061	0.056809	0.230402	0.074864	0.043019
	0.4	0.485771	0.112713	0.061139	0.262504	0.083458	0.047777
	0.5	0.524528	0.125141	0.068196	0.318186	0.098015	0.055805
	0.6	0.554000	0.134751	0.073673	0.363988	0.109727	0.062240
	0.7	0.577119	0.142377	0.078029	0.401889	0.119274	0.067473
	0.8	0.595719	0.148562	0.081568	0.433580	0.127170	0.071793
	0.9	0.610994	0.153674	0.084497	0.460378	0.133792	0.075412
	1.0	0.623758	0.157966	0.086959	0.483286	0.139416	0.078482
	0	0	0	0	0	0	0
	0.050	0.152226	0.022881	0.011724	0.017121	0.007761	0.004801
	0.075	0.199694	0.032998	0.017060	0.027163	0.012106	0.007449
	0.10	0.239781	0.042394	0.022088	0.038557	0.016806	0.010279
	0.15	0.305354	0.059288	0.031281	0.068466	0.027237	0.016085
	0.20	0.357505	0.073919	0.039375	0.098681	0.038282	0.022838
	0.25	0.400139	0.086550	0.046442	0.133718	0.049349	0.029162
0.8	0.30	0.435606	0.097452	0.052589	0.168847	0.059894	0.035133
	0.35	0.465515	0.106891	0.057941	0.202669	0.069688	0.040644
	0.4	0.491033	0.115102	0.062617	0.234490	0.078663	0.045671
	0.5	0.532172	0.128612	0.070344	0.291316	0.094255	0.054362
	0.6	0.563805	0.139203	0.076428	0.339434	0.107123	0.061502
	0.7	0.588833	0.147696	0.081321	0.380091	0.117807	0.067413
	0.8	0.609105	0.154641	0.085330	0.414622	0.126767	0.072359
	0.9	0.625845	0.160418	0.088671	0.444179	0.134364	0.076546
	1.0	0.639895	0.165295	0.091495	0.469691	0.140874	0.080129
	0	0	0	0	0	0	0
	0.050	0.152226	0.022881	0.011724	0.014278	0.006752	0.004243
	0.075	0.199694	0.032998	0.017061	0.022601	0.010524	0.006582
	0.10	0.239787	0.042397	0.022090	0.031907	0.014606	0.009084
	0.15	0.305455	0.059336	0.031311	0.054473	0.023719	0.014568
	0.20	0.357918	0.074114	0.039498	0.082075	0.033765	0.020486
	0.25	0.401101	0.087005	0.046728	0.112943	0.044137	0.026503
0.9	0.30	0.437295	0.098252	0.053091	0.145058	0.054334	0.032356
	0.35	0.468042	0.108087	0.058693	0.176942	0.064054	0.037893
	0.4	0.494452	0.116720	0.063634	0.207698	0.073154	0.043047
	0.5	0.537393	0.131084	0.071898	0.264216	0.089357	0.052172
	0.6	0.570729	0.142482	0.078489	0.313464	0.103068	0.059853
	0.7	0.597305	0.151708	0.083843	0.355945	0.114663	0.066324
	0.8	0.618961	0.159309	0.088265	0.392594	0.124523	0.071813
	0.9	0.636932	0.165669	0.091972	0.424348	0.132974	0.076508
	1.0	0.652077	0.171065	0.095122	0.452026	0.140279	0.080560
	0	0	0	0	0	0	0
	0.050	0.152226	0.022881	0.011724	0.011995	0.005886	0.003753
	0.075	0.199694	0.032998	0.017061	0.018913	0.009169	0.005820
	0.10	0.239789	0.042393	0.022090	0.026640	0.012718	0.008032
	0.15	0.305499	0.059357	0.031325	0.045254	0.020689	0.012913
	0.20	0.358128	0.074217	0.039564	0.068463	0.029673	0.018294
	0.25	0.401642	0.087270	0.046897	0.095302	0.039226	0.023918
1.0	0.30	0.438312	0.098751	0.053410	0.124174	0.048886	0.029643
	0.35	0.469638	0.108870	0.059193	0.153680	0.058322	0.034976
	0.4	0.496690	0.117819	0.064335	0.182835	0.067339	0.040140
	0.5	0.540985	0.132848	0.073023	0.237930	0.083785	0.049494
	0.6	0.575654	0.144902	0.080032	0.287311	0.098048	0.057556
	0.7	0.603477	0.154740	0.085776	0.330795	0.110330	0.064468
	0.8	0.626270	0.162900	0.090555	0.368898	0.120921	0.070411
	0.9	0.645270	0.169765	0.094585	0.402316	0.130097	0.075548
	1.0	0.661341	0.175616	0.098025	0.431730	0.138098	0.080018

TABLE 2

τ	G_{11}	G_{11}	G_{12}	G_{12}	G_{13}	G_{13}	G_{14}	G_{14}
0	0	0	0	0	0	0	0	0
0.005	0.168785	0.168811	0.028209	0.028118	0.014244	0.014232	0.009507	0.009501
0.01	0.263256	0.246906	0.049063	0.048819	0.024942	0.024894	0.016664	0.016639
0.02	0.397343	0.364841	0.083120	0.082286	0.042704	0.042515	0.028585	0.028488
0.03	0.496273	0.447813	0.111327	0.109634	0.057683	0.057267	0.038681	0.038464
0.04	0.575484	0.511260	0.135736	0.132959	0.070841	0.070118	0.047584	0.047204
0.05	0.641593	0.561795	0.157345	0.153289	0.082643	0.081537	0.055599	0.055013
0.06	0.698207	0.603022	0.176755	0.171247	0.093369	0.091807	0.062908	0.062075
0.07	0.747557	0.637171	0.194367	0.187252	0.103204	0.101118	0.069633	0.068513
0.08	0.791142	0.665739	0.210469	0.201607	0.112284	0.109610	0.075862	0.074417
0.09	0.830028	0.689788	0.225279	0.214543	0.120711	0.117386	0.081660	0.079853
0.10	0.865001	0.710104	0.238964	0.226240	0.128564	0.124531	0.087080	0.084876
0.11	0.896665	0.727287	0.251662	0.236843	0.135908	0.131111	0.092163	0.089529
0.12	0.925492	0.741807	0.263482	0.246473	0.142795	0.137182	0.096944	0.093846
0.13	0.951860	0.754040	0.274518	0.255229	0.149271	0.142791	0.101451	0.097858
0.14	0.976076	0.764292	0.284847	0.263196	0.155372	0.147979	0.105708	0.101590
0.15	0.998397	0.772816	0.294535	0.270448	0.161131	0.152780	0.109738	0.105066
0.16	1.019034	0.779823	0.303640	0.277049	0.166577	0.157224	0.113557	0.108304
0.17	1.038169	0.785490	0.312212	0.283053	0.171734	0.161339	0.117183	0.111322
0.18	1.055953	0.789970	0.320296	0.288510	0.176624	0.165148	0.120630	0.114134
0.19	1.072521	0.793392	0.327929	0.293463	0.181267	0.168674	0.123909	0.116755
0.20	1.087986	0.795870	0.335147	0.297951	0.185680	0.171934	0.127033	0.119197
0.25	1.151897	0.797144	0.365978	0.314528	0.204792	0.184820	0.140649	0.129079
0.30	1.199245	0.785521	0.389965	0.323535	0.219982	0.193149	0.151578	0.135819
0.35	1.235256	0.766006	0.408958	0.327124	0.232233	0.198065	0.160474	0.140160
0.40	1.263184	0.741645	0.424196	0.326762	0.242224	0.200376	0.167788	0.142638
0.45	1.285172	0.714391	0.436550	0.323491	0.250443	0.200679	0.173852	0.143651
0.5	1.302696	0.685535	0.446652	0.318074	0.257252	0.199425	0.178912	0.143504
0.6	1.328250	0.626250	0.461861	0.302940	0.267681	0.193566	0.186735	0.140643
0.7	1.345309	0.568008	0.472400	0.284483	0.275057	0.184794	0.192335	0.135453
0.8	1.356967	0.512808	0.479829	0.264568	0.280351	0.174383	0.196395	0.128850
0.9	1.365081	0.461539	0.485138	0.244333	0.284192	0.163162	0.199371	0.121451
1.0	1.370811	0.414509	0.488975	0.224475	0.287007	0.151682	0.201570	0.113678

τ	G_{15}	G_{15}	G_{22}	G_{22}	G_{23}	G_{23}	G_{24}	G_{24}
0	0	0	0	0	0	0	0	0
0.005	0.007133	0.007129	0.004846	0.004845	0.002449	0.002449	0.001635	0.001635
0.01	0.012507	0.012491	0.009457	0.009453	0.004814	0.004814	0.003217	0.003217
0.02	0.021470	0.021405	0.018122	0.018099	0.009335	0.009329	0.006251	0.006249
0.03	0.029072	0.028927	0.026170	0.026106	0.013609	0.013593	0.009132	0.009124
0.04	0.035786	0.035531	0.033694	0.033564	0.017665	0.017631	0.011877	0.011859
0.05	0.041840	0.041445	0.040759	0.040535	0.021524	0.021463	0.014498	0.014465
0.06	0.047369	0.046806	0.047415	0.047069	0.025202	0.025104	0.017004	0.016951
0.07	0.052463	0.051706	0.053701	0.053203	0.028713	0.028566	0.019404	0.019326
0.08	0.057188	0.056209	0.059650	0.058970	0.032068	0.031861	0.021706	0.021594
0.09	0.061593	0.060366	0.065291	0.064399	0.035277	0.034999	0.023914	0.023763
0.10	0.065716	0.064217	0.070647	0.069512	0.038351	0.037989	0.026036	0.025838
0.11	0.069587	0.067793	0.075739	0.074331	0.041296	0.040837	0.028074	0.027822
0.12	0.073233	0.071120	0.080586	0.078876	0.044120	0.043552	0.030034	0.029721
0.13	0.076675	0.074221	0.085205	0.083163	0.046830	0.046139	0.031921	0.031537
0.14	0.079930	0.077113	0.089610	0.087207	0.049433	0.048606	0.033737	0.033276
0.15	0.083015	0.079815	0.093815	0.091023	0.051933	0.050957	0.035486	0.034939
0.16	0.085943	0.082339	0.097832	0.094623	0.054335	0.053197	0.037171	0.036531
0.17	0.088725	0.084698	0.101673	0.098019	0.056646	0.055333	0.038795	0.038054
0.18	0.091373	0.086905	0.105347	0.101222	0.058869	0.057367	0.040362	0.039511
0.19	0.093896	0.088968	0.108864	0.104242	0.061008	0.059305	0.041873	0.040904
0.20	0.096302	0.090897	0.112233	0.107089	0.063067	0.061150	0.043331	0.042237
0.25	0.106821	0.098793	0.127110	0.119001	0.072290	0.069113	0.049902	0.048059
0.30	0.115310	0.104313	0.139264	0.127665	0.079987	0.075255	0.055440	0.052657
0.35	0.122253	0.108005	0.149282	0.133752	0.086450	0.079897	0.060133	0.056230
0.40	0.127989	0.110264	0.157597	0.137774	0.091902	0.083292	0.064125	0.058940
0.45	0.132766	0.111386	0.164539	0.140131	0.096520	0.085650	0.067532	0.060920
0.5	0.136768	0.111600	0.170362	0.141142	0.100446	0.087143	0.070449	0.062283
0.6	0.142992	0.109983	0.179416	0.140107	0.106654	0.088078	0.075106	0.063520
0.7	0.147479	0.106472	0.185927	0.136207	0.111212	0.086988	0.078566	0.063249
0.8	0.150754	0.101772	0.190662	0.130485	0.114585	0.084510	0.081154	0.061911
0.9	0.153169	0.096364	0.194136	0.123662	0.117099	0.081106	0.083101	0.059831
1.0	0.154964	0.090583	0.196704	0.116237	0.118984	0.077114	0.084573	0.057253

TABLE 2-Continued

τ	G_{25}	G_{25}	G_{33}	G_{33}	G_{34}	G_{34}	G_{35}	G_{35}
0	0	0	0	0	0	0	0	0
0.005	0.001226	0.001226	0.001238	0.001238	0.000826	0.000826	0.000620	0.000620
0.01	0.002415	0.002415	0.002451	0.002451	0.001638	0.001638	0.001230	0.001230
0.02	0.004696	0.004694	0.004809	0.004808	0.003221	0.003220	0.002420	0.002419
0.03	0.006865	0.006860	0.007080	0.007076	0.004751	0.004749	0.003572	0.003570
0.04	0.008935	0.008923	0.009266	0.009258	0.006231	0.006226	0.004688	0.004685
0.05	0.010915	0.010893	0.011374	0.011357	0.007662	0.007653	0.005769	0.005763
0.06	0.012811	0.012775	0.013407	0.013379	0.009047	0.009032	0.006817	0.006806
0.07	0.014629	0.014576	0.015367	0.015324	0.010388	0.010365	0.007832	0.007816
0.08	0.016375	0.016299	0.017259	0.017196	0.011686	0.011652	0.008817	0.008794
0.09	0.018053	0.017950	0.019086	0.018999	0.012942	0.012895	0.009771	0.009739
0.10	0.019666	0.019531	0.020849	0.020733	0.014160	0.014096	0.010697	0.010654
0.11	0.021219	0.021047	0.022552	0.022403	0.015339	0.015256	0.011595	0.011539
0.12	0.022714	0.022500	0.024198	0.024009	0.016481	0.016376	0.012466	0.012395
0.13	0.024154	0.023892	0.025788	0.025554	0.017588	0.017458	0.013312	0.013223
0.14	0.025543	0.025227	0.027325	0.027041	0.018660	0.018502	0.014132	0.014023
0.15	0.026882	0.026507	0.028812	0.028470	0.019700	0.019509	0.014928	0.014797
0.16	0.028173	0.027735	0.030249	0.029845	0.020708	0.020481	0.015700	0.015545
0.17	0.029420	0.028911	0.031639	0.031166	0.021685	0.021419	0.016450	0.016267
0.18	0.030624	0.030038	0.032984	0.032436	0.022633	0.022323	0.017179	0.016965
0.19	0.031786	0.031119	0.034285	0.033656	0.023552	0.023195	0.017885	0.017639
0.20	0.032909	0.032154	0.035544	0.034828	0.024444	0.024035	0.018572	0.018290
0.25	0.037986	0.036706	0.041262	0.040015	0.028517	0.027794	0.021719	0.021217
0.30	0.042288	0.040344	0.046137	0.044203	0.032025	0.030888	0.024444	0.023650
0.35	0.045950	0.043211	0.050305	0.047536	0.035052	0.033402	0.026807	0.025649
0.40	0.049081	0.045425	0.053880	0.050134	0.037670	0.035413	0.028859	0.027268
0.45	0.051765	0.047081	0.056953	0.052103	0.039936	0.036985	0.030645	0.028554
0.5	0.054072	0.048263	0.059599	0.053530	0.041903	0.038176	0.032201	0.029549
0.6	0.057777	0.049472	0.063857	0.055057	0.045097	0.039605	0.034742	0.030804
0.7	0.060550	0.049495	0.067047	0.055213	0.047519	0.040031	0.036682	0.031277
0.8	0.062637	0.048664	0.069451	0.054374	0.049363	0.039708	0.038170	0.031157
0.9	0.064218	0.047227	0.071270	0.052826	0.050772	0.038836	0.039314	0.030596
1.0	0.065420	0.045373	0.072653	0.050786	0.051852	0.037568	0.040196	0.029710

τ	G_{44}	G_{44}	G_{45}	G_{45}	G_{55}	G_{55}	$E_1^{(2)}$
0	0	0	0	0	0	0	∞
0.005	0.000551	0.000551	0.000414	0.000414	0.000310	0.000310	11.139404
0.01	0.001095	0.001095	0.000822	0.000822	0.000617	0.000617	8.924688
0.02	0.002157	0.002157	0.001620	0.001620	0.001217	0.001217	6.362987
0.03	0.003188	0.003187	0.002397	0.002396	0.001802	0.001802	5.083102
0.04	0.004190	0.004187	0.003152	0.003150	0.002372	0.002370	4.271850
0.05	0.005162	0.005157	0.003886	0.003883	0.002926	0.002924	3.697389
0.06	0.006106	0.006098	0.004600	0.004595	0.003466	0.003463	3.263197
0.07	0.007022	0.007010	0.005295	0.005286	0.003992	0.003987	2.920528
0.08	0.007913	0.007894	0.005970	0.005958	0.004505	0.004496	2.641609
0.09	0.008777	0.008752	0.006627	0.006610	0.005004	0.004992	2.409252
0.10	0.009617	0.009583	0.007266	0.007243	0.005490	0.005474	2.212149
0.11	0.010433	0.010388	0.007888	0.007857	0.005963	0.005942	2.042502
0.12	0.011226	0.011169	0.008492	0.008453	0.006424	0.006397	1.894734
0.13	0.011997	0.011925	0.009081	0.009031	0.006873	0.006840	1.764732
0.14	0.012745	0.012657	0.009653	0.009592	0.007311	0.007270	1.649385
0.15	0.013473	0.013366	0.010210	0.010137	0.007737	0.007687	1.546292
0.16	0.014180	0.014052	0.010752	0.010664	0.008153	0.008093	1.453564
0.17	0.014867	0.014716	0.011279	0.011176	0.008557	0.008486	1.369694
0.18	0.015535	0.015359	0.011792	0.011671	0.008952	0.008869	1.293459
0.19	0.016184	0.015981	0.012292	0.012152	0.009336	0.009239	1.223859
0.20	0.016815	0.016582	0.012778	0.012617	0.009710	0.009599	1.160064
0.25	0.019717	0.019298	0.015020	0.014729	0.011443	0.011241	0.907304
0.30	0.022242	0.021572	0.016981	0.016513	0.012966	0.012639	0.729650
0.35	0.024440	0.023456	0.018696	0.018006	0.014305	0.013821	0.598701
0.40	0.026356	0.024996	0.020199	0.019240	0.015483	0.014807	0.498831
0.45	0.028028	0.026232	0.021517	0.020244	0.016521	0.015620	0.420678
0.5	0.029490	0.027201	0.022672	0.021044	0.017435	0.016277	0.358275
0.6	0.031886	0.028458	0.024579	0.022121	0.018952	0.017190	0.265984
0.7	0.033724	0.028985	0.026052	0.022631	0.020133	0.017663	0.202258
0.8	0.035139	0.028955	0.027193	0.022701	0.021053	0.017790	0.156636
0.9	0.036230	0.028504	0.028079	0.022435	0.021772	0.017650	0.123071
1.0	0.037074	0.027741	0.028768	0.021916	0.022335	0.017304	0.097843

where $\gamma = 0.5772156 \ldots$ is the Euler-Mascheroni constant. The functions of higher order can then be found from F_1 in accordance with the recursion formula

$$F_{j+1}(\tau, \mu) = \mu\left[F_j(\tau, \mu) - \frac{1}{j} + e^{\tau/\mu} E_{j+1}(\tau)\right].$$

The function $G_{n,m}(\tau)$ is expressible in terms of the exponential integrals and the F-functions. We have

$$(m+n-1) G_{n,m}(\tau) = \tau E_n(\tau) E_m(\tau) + F_n(\tau, -1) + F_m(\tau, -1).$$

The function G'_{11} is given by

$$G'_{11}(\tau) = 2\left[E_1(\tau) + (\log \tau + \gamma) E_2(\tau) - \tau E_1^{(2)}(\tau)\right],$$

where

$$E_1^{(2)}(\tau) = \int_\tau^\infty E_1(t) \frac{dt}{t}.$$

(This function, $E_1^{(2)}(\tau)$ is also included in Table 2.) The functions $G'_{n,m}$ of higher order can be derived from G'_{11} by successive applications of the recursion formula

$$\begin{aligned} G'_{n,m}(\tau) &= G'_{n-1,m+1}(\tau) - \frac{1}{n-1} E_{m+1}(\tau) + \frac{1}{m} E_n(\tau) \\ &= G'_{n+1,m-1}(\tau) - \frac{1}{m-1} E_{n+1}(\tau) + \frac{1}{n} E_m(\tau). \end{aligned}$$

The calculations were carried out with eight decimals; the six decimals retained in our tabulations should therefore be reliable.

Finally, we wish to acknowledge that our calculations were enormously facilitated by G. Placzek's extensive tabulations of the exponential integrals.[13]

[13] *The Functions* $E_n(x) = \int_1^\infty e^{-xu} u^{-n} du$ (National Research Council of Canada, Division of Atomic Energy, Doc. No. MT – 1 [1946]).