ON THE RADIATIVE EQUILIBRIUM OF A STELLAR ATMOSPHERE. XXII (*Concluded*)*

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III. SCATTERING IN ACCORDANCE WITH RAYLEIGH'S PHASE FUNCTION

11. *The equations of the problem.*—We have already indicated in Paper XVII, § 6, how the functional equations governing the angular distributions of the reflected and the transmitted radiations from an atmosphere scattering according to a general phase function, expressible as a series in Legendre polynomials, can be reduced to independent systems of functional equations.

In the case of scattering according to Rayleigh's phase function, we can express the reflected and the transmitted intensities in the forms (cf. Paper XIV, eq. [231])

$$\begin{aligned} I(0;\mu,\varphi;\mu_0,\varphi_0) = \frac{3}{32\mu}F[&S^{(0)}(\mu,\mu_0) - 4\mu\mu_0(1-\mu^2)^{\frac{1}{2}}(1-\mu_0^2)^{\frac{1}{2}} \\ &\times S^{(1)}(\mu,\mu_0)\cos(\varphi-\varphi_0) + (1-\mu^2)(1-\mu_0^2)S^{(2)}(\mu,\mu_0)\cos 2(\varphi-\varphi_0)] \end{aligned} \tag{214}$$

and

$$\begin{aligned} I(\tau_1;-\mu,\varphi;\mu_0,\varphi_0) = \frac{3}{32\mu}F[&T^{(0)}(\mu,\mu_0) + 4\mu\mu_0(1-\mu^2)^{\frac{1}{2}}(1-\mu_0^2)^{\frac{1}{2}} \\ &\times T^{(1)}(\mu,\mu_0)\cos(\varphi-\varphi_0) + (1-\mu^2)(1-\mu_0^2)T^{(2)}(\mu,\mu_0)\cos 2(\varphi-\varphi_0)]\,, \end{aligned} \tag{215}$$

and the functions of the different orders (distinguished by the superscripts) satisfy independent systems of equations. Of these systems, the two governing the functions of order one and two are directly reducible to the standard forms considered in Section I. And the terms in the reflected and the transmitted intensities proportional to $\cos(\varphi - \varphi_0)$ and $\cos 2(\varphi - \varphi_0)$ are of exactly the same forms as those given in Paper XXI, equations (223) and (224); only the functions $X^{(1)}$, $Y^{(1)}$, and $X^{(2)}$, $Y^{(2)}$, must now be redefined in terms of the functional equations which they satisfy. These terms require, therefore, no further consideration.

Turning to the functions $S^{(0)}(\mu, \mu_0)$ and $T^{(0)}(\mu, \mu_0)$ of zero order, we find that these functions must be expressible in the forms (cf. Paper XIV, eqs. [241]–[245])

$$\left(\frac{1}{\mu_0}+\frac{1}{\mu}\right)S^{(0)}(\mu,\mu_0) = \tfrac{1}{3}[\psi(\mu)\psi(\mu_0) - \chi(\mu)\chi(\mu_0)] + \tfrac{8}{3}[\phi(\mu)\phi(\mu_0) - \zeta(\mu)\zeta(\mu_0)] \tag{216}$$

and

$$\left(\frac{1}{\mu_0}-\frac{1}{\mu}\right)T^{(0)}(\mu,\mu_0) = \tfrac{1}{3}[\chi(\mu)\psi(\mu_0) - \psi(\mu)\chi(\mu_0)] + \tfrac{8}{3}[\zeta(\mu)\phi(\mu_0) - \phi(\mu)\zeta(\mu_0)]\,, \tag{217}$$

* Sections I and II of this paper have already appeared in *Ap. J.*, **107**, 48, 1948. The remaining Sections III–V of the paper are published here. The numbering of the sections and equations continue from those of the earlier part.

where

$$\psi(\mu) = 3 - \mu^2 + \tfrac{3}{16}\int_0^1 (3 - \mu'^2)\, S^{(0)}(\mu, \mu') \frac{d\mu'}{\mu'}, \tag{218}$$

$$\phi(\mu) = \mu^2 + \tfrac{3}{16}\int_0^1 \mu'^2 S^{(0)}(\mu, \mu') \frac{d\mu'}{\mu'}, \tag{219}$$

$$\chi(\mu) = (3 - \mu^2)\, e^{-\tau_1/\mu} + \tfrac{3}{16}\int_0^1 (3 - \mu'^2)\, T^{(0)}(\mu, \mu') \frac{d\mu'}{\mu'} \tag{220}$$

and

$$\zeta(\mu) = \mu^2 e^{-\tau_1/\mu} + \tfrac{3}{16}\int_0^1 \mu'^2 T^{(0)}(\mu, \mu') \frac{d\mu'}{\mu'}. \tag{221}$$

Further, we must also have

$$\frac{\partial S^{(0)}}{\partial \tau_1} = \tfrac{1}{3}\chi(\mu)\,\chi(\mu_0) + \tfrac{8}{3}\zeta(\mu)\,\zeta(\mu_0) \tag{222}$$

and

$$\left(\frac{1}{\mu_0} - \frac{1}{\mu}\right)\frac{\partial T^{(0)}}{\partial \tau_1} = \frac{1}{\mu_0}\left[\tfrac{1}{3}\psi(\mu)\,\chi(\mu_0) + \tfrac{8}{3}\phi(\mu)\,\zeta(\mu_0)\right] - \frac{1}{\mu}\left[\tfrac{1}{3}\chi(\mu)\,\psi(\mu_0) + \tfrac{8}{3}\zeta(\mu)\,\phi(\mu_0)\right]. \tag{223}$$

Substituting for $S^{(0)}$ and $T^{(0)}$ according to equations (216) and (217) in equations (218)–(221), we obtain the following system of functional equations of fourth order:

$$\begin{aligned}\psi(\mu) = 3 - \mu^2 &+ \tfrac{1}{16}\mu\int_0^1 \frac{3 - \mu'^2}{\mu + \mu'}\left[\psi(\mu)\,\psi(\mu') - \chi(\mu)\,\chi(\mu')\right] d\mu' \\ &+ \tfrac{1}{2}\mu\int_0^1 \frac{3 - \mu'^2}{\mu + \mu'}\left[\phi(\mu)\,\phi(\mu') - \zeta(\mu)\,\zeta(\mu')\right] d\mu',\end{aligned} \tag{224}$$

$$\begin{aligned}\phi(\mu) = \mu^2 &+ \tfrac{1}{16}\mu\int_0^1 \frac{\mu'^2}{\mu + \mu'}\left[\psi(\mu)\,\psi(\mu') - \chi(\mu)\,\chi(\mu')\right] d\mu' \\ &+ \tfrac{1}{2}\mu\int_0^1 \frac{\mu'^2}{\mu + \mu'}\left[\phi(\mu)\,\phi(\mu') - \zeta(\mu)\,\zeta(\mu')\right] d\mu',\end{aligned} \tag{225}$$

$$\begin{aligned}\chi(\mu) = (3 - \mu^2)\, e^{-\tau_1/\mu} &+ \tfrac{1}{16}\mu\int_0^1 \frac{3 - \mu'^2}{\mu - \mu'}\left[\chi(\mu)\,\psi(\mu') - \psi(\mu)\,\chi(\mu')\right] d\mu' \\ &+ \tfrac{1}{2}\mu\int_0^1 \frac{3 - \mu'^2}{\mu - \mu'}\left[\zeta(\mu)\,\phi(\mu') - \phi(\mu)\,\zeta(\mu')\right] d\mu',\end{aligned} \tag{226}$$

and

$$\begin{aligned}\zeta(\mu) = \mu^2 e^{-\tau_1/\mu} &+ \tfrac{1}{16}\mu\int_0^1 \frac{\mu'^2}{\mu - \mu'}\left[\chi(\mu)\,\psi(\mu') - \psi(\mu)\,\chi(\mu')\right] d\mu' \\ &+ \tfrac{1}{2}\mu\int_0^1 \frac{\mu'^2}{\mu - \mu'}\left[\zeta(\mu)\,\phi(\mu') - \phi(\mu)\,\zeta(\mu')\right] d\mu'.\end{aligned} \tag{227}$$

12. *The form of the solution.*—In solving systems of functional equations of the type of equations (224)–(227), we shall be guided by the forms of the solutions obtained in the direct solution of the equations of transfer in a general finite approximation and the correspondence enunciated in theorem 9 between the X- and Y-functions occurring in such approximate solutions and the exact functions defined in terms of functional equations they satisfy.

Accordingly, in the present instance, we shall assume that $S^{(0)}(\mu, \mu')$ and $T^{(0)}(\mu, \mu')$ are of the forms (cf. Paper XXI, eqs. [221] and [222])

$$\begin{aligned}\left(\frac{1}{\mu'}+\frac{1}{\mu}\right) S^{(0)}(\mu, \mu') = {} & X(\mu) X(\mu')[3+c_1(\mu+\mu')+\mu\mu'] \\ & - Y(\mu) Y(\mu')[3-c_1(\mu+\mu')+\mu\mu'] \\ & + c_2(\mu+\mu')[X(\mu) Y(\mu')+Y(\mu) X(\mu')]\end{aligned} \tag{228}$$

and

$$\begin{aligned}\left(\frac{1}{\mu'}-\frac{1}{\mu}\right) T^{(0)}(\mu, \mu') = {} & Y(\mu) X(\mu')[3-c_1(\mu-\mu')-\mu\mu'] \\ & - X(\mu) Y(\mu')[3+c_1(\mu-\mu')-\mu\mu'] \\ & - c_2(\mu-\mu')[X(\mu) X(\mu')+Y(\mu) Y(\mu')],\end{aligned} \tag{229}$$

where c_1 and c_2 are certain constants unspecified for the present, and $X(\mu)$ and $Y(\mu)$ are the *standard solutions* of the equations

$$X(\mu) = 1+\tfrac{3}{16}\mu \int_0^1 \frac{3-\mu'^2}{\mu+\mu'}[X(\mu) X(\mu')-Y(\mu) Y(\mu')]\,d\mu' \tag{230}$$

and

$$Y(\mu) = e^{-\tau_1/\mu}+\tfrac{3}{16}\mu \int_0^1 \frac{3-\mu'^2}{\mu-\mu'}[Y(\mu) X(\mu')-X(\mu) Y(\mu')]\,d\mu', \tag{231}$$

having the property

$$\tfrac{3}{16}\int_0^1 (3-\mu^2) X(\mu)\,d\mu = \tfrac{3}{16}(3\alpha_0-\alpha_2) = 1 \tag{232}$$

and

$$\int_0^1 (3-\mu^2) Y(\mu)\,d\mu = (3\beta_0-\beta_2) = 0 \tag{233}$$

where α_n and β_n have their usual meanings (cf. eq. [11]).

An alternative form of equations (228) and (229) which we shall find useful may be noted here:

$$\begin{aligned}S^{(0)}(\mu, \mu') = \{ & (3-\mu^2)[X(\mu) X(\mu')-Y(\mu) Y(\mu')] \\ & + (\mu+\mu') X(\mu)[(c_1+\mu) X(\mu')+c_2 Y(\mu')] \\ & + (\mu+\mu') Y(\mu)[c_2 X(\mu')+(c_1-\mu) Y(\mu')]\} \frac{\mu\mu'}{\mu+\mu'}\end{aligned} \tag{234}$$

and

$$\begin{aligned}T^{(0)}(\mu, \mu') = \{ & (3-\mu^2)[Y(\mu) X(\mu')-X(\mu) Y(\mu')] \\ & - (\mu-\mu') X(\mu)[(c_1+\mu) Y(\mu')+c_2 X(\mu')] \\ & - (\mu-\mu') Y(\mu)[c_2 Y(\mu')+(c_1-\mu) X(\mu')]\} \frac{\mu\mu'}{\mu-\mu'}.\end{aligned} \tag{235}$$

13. *Verification of the solution and a relation between the constants* c_1 *and* c_2.—The verification that the solutions for $S^{(0)}$ and $T^{(0)}$ have the forms assumed in § 12 will consist in first evaluating ψ, ϕ, χ, and ζ according to equations (218)–(221) and then showing that when the resulting expressions for ψ, ϕ, χ, and ζ are substituted back into

equations (216) and (217) we shall recover the form of the solutions assumed. In general, such a procedure will lead to certain conditions which the constants introduced into the solution (such as c_1 and c_2 in the present instance) must satisfy. We shall see that, in the particular case under discussion, the conditions derived in the manner indicated do not suffice to determine c_1 and c_2 without an ambiguity and an arbitrariness. This is a further example of the nonuniqueness of the solution, in conservative cases, of the functional equations incorporating the invariances of the problem. But, again, an appeal to the integrals of the problem resolves the ambiguity and the arbitrariness.

Our first step, then, is to evaluate ψ, ϕ, χ, and ζ according to equations (218)–(221), when $S^{(0)}(\mu, \mu')$ and $T^{(0)}(\mu, \mu')$ have the forms given by equations (234) and (235). The evaluation of the integrals defining ψ, ϕ, etc., is fairly straightforward if appropriate use is made of the various integral properties of the standard solutions of equations (230) and (231). It may be noted that, in addition to equations (232) and (233), use must also be made of the relations (cf. theorem 4, eqs. [44]–[46])

$$\alpha_0 = 1 + \tfrac{3}{32}\left[3\left(\alpha_0^2 - \beta_0^2\right) - \left(\alpha_1^2 - \beta_1^2\right)\right], \tag{236}$$

$$\begin{aligned}(3-\mu^2)\int_0^1 \frac{X(\mu)\,X(\mu') - Y(\mu)\,Y(\mu')}{\mu+\mu'}\,d\mu' &= \frac{X(\mu)-1}{\tfrac{3}{16}\mu} \\ &\quad + (\alpha_1 - \mu\alpha_0)\,X(\mu) - (\beta_1 - \mu\beta_0)\,Y(\mu)\end{aligned} \tag{237}$$

and

$$\begin{aligned}(3-\mu^2)\int_0^1 \frac{Y(\mu)\,X(\mu') - X(\mu)\,Y(\mu')}{\mu-\mu'}\,d\mu' &= \frac{Y(\mu) - e^{-\tau_1/\mu}}{\tfrac{3}{16}\mu} \\ &\quad + (\beta_1 + \mu\beta_0)\,X(\mu) - (\alpha_1 + \mu\alpha_0)\,Y(\mu)\,.\end{aligned} \tag{238}$$

Evaluating ψ, ϕ, χ, and ζ in the manner indicated, we find that

$$\psi(\mu) = (3 + c_1\mu)\,X(\mu) + c_2\mu\,Y(\mu)\,, \tag{239}$$

$$\chi(\mu) = (3 - c_1\mu)\,Y(\mu) - c_2\mu X(\mu) \tag{240}$$

$$\phi(\mu) = +\mu\left[q_1 X(\mu) + q_2 Y(\mu)\right], \tag{241}$$

and

$$\zeta(\mu) = -\mu\left[q_2 X(\mu) + q_1 Y(\mu)\right] \tag{242}$$

where

$$q_1 = \tfrac{3}{16}\left(c_1\alpha_2 + c_2\beta_2 + 3\alpha_1\right) \tag{243}$$

and

$$q_2 = \tfrac{3}{16}\left(c_1\beta_2 + c_2\alpha_2 - 3\beta_1\right). \tag{244}$$

Using the expressions (239)–(242) for ψ, χ, ϕ, and ζ, we next evaluate $S^{(0)}$ and $T^{(0)}$ according to equations (216) and (217). We find

$$\begin{aligned}\left(\frac{1}{\mu'}+\frac{1}{\mu}\right) S^{(0)}(\mu,\mu') &= X(\mu)\,X(\mu')\left[3 + c_1(\mu+\mu') + \tfrac{1}{3}\left\{c_1^2 - c_2^2 + 8\left(q_1^2 - q_2^2\right)\right\}\mu\mu'\right] \\ &\quad - Y(\mu)\,Y(\mu')\left[3 - c_1(\mu+\mu') + \tfrac{1}{3}\left\{c_1^2 - c_2^2 + 8\left(q_1^2 - q_2^2\right)\right\}\mu\mu'\right] \\ &\quad + c_2(\mu+\mu')\left[X(\mu)\,Y(\mu') + Y(\mu)\,X(\mu')\right]\end{aligned} \tag{245}$$

and

$$\left(\frac{1}{\mu'}-\frac{1}{\mu}\right)T^{(0)}(\mu,\mu')$$
$$\begin{aligned}= Y(\mu)\,X(\mu')\,[3-c_1(\mu-\mu')-\tfrac{1}{3}\{c_1^2-c_2^2+8(q_1^2-q_2^2)\}\mu\mu'] \\ - X(\mu)\,Y(\mu')\,[3+c_1(\mu-\mu')-\tfrac{1}{3}\{c_1^2-c_2^2+8(q_1^2-q_2^2)\}\mu\mu'] \\ - c_2(\mu-\mu')\,[X(\mu)\,X(\mu')+Y(\mu)\,Y(\mu')]\,.\end{aligned} \tag{246}$$

A comparison of equations (245) and (246) and (228) and (229) now shows that, among the constants c_1, c_2, q_1, and q_2, we must require that there exist the relation (cf. Paper XIV, eq. [250])

$$c_1^2-c_2^2+8(q_1^2-q_2^2)=3\,. \tag{247}$$

Substituting for q_1 and q_2 according to equations (243) and (244) in equation (247), we obtain

$$\begin{aligned}32(c_1^2-c_2^2)+9[(c_1+c_2)(\alpha_2+\beta_2)+3(\alpha_1-\beta_1)] \\ \times[(c_1-c_2)(\alpha_2-\beta_2)+3(\alpha_1+\beta_1)]-96=0\,.\end{aligned} \tag{248}$$

After some minor rearranging of the terms, the foregoing equation can be reduced to the form

$$\begin{aligned}[32+9(\alpha_2^2-\beta_2^2)](c_1^2-c_2^2)+27(\alpha_1+\beta_1)(\alpha_2+\beta_2)(c_1+c_2) \\ +27(\alpha_1-\beta_1)(\alpha_2-\beta_2)(c_1-c_2)+81(\alpha_1^2-\beta_1^2)-96=0\,.\end{aligned} \tag{249}$$

On the other hand, according to equations (232), (233), and (236),

$$\begin{aligned}32+9(\alpha_2^2-\beta_2^2) &= 32+(9\alpha_0-16)^2-81\beta_0^2 \\ &= 288(1-\alpha_0)+81(\alpha_0^2-\beta_0^2) \\ &= 27(\alpha_1^2-\beta_1^2)\,.\end{aligned} \tag{250}$$

Equation (249) therefore becomes

$$\begin{aligned}(\alpha_1^2-\beta_1^2)(c_1^2-c_2^2)+(\alpha_1+\beta_1)(\alpha_2+\beta_2)(c_1+c_2) \\ +(\alpha_1-\beta_1)(\alpha_2-\beta_2)(c_1-c_2)+(\alpha_2^2-\beta_2^2)=0\,.\end{aligned} \tag{251}$$

Hence

$$[(\alpha_1+\beta_1)(c_1+c_2)+(\alpha_2-\beta_2)][(\alpha_1-\beta_1)(c_1-c_2)+(\alpha_2+\beta_2)]=0\,. \tag{252}$$

It is apparent that one of the two factors in equation (252) must vanish. But within the framework of equations (224)–(227) it is impossible to decide which of the two it must be; and in either case we shall have only one relation between the two constants c_1 and c_2. The problem is therefore characterized by an ambiguity and an arbitrariness. We shall show in the following section how this can be resolved.

14. *The resolution of the ambiguity and the arbitrariness in the solution.*—It can be readily verified that the problem of diffuse reflection and transmission in accordance with Rayleigh's phase function admits, as in the conservative isotropic case, the flux and the K-integrals. The emergent values of F and K must therefore be given by equations of the form (cf. eqs. [190]–[193])

$$F(0) = \mu_0 F(1+\gamma_1)\,; \qquad F(\tau_1) = \mu_0 F(e^{-\tau_1/\mu_0}+\gamma_1)\,, \tag{253}$$

$$K(0) = \tfrac{1}{4}\mu_0 F(-\mu_0+\gamma_2)\,, \tag{254}$$

and

$$K(\tau_1) = \tfrac{1}{4}\mu_0 F(-\mu_0 e^{-\tau_1/\mu_0}+\gamma_1\tau_1+\gamma_2)\,, \tag{255}$$

where γ_1 and γ_2 are constants.

It is evident that only the azimuth independent terms in the intensity will contribute to F and K. We have, accordingly, to evaluate $F(0)$, $F(\tau_1)$, $K(0)$, and $K(\tau_1)$ for emergent intensities of the forms (cf. eqs. [214], [215], [234], and [235])

$$\begin{aligned} I(0,\mu) = \tfrac{1}{2}\mu_0 F\Big\{ &\tfrac{3}{16}\frac{3-\mu^2}{\mu_0+\mu}[X(\mu_0)X(\mu)-Y(\mu_0)Y(\mu)] \\ &+\tfrac{3}{16}X(\mu_0)[(c_1+\mu)X(\mu)+c_2Y(\mu)] \\ &+\tfrac{3}{16}Y(\mu_0)[c_2X(\mu)+(c_1-\mu)Y(\mu)]\Big\} \end{aligned} \tag{256}$$

and

$$\begin{aligned} I(\tau_1,-\mu) = \tfrac{1}{2}\mu_0 F\Big\{ &\tfrac{3}{16}\frac{3-\mu^2}{\mu_0-\mu}[Y(\mu_0)X(\mu)-X(\mu_0)Y(\mu)] \\ &-\tfrac{3}{16}X(\mu_0)[c_2X(\mu)+(c_1-\mu)Y(\mu)] \\ &-\tfrac{3}{16}Y(\mu_0)[(c_1+\mu)X(\mu)+c_2Y(\mu)]\Big\}\,. \end{aligned} \tag{257}$$

With $I(0,\mu)$ and $I(\tau_1,-\mu)$ given by equations (256) and (257), the integrals defining $F(0)$, $F(\tau_1)$, $K(0)$, and $K(\tau_1)$ can all be evaluated quite simply by using the various relations given in theorem 8 (eqs. [81]–[86]) and remembering that in the present case

$$x_1 = \tfrac{3}{16}(3\alpha_1-\alpha_3) \qquad \text{and} \qquad y_1 = \tfrac{3}{16}(3\beta_1-\beta_3)\,. \tag{258}$$

We thus find

$$F(0) = \mu_0 F\{1+\tfrac{3}{16}X(\mu_0)(c_1\alpha_1+c_2\beta_1+\alpha_2)+\tfrac{3}{16}Y(\mu_0)(c_1\beta_1+c_2\alpha_1-\beta_2)\}\,, \tag{259}$$

$$\begin{aligned} F(\tau_1) = \mu_0 F\{ &e^{-\tau_1/\mu_0}+\tfrac{3}{16}X(\mu_0)(c_1\beta_1+c_2\alpha_1-\beta_2) \\ &+\tfrac{3}{16}Y(\mu_0)(c_1\alpha_1+c_2\beta_1+\alpha_2)\}\,, \end{aligned} \tag{260}$$

$$\begin{aligned} K(0) = \tfrac{1}{4}\mu_0 F\{ &-\mu_0+\tfrac{3}{16}X(\mu_0)(c_1\alpha_2+c_2\beta_2+3\alpha_1) \\ &+\tfrac{3}{16}Y(\mu_0)(c_1\beta_2+c_2\alpha_2-3\beta_1)\}\,, \end{aligned} \tag{261}$$

and

$$\begin{aligned} K(\tau_1) = \tfrac{1}{4}\mu_0 F\{ &-\mu_0 e^{-\tau_1/\mu_0}-\tfrac{3}{16}X(\mu_0)(c_1\beta_2+c_2\alpha_2-3\beta_1) \\ &-\tfrac{3}{16}Y(\mu_0)(c_1\alpha_2+c_2\beta_2+3\alpha_1)\}\,. \end{aligned} \tag{262}$$

Comparing the reflected and the transmitted fluxes given by equations (259) and (260) with those given by the flux integral (eq. [253]), we find that

$$\gamma_1 = \tfrac{3}{16} X(\mu_0)(c_1\alpha_1 + c_2\beta_1 + \alpha_2) + \tfrac{3}{16} Y(\mu_0)(c_1\beta_1 + c_2\alpha_1 - \beta_2) \tag{263}$$

and also that

$$\gamma_1 = \tfrac{3}{16} X(\mu_0)(c_1\beta_1 + c_2\alpha_1 - \beta_2) + \tfrac{3}{16} Y(\mu_0)(c_1\alpha_1 + c_2\beta_1 + \alpha_2)\,. \tag{264}$$

We must therefore require that

$$c_1\alpha_1 + c_2\beta_1 + \alpha_2 = c_1\beta_1 + c_2\alpha_1 - \beta_2\,, \tag{265}$$

or

$$(c_1 - c_2)(\alpha_1 - \beta_1) + \alpha_2 + \beta_2 = 0\,; \tag{266}$$

but this is one of the factors in equation (252). The appeal to the flux integral has therefore decided which of the two factors in equation (252) must be set equal to zero.

In view of equation (266) we can combine equations (263) and (264) to give

$$\gamma_1 = \tfrac{3}{32}[(c_1 + c_2)(\alpha_1 + \beta_1) + (\alpha_2 - \beta_2)][X(\mu_0) + Y(\mu_0)]\,. \tag{267}$$

Next, from equations (254) and (261) we find that

$$\gamma_2 = \tfrac{3}{16} X(\mu_0)(c_1\alpha_2 + c_2\beta_2 + 3\alpha_1) + \tfrac{3}{16} Y(\mu_0)(c_1\beta_2 + c_2\alpha_2 - 3\beta_1)\,. \tag{268}$$

And, finally, from equations (255) and (262) we obtain

$$\gamma_1\tau_1 + \gamma_2 = -\tfrac{3}{16} X(\mu_0)(c_1\beta_2 + c_2\alpha_2 - 3\beta_1) - \tfrac{3}{16} Y(\mu_0)(c_1\alpha_2 + c_2\beta_2 + 3\alpha_1)\,. \tag{269}$$

Now, substituting for γ_1 and γ_2 according to equations (267) and (268) in equation (269), we find

$$[(\alpha_1 + \beta_1)(c_1 + c_2) + (\alpha_2 - \beta_2)]\,\tau_1 = -2\,[(\alpha_2 + \beta_2)(c_1 + c_2) + 3(\alpha_1 - \beta_1)]\,; \tag{270}$$

or, solving for $(c_1 + c_2)$, we have

$$c_1 + c_2 = -\frac{(\alpha_2 - \beta_2)\,\tau_1 + 6(\alpha_1 - \beta_1)}{(\alpha_1 + \beta_1)\,\tau_1 + 2(\alpha_2 + \beta_2)}\,. \tag{271}$$

Since we have already shown that (cf. eq. [266])

$$c_1 - c_2 = -\frac{\alpha_2 + \beta_2}{\alpha_1 - \beta_1}\,, \tag{272}$$

the solution to the problem is completed.

IV. SCATTERING IN ACCORDANCE WITH THE PHASE FUNCTION $\lambda(1 + x\cos\Theta)$

15. *The equations of the problem.*—In the problem of diffuse reflection and transmission according to the phase function $\lambda(1 + x\cos\Theta)(\lambda < 1,\ 1 \geqslant x \geqslant -1)$ we can express the reflected and the transmitted intensities in the following forms (cf. Paper XIV, eqs. [196] and [199]):

$$I(0;\mu,\varphi;\mu_0,\varphi_0)=\frac{\lambda}{4\mu}F[S^{(0)}(\mu,\mu_0)+x(1-\mu^2)^{\frac{1}{2}}(1-\mu_0^2)^{\frac{1}{2}}\times S^{(1)}(\mu,\mu_0)\cos(\varphi-\varphi_0)] \tag{273}$$

and

$$I(\tau_1;-\mu,\varphi;\mu_0,\varphi_0)=\frac{\lambda}{4\mu}F[T^{(0)}(\mu,\mu_0)+x(1-\mu^2)^{\frac{1}{2}}(1-\mu_0^2)^{\frac{1}{2}}\times T^{(1)}(\mu,\mu_0)\cos(\varphi-\varphi_0)]\,. \tag{274}$$

The system of equations governing $S^{(1)}$ and $T^{(1)}$ are directly reducible to the standard forms considered in Section I. And the terms in the emergent intensities proportional to $\cos(\varphi-\varphi_0)$ are of exactly the same forms as those given in Paper XXI, equations (278) and (279); only the functions $X^{(1)}$ and $Y^{(1)}$ must now be redefined in terms of the functional equations which they satisfy.

Turning to the "zero-order" functions $S^{(0)}(\mu,\mu_0)$ and $T^{(0)}(\mu,\mu_0)$, we find that these functions must be expressible in the forms (cf. Paper XIV, eqs. [205]–[209])

$$\left(\frac{1}{\mu_0}+\frac{1}{\mu}\right)S^{(0)}(\mu,\mu_0)=\psi(\mu)\psi(\mu_0)-\chi(\mu)\chi(\mu_0)-x[\phi(\mu)\phi(\mu_0)-\zeta(\mu)\zeta(\mu_0)] \tag{275}$$

and

$$\left(\frac{1}{\mu_0}-\frac{1}{\mu}\right)T^{(0)}(\mu,\mu_0)=\chi(\mu)\psi(\mu_0)-\psi(\mu)\chi(\mu_0)+x[\zeta(\mu)\phi(\mu_0)-\phi(\mu)\zeta(\mu_0)]\,, \tag{276}$$

where ψ, ϕ, χ, and ζ are defined in terms of $S^{(0)}$ and $T^{(0)}$ in the following manner:

$$\psi(\mu)=1+\tfrac{1}{2}\lambda\int_0^1 S^{(0)}(\mu,\mu')\frac{d\mu'}{\mu'}\,, \tag{277}$$

$$\phi(\mu)=\mu-\tfrac{1}{2}\lambda\int_0^1 S^{(0)}(\mu,\mu')\,d\mu'\,, \tag{278}$$

$$\chi(\mu)=e^{-\tau_1/\mu}+\tfrac{1}{2}\lambda\int_0^1 T^{(0)}(\mu,\mu')\frac{d\mu'}{\mu'}\,, \tag{279}$$

and

$$\zeta(\mu)=\mu\,e^{-\tau_1/\mu}+\tfrac{1}{2}\lambda\int_0^1 T^{(0)}(\mu,\mu')\,d\mu'\,. \tag{280}$$

Further, we must also have

$$\frac{\partial S^{(0)}}{\partial\tau_1}=\chi(\mu)\chi(\mu_0)-x\zeta(\mu)\zeta(\mu_0) \tag{281}$$

and

$$\left(\frac{1}{\mu_0}-\frac{1}{\mu}\right)\frac{\partial T^{(0)}}{\partial\tau_1}=\frac{1}{\mu_0}[\psi(\mu)\chi(\mu_0)+x\phi(\mu)\zeta(\mu_0)]-\frac{1}{\mu}[\chi(\mu)\psi(\mu_0)+x\zeta(\mu)\phi(\mu_0)]\,. \tag{282}$$

Substituting for $S^{(0)}$ and $T^{(0)}$ according to equations (275) and (276) in equations (277)–(280), we obtain the following system of functional equations of fourth order:

$$\begin{aligned}\psi(\mu) &= 1+\tfrac{1}{2}\lambda\mu\int_0^1 \frac{d\mu'}{\mu+\mu'}[\psi(\mu)\psi(\mu')-\chi(\mu)\chi(\mu')]\\ &\qquad -\tfrac{1}{2}x\lambda\mu\int_0^1\frac{d\mu'}{\mu+\mu'}[\phi(\mu)\phi(\mu')-\zeta(\mu)\zeta(\mu')]\,,\end{aligned}\tag{283}$$

$$\begin{aligned}\phi(\mu) &= \mu-\tfrac{1}{2}\lambda\mu\int_0^1 \frac{\mu' d\mu'}{\mu+\mu'}[\psi(\mu)\psi(\mu')-\chi(\mu)\chi(\mu')]\\ &\qquad +\tfrac{1}{2}x\lambda\mu\int_0^1\frac{\mu' d\mu'}{\mu+\mu'}[\phi(\mu)\phi(\mu')-\zeta(\mu)\zeta(\mu')]\,,\end{aligned}\tag{284}$$

$$\begin{aligned}\chi(\mu) &= e^{-\tau_1/\mu}+\tfrac{1}{2}\lambda\mu\int_0^1 \frac{d\mu'}{\mu-\mu'}[\chi(\mu)\psi(\mu')-\psi(\mu)\chi(\mu')]\\ &\qquad +\tfrac{1}{2}x\lambda\mu\int_0^1\frac{d\mu'}{\mu-\mu'}[\zeta(\mu)\phi(\mu')-\phi(\mu)\zeta(\mu')]\,,\end{aligned}\tag{285}$$

and

$$\begin{aligned}\zeta(\mu) &= \mu\, e^{-\tau_1/\mu}+\tfrac{1}{2}\lambda\mu\int_0^1 \frac{\mu' d\mu'}{\mu-\mu'}[\chi(\mu)\psi(\mu')-\psi(\mu)\chi(\mu')]\\ &\qquad +\tfrac{1}{2}x\lambda\mu\int_0^1\frac{\mu' d\mu'}{\mu-\mu'}[\zeta(\mu)\phi(\mu')-\phi(\mu)\zeta(\mu')]\,.\end{aligned}\tag{286}$$

16. *The form of the solution.*—The solutions for the reflected and the transmitted intensities in a general finite approximation have been found in Paper XXI (eqs. [276] and [277]). Applying to these solutions the correspondence enunciated in theorem 9, we are led to assume for $S^{(0)}(\mu, \mu_0)$ and $T^{(0)}(\mu, \mu_0)$ the following forms:

$$\begin{aligned}S^{(0)}(\mu,\mu') &= \{X(\mu)X(\mu')[1-x(1-\lambda)c_1(\mu+\mu')-x(1-\lambda)\mu\mu']\\ &\qquad -Y(\mu)Y(\mu')[1+x(1-\lambda)c_1(\mu+\mu')-x(1-\lambda)\mu\mu']\\ &\qquad -x(1-\lambda)c_2(\mu+\mu')[X(\mu)Y(\mu')+Y(\mu)X(\mu')]\}\frac{\mu\mu'}{\mu+\mu'}\end{aligned}\tag{287}$$

and

$$\begin{aligned}T^{(0)}(\mu,\mu') &= \{Y(\mu)X(\mu')[1+x(1-\lambda)c_1(\mu-\mu')+x(1-\lambda)\mu\mu']\\ &\qquad -X(\mu)Y(\mu')[1-x(1-\lambda)c_1(\mu-\mu')+x(1-\lambda)\mu\mu']\\ &\qquad +x(1-\lambda)c_2(\mu-\mu')[X(\mu)X(\mu')+Y(\mu)Y(\mu')]\}\frac{\mu\mu'}{\mu-\mu'}\,,\end{aligned}\tag{288}$$

where c_1 and c_2 are certain constants unspecified for the present and $X(\mu)$ and $Y(\mu)$ are solutions of the equations

$$X(\mu) = 1+\tfrac{1}{2}\lambda\mu\int_0^1\frac{1+x(1-\lambda)\mu'^2}{\mu+\mu'}[X(\mu)X(\mu')-Y(\mu)Y(\mu')]\,d\mu'\tag{289}$$

and

$$Y(\mu) = e^{-\tau_1/\mu}+\tfrac{1}{2}\lambda\mu\int_0^1\frac{1+x(1-\lambda)\mu'^2}{\mu-\mu'}[Y(\mu)X(\mu')-X(\mu)Y(\mu')]\,d\mu'\,.\tag{290}$$

17. *Verification of the solution and the evaluation of the constants in the solution in terms of the moments of* X(μ) *and* Y(μ).—The verification that the solutions for $S^{(0)}(\mu, \mu')$

and $T^{(0)}(\mu, \mu')$ have the forms assumed in § 16, will consist in first evaluating ψ, ϕ, χ, and ζ according to equations (277)–(280); then requiring that, when the resulting expressions for ψ, ϕ, χ, and ζ are substituted back into equations (275) and (276), we shall recover the form of the solutions assumed; and, finally, showing that the various requirements can be met. In the present case it will appear that the procedure outlined makes the solution determinate.

The evaluation of ψ, ϕ, χ, and ζ according to equations (277)–(280) for $S^{(0)}(\mu, \mu')$ and $T^{(0)}(\mu, \mu')$ given by equations (287) and (288) is straightforward if proper use is made of the integral properties of the functions $X(\mu)$ and $Y(\mu)$. Since these functions are defined in terms of the characteristic function

$$\Psi(\mu) = \tfrac{1}{2}\lambda\left[1 + x(1-\lambda)\mu^2\right], \tag{291}$$

we have (cf. theorem 4, eqs. [44]–[46])

$$\alpha_0 = 1 + \tfrac{1}{4}\lambda\left[\alpha_0^2 - \beta_0^2 + x(1-\lambda)(\alpha_1^2 - \beta_1^2)\right], \tag{292}$$

$$\begin{aligned}\left[1 + x(1-\lambda)\mu^2\right]\int_0^1 \frac{X(\mu)X(\mu') - Y(\mu)Y(\mu')}{\mu+\mu'}\,d\mu' &= \frac{X(\mu)-1}{\tfrac{1}{2}\lambda\mu} \\ &- x(1-\lambda)\left[(\alpha_1 - \mu\alpha_0)X(\mu) - (\beta_1 - \mu\beta_0)Y(\mu)\right],\end{aligned} \tag{293}$$

and

$$\begin{aligned}\left[1 + x(1-\lambda)\mu^2\right]\int_0^1 \frac{Y(\mu)X(\mu') - X(\mu)Y(\mu')}{\mu-\mu'}\,d\mu' &= \frac{Y(\mu)-e^{-\tau_1/\mu}}{\tfrac{1}{2}\lambda\mu} \\ &- x(1-\lambda)\left[(\beta_1 + \mu\beta_0)X(\mu) - (\alpha_1 + \mu\alpha_0)Y(\mu)\right].\end{aligned} \tag{294}$$

Evaluating ψ, ϕ, χ, and ζ in the manner indicated, we find

$$\psi(\mu) = (1 - q_0\mu)X(\mu) - p_0\mu Y(\mu), \tag{295}$$

$$\chi(\mu) = (1 + q_0\mu)Y(\mu) + p_0\mu X(\mu), \tag{296}$$

$$\phi(\mu) = \mu\left[q_1 X(\mu) + p_1 Y(\mu)\right], \tag{297}$$

and

$$\zeta(\mu) = \mu\left[p_1 X(\mu) + q_1 Y(\mu)\right], \tag{298}$$

where

$$q_0 = \tfrac{1}{2}x\lambda(1-\lambda)(c_1\alpha_0 + c_2\beta_0 + \alpha_1), \tag{299}$$

$$p_0 = \tfrac{1}{2}x\lambda(1-\lambda)(c_1\beta_0 + c_2\alpha_0 - \beta_1), \tag{300}$$

$$q_1 = 1 + \tfrac{1}{2}\lambda\left[x(1-\lambda)(c_1\alpha_1 + c_2\beta_1) - \alpha_0\right], \tag{301}$$

and

$$p_1 = \tfrac{1}{2}\lambda\left[x(1-\lambda)(c_1\beta_1 + c_2\alpha_1) + \beta_0\right]. \tag{302}$$

Using the expressions (295)–(298) for ψ, χ, ϕ, and ζ in equations (275) and (276) for $S^{(0)}$ and $T^{(0)}$, we obtain

$$\begin{aligned}\left(\frac{1}{\mu'} + \frac{1}{\mu}\right)S^{(0)}(\mu, \mu') &= X(\mu)X(\mu')\left[1 - q_0(\mu+\mu') + \{q_0^2 - p_0^2 - x(q_1^2 - p_1^2)\}\mu\mu'\right] \\ &- Y(\mu)Y(\mu')\left[1 + q_0(\mu+\mu') + \{q_0^2 - p_0^2 - x(q_1^2 - p_1^2)\}\mu\mu'\right] \\ &- p_0(\mu+\mu')\left[X(\mu)Y(\mu') + Y(\mu)X(\mu')\right]\end{aligned} \tag{303}$$

and

$$\left(\frac{1}{\mu'}-\frac{1}{\mu}\right)T^{(0)}(\mu,\mu')$$
$$\begin{aligned}&= Y(\mu)\,X(\mu')\,[1+q_0(\mu-\mu')-\{q_0^2-p_0^2-x(q_1^2-p_1^2)\}\mu\mu']\\&-X(\mu)\,Y(\mu')\,[1-q_0(\mu-\mu')-\{q_0^2-p_0^2-x(q_1^2-p_1^2)\}\mu\mu']\\&+p_0(\mu-\mu')\,[X(\mu)\,X(\mu')+Y(\mu)\,Y(\mu')]\,.\end{aligned} \tag{304}$$

Now, comparing equations (303) and (304) and (287) and (288), we observe that we must have

$$q_0=x(1-\lambda)\,c_1 \quad \text{and} \quad p_0=x(1-\lambda)\,c_2\,; \tag{305}$$

further, we must also require that

$$q_0^2-p_0^2-x(q_1^2-p_1^2)=-x(1-\lambda)\,. \tag{306}$$

According to equations (299), (300), and (305), we have

$$c_1=\tfrac{1}{2}\lambda\,(c_1\alpha_0+c_2\beta_0+\alpha_1) \tag{307}$$

and

$$c_2=\tfrac{1}{2}\lambda\,(c_1\beta_0+c_2\alpha_0-\beta_1)\,. \tag{308}$$

Solving these equations for c_1 and c_2, we obtain

$$c_1=\frac{q_0}{x(1-\lambda)}=\lambda\,\frac{(2-\lambda\alpha_0)\,\alpha_1-\lambda\beta_0\beta_1}{(2-\lambda\alpha_0)^2-\lambda^2\beta_0^2} \tag{309}$$

and

$$c_2=\frac{p_0}{x(1-\lambda)}=\lambda\,\frac{-(2-\lambda\alpha_0)\,\beta_1+\lambda\beta_0\alpha_1}{(2-\lambda\alpha_0)^2-\lambda^2\beta_0^2}\,. \tag{310}$$

Inserting for c_1 and c_2 from equations (309) and (310) in equation (301) for q_1, we find

$$q_1=\tfrac{1}{2}(2-\lambda\alpha_0)\left[1+\lambda\,\frac{x\lambda(1-\lambda)(\alpha_1^2-\beta_1^2)}{(2-\lambda\alpha_0)^2-\lambda^2\beta_0^2}\right]. \tag{311}$$

Using the relation (cf. eq. [292])

$$4(\alpha_0-1)-\lambda(\alpha_0^2-\beta_0^2)=x\lambda(1-\lambda)(\alpha_1^2-\beta_1^2)\,, \tag{312}$$

we can now reduce equation (311) to the form

$$q_1=\frac{2(1-\lambda)(2-\lambda\alpha_0)}{(2-\lambda\alpha_0)^2-\lambda^2\beta_0^2}\,. \tag{313}$$

Similarly,

$$p_1=\frac{2(1-\lambda)\,\lambda\beta_0}{(2-\lambda\alpha_0)^2-\lambda^2\beta_0^2}\,. \tag{314}$$

It remains to verify that equation (306) is valid for q_0, p_0, q_1, and p_1 given by equations (309), (310), (313), and (314). To show that this is the case, we first observe that, according to equations (309) and (310),

$$q_0^2-p_0^2=x^2\lambda^2(1-\lambda)^2\,\frac{\alpha_1^2-\beta_1^2}{(2-\lambda\alpha_0)^2-\lambda^2\beta_0^2}\,, \tag{315}$$

while, according to equations (313) and (314),

$$q_1^2 - p_1^2 = \frac{4(1-\lambda)^2}{(2-\lambda\alpha_0)^2 - \lambda^2\beta_0^2}. \tag{316}$$

Hence

$$q_0^2 - p_0^2 - x(q_1^2 - p_1^2) = \frac{x(1-\lambda)}{(2-\lambda\alpha_0)^2 - \lambda^2\beta_0^2}[x\lambda^2(1-\lambda)(\alpha_1^2 - \beta_1^2) - 4(1-\lambda)]; \tag{317}$$

or, using equation (312), we have

$$\begin{aligned} q_0^2 - p_0^2 - x(q_1^2 - p_1^2) &= \frac{x(1-\lambda)}{(2-\lambda\alpha_0)^2 - \lambda^2\beta_0^2}[4\lambda(\alpha_0 - 1) - \lambda^2(\alpha_0^2 - \beta_0^2) - 4(1-\lambda)] \\ &= \frac{x(1-\lambda)}{(2-\lambda\alpha_0)^2 - \lambda^2\beta_0^2}[\lambda^2\beta_0^2 - 4 + 4\lambda\alpha_0 - \lambda^2\alpha_0^2] \\ &= -x(1-\lambda). \end{aligned} \tag{318}$$

The constants q_0, p_0, q_1, and p_1, as defined by equations (309), (310), (313), and (314), are therefore related, as required.

This completes the verification.

V. RAYLEIGH SCATTERING

18. *The equations of the problem.*—When proper allowance is made for the polarization characteristics of the radiation field, the laws of diffuse reflection and transmission are best formulated in terms of a scattering matrix S and a transmission matrix T (cf. Paper XXI, eqs. [533] and [534]). And, as we have already indicated in Paper XVII (the last paragraph of § 5 on p. 455), the equations governing S and T are of the same forms as equations (85)–(88) of Paper XVII, provided that these equations are interpreted as matrix equations in which a *phase matrix* plays the same role as the *phase function* in the more conventional problems. For the particular case of Rayleigh scattering the phase matrix is explicitly known (cf. Paper XIV, eq. [10]), and the required equations for the field quantities I_l, I_r, U, and V[14] can be written down. However, in view of the form of the solutions for U, V, and the azimuth dependent terms in I_l and I_r found in Paper XXI (eqs. [290]–[303] and [548]–[551]) it is evident that the exact solutions for these terms in the scattering and the transmission matrices must be of identically the same forms: only the various X- and Y-functions occurring in the solutions must be redefined in terms of the exact functional equations which they satisfy. Consequently, it is sufficient to confine our detailed considerations to the azimuth independent terms in I_l and I_r which we shall now regard as the components of a *two-dimensional* vector

$$I = (I_l, I_r). \tag{319}$$[15]

Let

$$F = (F_l, F_r), \tag{320}$$

[14] For a definition of these quantities see Paper XXI, § 15, and the references there given.

[15] Strictly, superscripts (0) should be attached to these and similar azimuth independent quantities describing the diffuse-radiation field. We have suppressed them for the sake of convenience. They should, however, be restored when writing down the complete solution (cf. Paper XXI, § 19, eqs. [540]–[546]).

where πF_l and πF_r are the incident fluxes in the intensities in the meridian plane and at right angles to it,[16] respectively. The reflected and the transmitted intensities can then be expressed in terms of a scattering and a transmission matrix (with two rows and columns) in the forms

$$I(0, \mu) = \frac{3}{16\mu} S(\mu, \mu_0) F \quad \text{and} \quad I(\tau_1, -\mu) = \frac{3}{16\mu} T(\mu, \mu_0) F . \tag{321}$$[15]

The equations governing S and T can be written down in analogy with equations (85)–(88) of Paper XVII by replacing $p(\mu, \varphi; \mu', \varphi')$ by the matrix

$$\tfrac{3}{4} J(\mu, \mu') = \tfrac{3}{4} \begin{pmatrix} 2(1-\mu^2)(1-\mu'^2) + \mu^2\mu'^2 & \mu^2 \\ \mu'^2 & 1 \end{pmatrix} . \tag{322}$$

The resulting equations can be written most compactly by adopting the following notation:

Let the "product" $[A, B]_{\mu,\mu'}$ of two matrices, $A(\mu, \mu')$ and $B(\mu, \mu')$, be defined by the formula

$$[A, B]_{\mu,\mu'} = \frac{3}{8} \int_0^1 A(\mu, \mu'') B(\mu'', \mu') \frac{d\mu''}{\mu''} , \tag{323}$$

where, under the integral sign, the ordinary matrix product is intended. With this product notation, the equations satisfied by S and T take the forms

$$\left(\frac{1}{\mu_0} + \frac{1}{\mu}\right) S + \frac{\partial S}{\partial \tau_1} = J + [J, S] + [S, J] + [[S, J], S] , \tag{324}$$

$$\frac{\partial S}{\partial \tau_1} = \exp\left\{-\tau_1\left(\frac{1}{\mu} + \frac{1}{\mu_0}\right)\right\} J + e^{-\tau_1/\mu}[J, T] + e^{-\tau_1/\mu_0}[T, J] + [[T, J], T] , \tag{325}$$

$$\frac{1}{\mu_0} T + \frac{\partial T}{\partial \tau_1} = e^{-\tau_1/\mu} J + e^{-\tau_1/\mu}[J, S] + [T, J] + [[T, J], S] , \tag{326}$$

and

$$\frac{1}{\mu} T + \frac{\partial T}{\partial \tau_1} = e^{-\tau_1/\mu_0} J + [J, T] + e^{-\tau_1/\mu_0}[S, J] + [[S, J], T] . \tag{327}$$

A discussion of equations (324)–(327) shows that S and T must be expressible in the forms

$$\begin{aligned} \left(\frac{1}{\mu'} + \frac{1}{\mu}\right) S(\mu, \mu') &= \begin{pmatrix} \psi(\mu) & 2^{\frac{1}{2}}\phi(\mu) \\ \chi(\mu) & 2^{\frac{1}{2}}\zeta(\mu) \end{pmatrix} \begin{pmatrix} \psi(\mu') & \chi(\mu') \\ 2^{\frac{1}{2}}\phi(\mu') & 2^{\frac{1}{2}}\zeta(\mu') \end{pmatrix} \\ &\quad - \begin{pmatrix} \xi(\mu) & 2^{\frac{1}{2}}\eta(\mu) \\ \sigma(\mu) & 2^{\frac{1}{2}}\theta(\mu) \end{pmatrix} \begin{pmatrix} \xi(\mu') & \sigma(\mu') \\ 2^{\frac{1}{2}}\eta(\mu') & 2^{\frac{1}{2}}\theta(\mu') \end{pmatrix} \end{aligned} \tag{328}$$

and

$$\begin{aligned} \left(\frac{1}{\mu'} - \frac{1}{\mu}\right) T(\mu, \mu') &= \begin{pmatrix} \xi(\mu) & 2^{\frac{1}{2}}\eta(\mu) \\ \sigma(\mu) & 2^{\frac{1}{2}}\theta(\mu) \end{pmatrix} \begin{pmatrix} \psi(\mu') & \chi(\mu') \\ 2^{\frac{1}{2}}\phi(\mu') & 2^{\frac{1}{2}}\zeta(\mu') \end{pmatrix} \\ &\quad - \begin{pmatrix} \psi(\mu) & 2^{\frac{1}{2}}\phi(\mu) \\ \chi(\mu) & 2^{\frac{1}{2}}\zeta(\mu) \end{pmatrix} \begin{pmatrix} \xi(\mu') & \sigma(\mu') \\ 2^{\frac{1}{2}}\eta(\mu') & 2^{\frac{1}{2}}\theta(\mu') \end{pmatrix} , \end{aligned} \tag{329}$$

[16] These directions are referred in the transverse plane containing the electric and the magnetic vectors.

where

$$\psi(\mu) = \mu^2 + \frac{3}{8}\int_0^1 \frac{d\mu'}{\mu'}[\mu'^2 S_{11}(\mu, \mu') + S_{12}(\mu, \mu')], \tag{330}$$

$$\phi(\mu) = 1 - \mu^2 + \frac{3}{8}\int_0^1 \frac{d\mu'}{\mu'}(1-\mu'^2) S_{11}(\mu, \mu'), \tag{331}$$

$$\chi(\mu) = 1 + \frac{3}{8}\int_0^1 \frac{d\mu'}{\mu'}[\mu'^2 S_{21}(\mu, \mu') + S_{22}(\mu, \mu')], \tag{332}$$

$$\zeta(\mu) = \frac{3}{8}\int_0^1 \frac{d\mu'}{\mu'}(1-\mu'^2) S_{21}(\mu, \mu'), \tag{333}$$

$$\xi(\mu) = \mu^2 e^{-\tau_1/\mu} + \frac{3}{8}\int_0^1 \frac{d\mu'}{\mu'}[\mu'^2 T_{11}(\mu, \mu') + T_{12}(\mu, \mu')], \tag{334}$$

$$\eta(\mu) = (1-\mu^2)\, e^{-\tau_1/\mu} + \frac{3}{8}\int_0^1 \frac{d\mu'}{\mu'}(1-\mu'^2) T_{11}(\mu, \mu'). \tag{335}$$

$$\sigma(\mu) = e^{-\tau_1/\mu} + \frac{3}{8}\int_0^1 \frac{d\mu'}{\mu'}[\mu'^2 T_{21}(\mu, \mu') + T_{22}(\mu, \mu')], \tag{336}$$

and

$$\theta(\mu) = \frac{3}{8}\int_0^1 \frac{d\mu'}{\mu'}(1-\mu'^2) T_{21}(\mu, \mu'). \tag{337}$$

Substituting for S_{11}, etc., according to equations (328) and (329) in equations (330)–(337), we shall obtain a simultaneous system of functional equations of order *eight*. It is, however, not necessary to write down these equations explicitly.

19. *The form of the solution.*—The solutions for $\boldsymbol{S}$ and $\boldsymbol{T}$ in a general finite approximation have already been found in Paper XXI (eqs. [539]–[546]). Applying to these solutions the correspondence enunciated in theorem 9, we are led to assume the following forms for $\boldsymbol{S}$ and $\boldsymbol{T}$:

$$\begin{aligned}\left(\frac{1}{\mu'}+\frac{1}{\mu}\right) S_{11}(\mu, \mu') = 2\{ & X_l(\mu) X_l(\mu')[1+\nu_4(\mu+\mu')+\mu\mu'] \\ & - Y_l(\mu) Y_l(\mu')[1-\nu_4(\mu+\mu')+\mu\mu'] \\ & -\nu_3(\mu+\mu')[X_l(\mu) Y_l(\mu') + Y_l(\mu) X_l(\mu')]\},\end{aligned} \tag{338}$$

$$\begin{aligned}\left(\frac{1}{\mu'}+\frac{1}{\mu}\right) S_{12}(\mu, \mu') = (\mu+\mu')\{ & \nu_1[Y_l(\mu) X_r(\mu') + X_l(\mu) Y_r(\mu')] \\ & -\nu_2[X_l(\mu) X_r(\mu') + Y_l(\mu) Y_r(\mu')] \\ & +Q(\nu_2-\nu_1)\mu'[X_l(\mu)+Y_l(\mu)][X_r(\mu')-Y_r(\mu')]\},\end{aligned} \tag{339}$$

$$\begin{aligned}\left(\frac{1}{\mu'}+\frac{1}{\mu}\right) S_{21}(\mu, \mu') = (\mu+\mu')\{ & \nu_1[X_r(\mu) Y_l(\mu') + Y_r(\mu) X_l(\mu')] \\ & -\nu_2[X_r(\mu) X_l(\mu') + Y_r(\mu) Y_l(\mu')] \\ & +Q(\nu_2-\nu_1)\mu[X_r(\mu)-Y_r(\mu)][X_l(\mu')+Y_l(\mu')]\},\end{aligned} \tag{340}$$

$$
\begin{aligned}
\left(\frac{1}{\mu'}+\frac{1}{\mu}\right) S_{22}(\mu,\mu') = {} & X_r(\mu)\,X_r(\mu')\,[1-u_4(\mu+\mu')+u_5\mu\mu'] \\
& - Y_r(\mu)\,Y_r(\mu')\,[1+u_4(\mu+\mu')+u_5\mu\mu'] \\
& + u_3(\mu+\mu')\,[X_r(\mu)\,Y_r(\mu')+Y_r(\mu)\,X_r(\mu')] \\
& - Qu_5\mu\mu'(\mu+\mu')\,[X_r(\mu)-Y_r(\mu)]\,[X_r(\mu')-Y_r(\mu')] \\
& + Q(u_4-u_3)\,\{\mu^2[X_r(\mu)-Y_r(\mu)]\,[X_r(\mu')+Y_r(\mu')] \\
& + \mu'^2[X_r(\mu)+Y_r(\mu)]\,[X_r(\mu')-Y_r(\mu')]\},
\end{aligned} \tag{341}
$$

$$
\begin{aligned}
\left(\frac{1}{\mu'}-\frac{1}{\mu}\right) T_{11}(\mu,\mu') = {} & 2\{Y_l(\mu)\,X_l(\mu')\,[1-\nu_4(\mu-\mu')-\mu\mu'] \\
& - X_l(\mu)\,Y_l(\mu')\,[1+\nu_4(\mu-\mu')-\mu\mu'] \\
& + \nu_3(\mu-\mu')\,[X_l(\mu)\,X_l(\mu')+Y_l(\mu)\,Y_l(\mu')]\},
\end{aligned} \tag{342}
$$

$$
\begin{aligned}
\left(\frac{1}{\mu'}-\frac{1}{\mu}\right) T_{12}(\mu,\mu') = {} & (\mu-\mu')\,\{\nu_2[X_l(\mu)\,Y_r(\mu')+Y_l(\mu)\,X_r(\mu')] \\
& - \nu_1[X_l(\mu)\,X_r(\mu')+Y_l(\mu)\,Y_r(\mu')] \\
& - Q(\nu_2-\nu_1)\,\mu'\,[X_l(\mu)+Y_l(\mu)]\,[X_r(\mu')-Y_r(\mu')]\},
\end{aligned} \tag{343}
$$

$$
\begin{aligned}
\left(\frac{1}{\mu'}-\frac{1}{\mu}\right) T_{21}(\mu,\mu') = {} & (\mu-\mu')\,\{\nu_2[X_r(\mu)\,Y_l(\mu')+Y_r(\mu)\,X_l(\mu')] \\
& - \nu_1[X_r(\mu)\,X_l(\mu')+Y_r(\mu)\,Y_l(\mu')] \\
& - Q(\nu_2-\nu_1)\,\mu\,[X_r(\mu)-Y_r(\mu)]\,[X_l(\mu')+Y_l(\mu')]\},
\end{aligned} \tag{344}
$$

and

$$
\begin{aligned}
\left(\frac{1}{u'}-\frac{1}{\mu}\right) T_{22}(\mu,\mu') = {} & Y_r(\mu)\,X_r(\mu')\,[1+u_4(\mu-\mu')-u_5\mu\mu'] \\
& - X_r(\mu)\,Y_r(\mu')\,[1-u_4(\mu-\mu')-u_5\mu\mu'] \\
& - u_3(\mu-\mu')\,[X_r(\mu)\,X_r(\mu')+Y_r(\mu)\,Y_r(\mu')] \\
& + Qu_5\mu\mu'(\mu-\mu')\,[X_r(\mu)-Y_r(\mu)]\,[X_r(\mu')-Y_r(\mu')] \\
& - Q(u_4-u_3)\,\{\mu^2[X_r(\mu)-Y_r(\mu)]\,[X_r(\mu')+Y_r(\mu')] \\
& - \mu'^2[X_r(\mu)+Y_r(\mu)]\,[X_r(\mu')-Y_r(\mu')]\},
\end{aligned} \tag{345}
$$

where ν_1, ν_2, ν_3, ν_4, u_3, u_4, and Q are certain constants, unspecified for the present;

$$
u_5 = 1 + 2Q(u_4-u_3)\,; \tag{346}
$$
[17]

$X_r(\mu)$ and $Y_r(\mu)$ are the solutions of the equations

$$
X_r(\mu) = 1 + \tfrac{3}{8}\mu \int_0^1 \frac{1-\mu'^2}{\mu+\mu'}\,[X_r(\mu)\,X_r(\mu') - Y_r(\mu)\,Y_r(\mu')]\,d\mu' \tag{347}
$$

and

$$
Y_r(\mu) = e^{-\tau_1/\mu} + \tfrac{3}{8}\mu \int_0^1 \frac{1-\mu'^2}{\mu-\mu'}\,[Y_r(\mu)\,X_r(\mu') - X_r(\mu)\,Y_r(\mu')]\,d\mu'\,; \tag{348}
$$

[17] Cf. Paper XXI, eqs. (512)–(514).

and $X_l(\mu)$ and $Y_l(\mu)$ are the *standard solutions* of the equations

$$X_l(\mu) = 1 + \tfrac{3}{4}\mu \int_0^1 \frac{1-\mu'^2}{\mu+\mu'}[X_l(\mu)X_l(\mu') - Y_l(\mu)Y_l(\mu')]\,d\mu' \tag{349}$$

and

$$Y_l(\mu) = e^{-\tau_1/\mu} + \tfrac{3}{4}\mu \int_0^1 \frac{1-\mu'^2}{\mu-\mu'}[Y_l(\mu)X_l(\mu') - X_l(\mu)Y_l(\mu')]\,d\mu', \tag{350}$$

having the property

$$\tfrac{3}{4}\int_0^1 (1-\mu^2)X_l(\mu)\,d\mu = \tfrac{3}{4}(\alpha_0 - \alpha_2) = 1 \tag{351}$$

and

$$\int_0^1 (1-\mu^2)Y_l(\mu)\,d\mu = (\beta_0 - \beta_2) = 0\,. \tag{352}$$

For the purposes of the various evaluations in §§ 20 and 21, it is convenient to have equations (338)–(345) re-written in the following forms:

$$\begin{aligned} S_{11}(\mu,\mu') = 2\mu\mu' \Big\{ & \frac{1-\mu^2}{\mu+\mu'}[X_l(\mu)X_l(\mu') - Y_l(\mu)Y_l(\mu')] \\ & + X_l(\mu)[(\nu_4+\mu)X_l(\mu') - \nu_3 Y_l(\mu')] \\ & + Y_l(\mu)[-\nu_3 X_l(\mu') + (\nu_4-\mu)Y_l(\mu')]\Big\}, \end{aligned} \tag{353}$$

$$\begin{aligned} S_{12}(\mu,\mu') = \mu\mu'\{ & \nu_1[X_l(\mu)Y_r(\mu') + Y_l(\mu)X_r(\mu')] \\ & - \nu_2[X_l(\mu)X_r(\mu') + Y_l(\mu)Y_r(\mu')] \\ & + Q(\nu_2-\nu_1)\mu'[X_l(\mu)+Y_l(\mu)][X_r(\mu')-Y_r(\mu')]\}, \end{aligned} \tag{354}$$

$$\begin{aligned} S_{21}(\mu,\mu') = \mu\mu'\{ & \nu_1[X_r(\mu)Y_l(\mu') + Y_r(\mu)X_l(\mu')] \\ & - \nu_2[X_r(\mu)X_l(\mu') + Y_r(\mu)Y_l(\mu')] \\ & + Q(\nu_2-\nu_1)\mu[X_r(\mu)-Y_r(\mu)][X_l(\mu')+Y_l(\mu')]\} \end{aligned} \tag{355}$$

$$\begin{aligned} S_{22}(\mu,\mu') = \mu\mu'\Big\{ & \frac{1-\mu^2}{\mu+\mu'}[X_r(\mu)X_r(\mu') - Y_r(\mu)Y_r(\mu')] \\ & + X_r(\mu)[(-u_4+\mu)X_r(\mu') + u_3 Y_r(\mu')] \\ & + Y_r(\mu)[u_3 X_r(\mu') - (u_4+\mu)Y_r(\mu')] \\ & - Qu_5\mu\mu'[X_r(\mu)-Y_r(\mu)][X_r(\mu')-Y_r(\mu')] \\ & + Q(u_4-u_3)(\mu+\mu')[X_r(\mu)X_r(\mu') - Y_r(\mu)Y_r(\mu')] \\ & + Q(u_4-u_3)(\mu-\mu')[X_r(\mu)Y_r(\mu') - Y_r(\mu)X_r(\mu')]\Big\}, \end{aligned} \tag{356}$$ [18]

$$\begin{aligned} T_{11}(\mu,\mu') = 2\mu\mu'\Big\{ & \frac{1-\mu^2}{\mu-\mu'}[Y_l(\mu)X_l(\mu') - X_l(\mu)Y_l(\mu')] \\ & - X_l(\mu)[-\nu_3 X_l(\mu') + (\nu_4+\mu)Y_l(\mu')] \\ & - Y_l(\mu)[(\nu_4-\mu)X_l(\mu') - \nu_3 Y_l(\mu')]\Big\}, \end{aligned} \tag{357}$$

[18] The reduction of eqs. (341) and (345) to the forms (356) and (360) requires the use of eq. (346).

$$T_{12}(\mu,\mu') = \mu\mu'\{\nu_2[X_l(\mu)\,Y_r(\mu') + Y_l(\mu)\,X_r(\mu')] \\ -\nu_1[X_l(\mu)\,X_r(\mu') + Y_l(\mu)\,Y_r(\mu')] \\ -Q(\nu_2-\nu_1)\,\mu'[X_l(\mu)+Y_l(\mu)][X_r(\mu')-Y_r(\mu')]\}, \tag{358}$$

$$T_{21}(\mu,\mu') = \mu\mu'\{\nu_2[X_r(\mu)\,Y_l(\mu') + Y_r(\mu)\,X_l(\mu')] \\ -\nu_1[X_r(\mu)\,X_l(\mu') + Y_r(\mu)\,Y_l(\mu')] \\ -Q(\nu_2-\nu_1)\,\mu[X_r(\mu)-Y_r(\mu)][X_l(\mu')+Y_l(\mu')]\}, \tag{359}$$

and

$$T_{22}(\mu,\mu') = \mu\mu'\Big\{\frac{1-\mu^2}{\mu-\mu'}[Y_r(\mu)\,X_r(\mu') - X_r(\mu)\,Y_r(\mu')] \\ - X_r(\mu)[u_3X_r(\mu') + (-u_4+\mu)\,Y_r(\mu')] \\ - Y_r(\mu)[-(u_4+\mu)\,X_r(\mu') + u_3Y_r(\mu')] \\ +Qu_5\mu\mu'[X_r(\mu)-Y_r(\mu)][X_r(\mu')-Y_r(\mu')] \\ -Q(u_4-u_3)(\mu-\mu')[X_r(\mu)\,Y_r(\mu') - Y_r(\mu)\,X_r(\mu')] \\ -Q(u_4-u_3)(\mu+\mu')[X_r(\mu)\,X_r(\mu') - Y_r(\mu)\,Y_r(\mu')]\Big\}. \tag{360}$$[18]

20. *The verification of the solution and the expression of the constants* ν_1, ν_2, ν_3, ν_4, u_3, *and* u_4 *in terms of the moments of* $X_l(\mu)$, $Y_l(\mu)$, $X_r(\mu)$, *and* $Y_r(\mu)$ *and a single arbitrary constant* Q.—We shall first evaluate ψ, ϕ, χ, ζ, ξ, η, σ, and θ according to equations (330)–(337) for S and T given by equations (353)–(360); then require that, when the resulting expressions for ψ, ϕ, etc., are substituted back into equations (328) and (329), we shall recover the form of the solutions assumed. As we should expect, this procedure will lead to several conditions[19] among the constants ν_1, ν_2, ν_3, ν_4, u_3, u_4, and Q introduced into the solution. We shall show that all these conditions can be met and that six of the constants (ν_1, ν_2, ν_3, ν_4, u_3, and u_4) can be expressed in terms of Q and the various moments of X_l, Y_l, X_r, and Y_r. The constant Q itself will be found to be left arbitrary. This is a further example of the one-parametric nature of the solutions of the functional equations incorporating the invariances of the problem in conservative cases. In § 21 we shall then finally show how this last arbitrariness in the solutions can be removed by appealing to the K-integrals of the problem.

The evaluation of ψ, ϕ, etc., according to equations (330)–(337) for S and T given by equations (353)–(360) is straightforward if proper use is made of the various integral properties of the functions X_l, Y_l, X_r, and Y_r. In addition to equations (351) and (352), use must also be made of the following relations (cf. theorem 4, eqs. [44]–[46]):

$$\alpha_0 = 1 + \tfrac{3}{8}[(\alpha_0^2-\beta_0^2) - (\alpha_1^2-\beta_1^2)], \tag{361}$$

$$(1-\mu^2)\int_0^1 \frac{X_l(\mu)\,X_l(\mu') - Y_l(\mu)\,Y_l(\mu')}{\mu+\mu'}\,d\mu' = \frac{X_l(\mu)-1}{\frac{3}{4}\mu} \\ + (\alpha_1-\mu\alpha_0)\,X_l(\mu) - (\beta_1-\mu\beta_0)\,Y_l(\mu), \tag{362}$$

[19] Actually, we shall see that there are twelve of them.

$$(1-\mu^2)\int_0^1 \frac{Y_l(\mu)\,X_l(\mu')-X_l(\mu)\,Y_l(\mu')}{\mu-\mu'}\,d\mu'=\frac{Y_l(\mu)-e^{-\tau_1/\mu}}{\frac{3}{4}\mu}+(\beta_1+\mu\beta_0)\,X_l(\mu)-(\alpha_1+\mu\alpha_0)\,Y_l(\mu)\,, \tag{363}$$

$$A_0=1+\tfrac{3}{16}\left[(A_0^2-B_0^2)-(A_1^2-B_1^2)\right], \tag{364}$$

$$(1-\mu^2)\int_0^1 \frac{X_r(\mu)\,X_r(\mu')-Y_r(\mu)\,Y_r(\mu')}{\mu+\mu'}\,d\mu'=\frac{X_r(\mu)-1}{\frac{3}{8}\mu}+(A_1-\mu A_0)\,X_r(\mu)-(B_1-\mu B_0)\,Y_r(\mu)\,, \tag{365}$$

and

$$(1-\mu^2)\int_0^1 \frac{Y_r(\mu)\,X_r(\mu')-X_r(\mu)\,Y_r(\mu')}{\mu-\mu'}\,d\mu'=\frac{Y_r(\mu)-e^{-\tau_1/\mu}}{\frac{3}{8}\mu}+(B_1+\mu B_0)\,X_r(\mu)-(A_1+\mu A_0)\,Y_r(\mu)\,, \tag{366}$$

where (cf. eq. [11])

$$\alpha_n=\int_0^1 X_l(\mu)\,\mu^n d\mu\,,\qquad \beta_n=\int_0^1 Y_l(\mu)\,\mu^n d\mu\,,$$
$$A_n=\int_0^1 X_r(\mu)\,\mu^n d\mu\,,\quad\text{and}\quad B_n=\int_0^1 Y_r(\mu)\,\mu^n d\mu\,. \tag{367}$$

Evaluating ψ, ϕ, etc., in the manner indicated, we find that

$$\psi(\mu)=+\mu\left[q_1X_l(\mu)+q_2Y_l(\mu)\right], \tag{368}$$

$$\xi(\mu)=-\mu\left[q_2X_l(\mu)+q_1Y_l(\mu)\right], \tag{369}$$

$$\phi(\mu)=(1+\nu_4\mu)\,X_l(\mu)-\nu_3\mu\,Y_l(\mu)\,, \tag{370}$$

$$\eta(\mu)=(1-\nu_4\mu)\,Y_l(\mu)+\nu_3\mu X_l(\mu)\,, \tag{371}$$

$$\chi(\mu)=(1+p_1\mu)\,X_r(\mu)+p_2\mu\,Y_r(\mu)-t\mu^2\left[X_r(\mu)-Y_r(\mu)\right], \tag{372}$$

$$\sigma(\mu)=(1-p_1\mu)\,Y_r(\mu)-p_2\mu X_r(\mu)+t\mu^2\left[X_r(\mu)-Y_r(\mu)\right], \tag{373}$$

$$\zeta(\mu)=-\tfrac{1}{2}\mu\left[\nu_2X_r(\mu)-\nu_1Y_r(\mu)\right]+\tfrac{1}{2}Q(\nu_2-\nu_1)\,\mu^2\left[X_r(\mu)-Y_r(\mu)\right], \tag{374}$$

and

$$\theta(\mu)=+\tfrac{1}{2}\mu\left[-\nu_1X_r(\mu)+\nu_2Y_r(\mu)\right]-\tfrac{1}{2}Q(\nu_2-\nu_1)\,\mu^2\left[X_r(\mu)-Y_r(\mu)\right], \tag{375}$$

where

$$q_1=\tfrac{3}{4}\left[\alpha_2\nu_4-\beta_0\nu_3+\alpha_1+\tfrac{1}{2}B_0\nu_1-\tfrac{1}{2}A_0\nu_2+\tfrac{1}{2}Q(\nu_2-\nu_1)(A_1-B_1)\right], \tag{376}$$

$$q_2=\tfrac{3}{4}\left[\beta_0\nu_4-\alpha_2\nu_3-\beta_1+\tfrac{1}{2}A_0\nu_1-\tfrac{1}{2}B_0\nu_2+\tfrac{1}{2}Q(\nu_2-\nu_1)(A_1-B_1)\right], \tag{377}$$

$$p_1=\tfrac{3}{8}\left[+A_1-A_0u_4+B_0u_3+Q(u_4-u_3)(A_1-B_1)+\beta_0\nu_1-\alpha_2\nu_2\right], \tag{378}$$

$$p_2=\tfrac{3}{8}\left[-B_1-B_0u_4+A_0u_3+Q(u_4-u_3)(A_1-B_1)+\alpha_2\nu_1-\beta_0\nu_2\right], \tag{379}$$

and

$$t=\tfrac{3}{8}Q\left[-(\alpha_2+\beta_0)(\nu_2-\nu_1)-(A_0+B_0)(u_4-u_3)+u_5(A_1-B_1)\right]. \tag{380}$$

We now substitute for ψ, ϕ, etc., according to equations (368)–(375) in equations (328) and (329) and compare them with the solutions (353)–(360), which were originally assumed in the evaluation of ψ, ϕ, etc.

Considering, first, T_{11}, we have

$$\left(\frac{1}{\mu'}-\frac{1}{\mu}\right)T_{11}(\mu,\mu')=\xi(\mu)\psi(\mu')-\psi(\mu)\xi(\mu')+2[\eta(\mu)\phi(\mu')-\phi(\mu)\eta(\mu')]$$
$$=2\{Y_l(\mu)X_l(\mu')[1-\nu_4(\mu-\mu')-\{\nu_4^2-\nu_3^2+\tfrac{1}{2}(q_1^2-q_2^2)\}\mu\mu']$$
$$-X_l(\mu)Y_l(\mu')[1+\nu_4(\mu-\mu')-\{\nu_4^2-\nu_3^2+\tfrac{1}{2}(q_1^2-q_2^2)\}\mu\mu']$$
$$+\nu_3(\mu-\mu')[X_l(\mu)X_l(\mu')+Y_l(\mu)Y_l(\mu')]\}. \tag{381}$$

Comparing this with equation (342), we conclude that we must have

$$\nu_4^2-\nu_3^2+\tfrac{1}{2}(q_1^2-q_2^2)=1\,. \tag{382}$$

The consideration of S_{11} leads to the same condition, (382).

Considering, next, $S_{21}(\mu,\mu')$, we have

$$\left(\frac{1}{\mu'}+\frac{1}{\mu}\right)S_{21}(\mu,\mu')=\chi(\mu)\psi(\mu')-\sigma(\mu)\xi(\mu')+2[\zeta(\mu)\phi(\mu')-\theta(\mu)\eta(\mu')]$$
$$=X_r(\mu)Y_l(\mu')[+\nu_1\mu+q_2\mu'+\{p_1q_2-p_2q_1+\nu_2\nu_3-\nu_1\nu_4-Q(\nu_2-\nu_1)\}\mu\mu']$$
$$+Y_r(\mu)X_l(\mu')[+\nu_1\mu+q_2\mu'-\{p_1q_2-p_2q_1+\nu_2\nu_3-\nu_1\nu_4-Q(\nu_2-\nu_1)\}\mu\mu']$$
$$+X_r(\mu)X_l(\mu')[-\nu_2\mu+q_1\mu'+\{p_1q_1-p_2q_2+\nu_1\nu_3-\nu_2\nu_4-Q(\nu_2-\nu_1)\}\mu\mu']$$
$$+Y_r(\mu)Y_l(\mu')[-\nu_2\mu+q_1\mu'-\{p_1q_1-p_2q_2+\nu_1\nu_3-\nu_2\nu_4-Q(\nu_2-\nu_1)\}\mu\mu']$$
$$+Q(\nu_2-\nu_1)\mu(\mu+\mu')[X_r(\mu)-Y_r(\mu)][X_l(\mu')+Y_l(\mu')]$$
$$+\mu^2\mu'[Q(\nu_2-\nu_1)(\nu_4+\nu_3)+t(q_2-q_1)][X_r(\mu)-Y_r(\mu)][X_l(\mu')-Y_l(\mu')]\,. \tag{383}$$

Comparing equations (340) and (383), we find that we must have

$$q_2=\nu_1\,;\qquad q_1=-\nu_2\,, \tag{384}$$
$$Q(\nu_2-\nu_1)(\nu_4+\nu_3)+t(q_2-q_1)=0\,, \tag{385}$$
$$p_1q_2-p_2q_1+\nu_2\nu_3-\nu_1\nu_4=Q(\nu_2-\nu_1)\,, \tag{386}$$

and

$$p_1q_1-p_2q_2+\nu_1\nu_3-\nu_2\nu_4=Q(\nu_2-\nu_1)\,. \tag{387}$$

The consideration of the other cross-terms, S_{12}, T_{12}, and T_{21}, leads to the same conditions as do equations (384)–(387).

Finally, considering $S_{22}(\mu,\mu')$, we have

$$\left(\frac{1}{\mu'}+\frac{1}{\mu}\right)S_{22}(\mu,\mu')=\chi(\mu)\chi(\mu')-\sigma(\mu)\sigma(\mu')+2[\zeta(\mu)\zeta(\mu')-\theta(\mu)\theta(\mu')]$$
$$=X_r(\mu)X_r(\mu')[1+p_1(\mu+\mu')+\{p_1^2-p_2^2+\tfrac{1}{2}(\nu_2^2-\nu_1^2)\}\mu\mu']$$
$$-Y_r(\mu)Y_r(\mu')[1-p_1(\mu+\mu')+\{p_1^2-p_2^2+\tfrac{1}{2}(\nu_2^2-\nu_1^2)\}\mu\mu']$$
$$+p_2(\mu+\mu')[X_r(\mu)Y_r(\mu')+Y_r(\mu)X_r(\mu')] \tag{388}$$
$$-\mu\mu'(\mu+\mu')[(p_1-p_2)t+\tfrac{1}{2}Q(\nu_2^2-\nu_1^2)][X_r(\mu)-Y_r(\mu)][X_r(\mu')-Y_r(\mu')]$$
$$-t\{\mu^2[X_r(\mu)-Y_r(\mu)][X_r(\mu')+Y_r(\mu')]$$
$$+\mu'^2[X_r(\mu)+Y_r(\mu)][X_r(\mu')-Y_r(\mu')]\}.$$

From equations (341) and (388) we now obtain the further conditions

$$p_1 = -u_4 ; \qquad p_2 = u_3 , \tag{389}$$

$$p_1^2 - p_2^2 + \tfrac{1}{2}(\nu_2^2 - \nu_1^2) = u_5 = 1 + 2Q(u_4 - u_3) , \tag{390}$$

$$(p_1 - p_2)\, t + \tfrac{1}{2}Q(\nu_2^2 - \nu_1^2) = Qu_5 , \tag{391}$$

and

$$t = -Q(u_4 - u_3) . \tag{392}$$

The consideration of $T_{22}(\mu, \mu')$ leads to the same set of conditions as the foregoing.

Collecting all the conditions among the constants that we have found and combining them with equations (376)–(380), we have

$$\nu_1 = q_2 = \tfrac{3}{4}[-\beta_1 + \beta_0\nu_4 - a_2\nu_3 + \tfrac{1}{2}A_0\nu_1 - \tfrac{1}{2}B_0\nu_2 + \tfrac{1}{2}Q(\nu_2 - \nu_1)(A_1 - B_1)] \tag{393}$$

$$-\nu_2 = q_1 = \tfrac{3}{4}[+a_1 + a_2\nu_4 - \beta_0\nu_3 + \tfrac{1}{2}B_0\nu_1 - \tfrac{1}{2}A_0\nu_2 + \tfrac{1}{2}Q(\nu_2 - \nu_1)(A_1 - B_1)] , \tag{394}$$

$$u_3 = p_2 = \tfrac{3}{8}[-B_1 - B_0u_4 + A_0u_3 + a_2\nu_1 - \beta_0\nu_2 + Q(u_4 - u_3)(A_1 - B_1)] , \tag{395}$$

$$-u_4 = p_1 = \tfrac{3}{8}[+A_1 - A_0u_4 + B_0u_3 + \beta_0\nu_1 - a_2\nu_2 + Q(u_4 - u_3)(A_1 - B_1)] , \tag{396}$$

$$\nu_4^2 - \nu_3^2 + \tfrac{1}{2}(\nu_2^2 - \nu_1^2) = 1 , \tag{397}$$

$$\nu_2(u_3 + \nu_3) - \nu_1(u_4 + \nu_4) = Q(\nu_2 - \nu_1) , \tag{398}$$

$$-\nu_1(u_3 - \nu_3) + \nu_2(u_4 - \nu_4) = Q(\nu_2 - \nu_1) , \tag{399}$$

$$Q(\nu_2 - \nu_1)(\nu_4 + \nu_3) + t(\nu_2 + \nu_1) = 0 , \tag{400}$$

$$u_4^2 - u_3^2 + \tfrac{1}{2}(\nu_2^2 - \nu_1^2) = u_5 = 1 + 2Q(u_4 - u_3) , \tag{401}$$

$$-t(u_4 + u_3) + \tfrac{1}{2}Q(\nu_2^2 - \nu_1^2) = Qu_5 , \tag{402}$$

$$t = -Q(u_4 - u_3) , \tag{403}$$

and

$$t = \tfrac{3}{8}Q[-(\nu_2 - \nu_1)(a_2 + \beta_0) - (u_4 - u_3)(A_0 + B_0) + u_5(A_1 - B_1)] . \tag{404}$$

In considering the foregoing set of twelve equations, we first observe that, according to equation (403), equations (401) and (402) are equivalent. Further (cf. eqs. [400] and [403]),

$$(\nu_2 - \nu_1)(\nu_4 + \nu_3) = (u_4 - u_3)(\nu_2 + \nu_1) . \tag{405}$$

Next, adding and subtracting equations (398) and (399), we obtain

$$\nu_2(u_4 + u_3 - \nu_4 + \nu_3) - \nu_1(u_4 + u_3 + \nu_4 - \nu_3) = 2Q(\nu_2 - \nu_1) \tag{406}$$

and

$$\nu_2(\nu_4 + \nu_3 - u_4 + u_3) - \nu_1(\nu_4 + \nu_3 + u_4 - u_3) = 0 , \tag{407}$$

or

$$(\nu_2 - \nu_1)(u_4 + u_3 - 2Q) = (\nu_2 + \nu_1)(\nu_4 - \nu_3) \tag{408}$$

and

$$(\nu_2 - \nu_1)(\nu_4 + \nu_3) = (\nu_2 + \nu_1)(u_4 - u_3) . \tag{409}$$

Equations (405), (408), and (409) can be combined in the form

$$\frac{\nu_2 + \nu_1}{\nu_2 - \nu_1} = \frac{\nu_4 + \nu_3}{u_4 - u_3} = \frac{u_4 + u_3 - 2Q}{\nu_4 - \nu_3} = \frac{1}{\lambda} \text{ (say)} . \tag{410}$$

It is now seen that equations (397) and (401) are equivalent; for, according to equation (410),

$$\nu_4^2-\nu_3^2=u_4^2-u_3^2-2Q\,(u_4-u_3)\,; \tag{411}$$

or, using equation (397), we have

$$1-\tfrac{1}{2}\,(\nu_2^2-\nu_1^2)\;=u_4^2-u_3^2-2Q\,(u_4-u_3)\,; \tag{412}$$

but this is the same as equation (401).

Now turning to equations (393)–(396) and rearranging the terms, we can re-write them in the forms

$$(3A_0-8)\,\nu_2-3B_0\nu_1=6\,(\alpha_1+\alpha_2\nu_4-\beta_0\nu_3)\;+3Q\,(\nu_2-\nu_1)\,(A_1-B_1)\,, \tag{413}$$

$$3B_0\nu_2-\;(3A_0-8)\,\nu_1=6\,(-\beta_1+\beta_0\nu_4-\alpha_2\nu_3)\;+3Q\,(\nu_2-\nu_1)\,(A_1-B_1)\,, \tag{414}$$

$$(3A_0-8)\,u_4-3B_0u_3=3\,(A_1+\beta_0\nu_1-\alpha_2\nu_2)\;+3Q\,(u_4-u_3)\,(A_1-B_1)\,, \tag{415}$$

and

$$3B_0u_4-\;(3A_0-8)\,u_3=3\,(-B_1+\alpha_2\nu_1-\beta_0\nu_2)\;+3Q\,(u_4-u_3)\,(A_1-B_1)\,. \tag{416}$$

From these equations the following set can be derived:

$$P_1\,(\nu_2-\nu_1)\;-2\varpi_1\,(\nu_4-\nu_3)\;=a_2\,. \tag{417}$$

$$P_2\,(\nu_2+\nu_1)\;-2\varpi_2\,(\nu_4+\nu_3)\;=a_1\,, \tag{418}$$

$$P_1\,(u_4-u_3)\;+\varpi_1\,(\nu_2-\nu_1)\;=b_2\,, \tag{419}$$

$$P_2\,(u_4+u_3)\;+\varpi_2\,(\nu_2+\nu_1)\;=b_1\,, \tag{420}$$

where we have used the abbreviations

$$\begin{aligned} 6\,(\alpha_1+\beta_1)\;&=a_1\,; & 3\,(A_1+B_1)\;&=b_1\,; & 3\,(\alpha_2+\beta_0)\;&=\varpi_1\,,\\ 6\,(\alpha_1-\beta_1)\;&=a_2\,; & 3\,(A_1-B_1)\;&=b_2\,; & 3\,(\alpha_2-\beta_0)\;&=\varpi_2\,, \end{aligned} \tag{421}$$

$$P_1=3\,(A_0+B_0)\;-8-2Qb_2\,,\qquad\text{and}\qquad P_2=3\,(A_0-B_0)\;-8\,. \tag{422}$$

Using equation (410), we can reduce equations (417)–(420) to the forms

$$P_1\,(\nu_2+\nu_1)\;-2\varpi_1\,(u_4+u_3)\;=\frac{a_2}{\lambda}-4\varpi_1Q\,, \tag{423}$$

$$\varpi_2\,(\nu_2+\nu_1)\;+P_2\,(u_4+u_3)\;=b_1\,, \tag{424}$$

$$P_1\,(\nu_4+\nu_3)\;+\varpi_1\,(\nu_2+\nu_1)\;=\frac{b_2}{\lambda}\,, \tag{425}$$

and

$$-2\varpi_2\,(\nu_4+\nu_3)\;+P_2\,(\nu_2+\nu_1)\;=a_1\,. \tag{426}$$

Solving equations (423) and (424) for $(\nu_2+\nu_1)$ and (u_4+u_3), we have

$$\Delta\,(\nu_2+\nu_1)\;=P_2\left(\frac{a_2}{\lambda}-4\varpi_1Q\right)+2\varpi_1b_1 \tag{427}$$

and

$$\Delta\,(u_4+u_3)\;=\;-\varpi_2\left(\frac{a_2}{\lambda}-4\varpi_1Q\right)+b_1P_1\,, \tag{428}$$

where

$$\Delta=P_1P_2+2\varpi_1\varpi_2\,. \tag{429}$$

Similarly, from equations (424) and (425) we find

$$\Delta(\nu_4+\nu_3) = P_2\frac{b_2}{\lambda} - a_1\varpi_1 \tag{430}$$

and

$$\Delta(\nu_2+\nu_1) = 2\varpi_2\frac{b_2}{\lambda} + a_1P_1 . \tag{431}$$

Equations (427) and (431) now determine λ; for, according to these equations, we must have

$$P_2\left(\frac{a_2}{\lambda} - 4\varpi_1 Q\right) + 2\varpi_1 b_1 = 2\varpi_2\frac{b_2}{\lambda} + a_1P_1 , \tag{432}$$

or

$$\lambda = \frac{P_2a_2 - 2b_2\varpi_2}{a_1P_1 - 2b_1\varpi_1 + 4\varpi_1P_2Q} . \tag{433}$$

It is now seen that equations (410), (427), (or [431]), (428), (430), and (433) determine the six constants ν_1, ν_2, ν_3, ν_4, u_3, and u_4 uniquely in terms of the various moments of $X_l(\mu)$, $Y_l(\mu)$, $X_r(\mu)$, and $Y_r(\mu)$ *and* the constant Q. It remains to verify that, with the constants determined in this fashion, equations (397) and (404) (which we have not used so far) are also satisfied.

Considering, first, condition (397), we observe that, according to equation (410), this is equivalent to

$$(\nu_4+\nu_3)\lambda(u_4+u_3-2Q) + \tfrac{1}{2}(\nu_2+\nu_1)\lambda(\nu_2+\nu_1) = 1 . \tag{434}$$

Substituting for $(\nu_4 + \nu_3)$, $(u_4 + u_3)$, and $(\nu_2 + \nu_1)$ from equations (427), (428), (430), and (431) in equation (434), we have

$$\begin{aligned}(b_2P_2 - a_1\varpi_1\lambda)&(-a_2\varpi_2 + b_1P_1\lambda + 4\varpi_1\varpi_2\lambda Q - 2\lambda Q\Delta)\\ &+ \tfrac{1}{2}(a_2P_2 + 2\varpi_1b_1\lambda - 4\varpi_1\lambda P_2Q)(a_1\lambda P_1 + 2\varpi_2b_2) = \lambda\Delta^2 .\end{aligned} \tag{435}$$

After some straightforward reductions, equation (435) becomes

$$\tfrac{1}{2}\lambda(a_1a_2 + 2b_1b_2)(P_1P_2 + 2\varpi_1\varpi_2) - 2b_2\lambda P_2Q\Delta = \lambda\Delta^2 , \tag{436}$$

or (cf. eq. [429])

$$\tfrac{1}{2}(a_1a_2 + 2b_1b_2) = \Delta + 2b_2P_2Q . \tag{437}$$

Hence we have only to verify the truth of equation (437).

Now (cf. eqs. [351], [361], and [364])

$$\begin{aligned}[3(A_0+B_0) - 8][3(A_0-B_0) - 8] &+ 18(\alpha_2^2 - \beta_0^2)\\ &= 9(A_0^2 - B_0^2) - 48A_0 + 64 + 18(\alpha_2^2 - \beta_0^2)\\ &= 9(A_1^2 - B_1^2) + 16 + 18[(\tfrac{4}{3} - \alpha_0)^2 - \beta_0^2]\\ &= 9(A_1^2 - B_1^2) + 48 + 18(\alpha_0^2 - \beta_0^2) - 48\alpha_0\\ &= 9(A_1^2 - B_1^2) + 18(\alpha_1^2 - \beta_1^2) = \tfrac{1}{2}(a_1a_2 + 2b_1b_2) .\end{aligned} \tag{438}$$

Hence

$$\begin{aligned}\tfrac{1}{2}(a_1a_2 + 2b_1b_2) &= P_2(P_1 + 2Qb_2) + 2\varpi_1\varpi_2\\ &= \Delta + 2b_2P_2Q ,\end{aligned} \tag{439}$$

as required.

Finally, considering equation (404), we can re-write this in the form (cf. eq. [403])

$$8(u_4-u_3)=3\left[(\nu_2-\nu_1)(a_2+\beta_0)+(u_4-u_3)(A_0+B_0)\right.\qquad\qquad \left.-(A_1-B_1)\{1+2Q(u_4-u_3)\}\right] \tag{440}$$

or

$$(u_4-u_3)[3(A_0+B_0)-8-6Q(A_1-B_1)]+3(\nu_2-\nu_1)(a_2+\beta_0)=3(A_1-B_1)\,. \tag{441}$$

With the abbreviations (421) and (422), the foregoing equation is equivalent to

$$P_1(u_4-u_3)+\varpi_1(\nu_2-\nu_1)=b_2\cdot \tag{442}$$

but this is the same as equation (419), which we have already satisfied. With this we have satisfied all the equations (393)–(404).

Substituting for λ according to equation (433) in equations (428), (430), and (431), we find that the solutions for the constants can be expressed in the following forms:

$$\nu_2+\nu_1=\frac{1}{\lambda}(\nu_2-\nu_1)=\frac{a_1a_2-4\varpi_1\varpi_2}{P_2a_2-2b_2\varpi_2}, \tag{443}$$

$$\nu_4+\nu_3=\frac{1}{\lambda}(u_4-u_3)=\frac{a_1b_2-2\varpi_1P_2}{P_2a_2-2b_2\varpi_2}, \tag{444}$$

$$u_4+u_3=\frac{a_2b_1-2\varpi_2P_1-4\varpi_2b_2Q}{P_2a_2-2b_2\varpi_2}, \tag{445}$$

$$u_4+u_3-2Q=\frac{1}{\lambda}(\nu_4-\nu_3)=\frac{a_2b_1-2\varpi_2P_1-2a_2P_2Q}{P_2a_2-2b_2\varpi_2}, \tag{446}$$

$$u_5=1+2Q(u_4-u_3)=\lambda\frac{a_1P_1-2\varpi_1b_1+2a_1b_2Q}{P_2a_2-2b_2\varpi_2}, \tag{447}$$

and

$$\frac{1}{\lambda}=\frac{a_1P_1-2\varpi_1b_1+4\varpi_1P_2Q}{P_2a_2-2b_2\varpi_2}, \tag{448}$$

where it may be recalled that a_1, a_2, b_1, b_2, ϖ_1, ϖ_2, P_1, and P_2 are defined in equations (421) and (422).

The constant Q is, however, left entirely arbitrary.

21. *The removal of the arbitrariness in the solution and the determination of the constant* Q.—In the preceding section we have verified that the solutions for the scattering and the transmission matrices are of the forms given by equations (338)–(345) (or, equivalently, [353]–[360]) and have further shown that the constants, ν_1, ν_2, ν_3, ν_4, u_3, u_4, and u_5, occurring in the solutions can all be expressed in a unique manner in terms of the constant Q (which is left arbitrary) and the moments of the functions $X_r(\mu)$, $Y_r(\mu)$, $X_l(\mu)$, and $Y_l(\mu)$—the latter two functions being the *standard solutions* of equations (349) and (350). The functional equations governing S and T therefore admit a one-parametric family of solutions. As in the two other cases of conservative scattering that we have considered (Secs. II and III), this arbitrariness in the solution of the equations incorporating the invariances of the problem can be removed by appealing to the flux and the K-integrals. However, in the present instance, there are (formally) two such sets of integrals corresponding to the fact that F_l and F_r can be specified independently of each other. Indeed, starting from the equations of transfer (Paper XIV, System I, p.

153) appropriate to the problem on hand, we can show that the problem admits the integrals

$$F_l(\tau) = 2\int_{-1}^{+1} [I_{ll}(\tau, \mu) + I_{rl}(\tau, \mu)]\, \mu d\mu = \mu_0 F_l [e^{-\tau/\mu_0} + \gamma_l^{(1)}] , \tag{449}$$

$$K_l(\tau) = \frac{1}{2}\int_{-1}^{+1} [I_{ll}(\tau, \mu) + I_{rl}(\tau, \mu)]\, \mu^2 d\mu = \tfrac{1}{4}\mu_0 F_l [-\mu_0 e^{-\tau/\mu_0} + \gamma_l^{(1)}\tau + \gamma_l^{(2)}] , \tag{450}$$

$$F_r(\tau) = 2\int_{-1}^{+1} [I_{lr}(\tau, \mu) + I_{rr}(\tau, \mu)]\, \mu d\mu = \mu_0 F_r [e^{-\tau/\mu_0} + \gamma_r^{(1)}] , \tag{451}$$

and

$$K_r(\tau) = \frac{1}{2}\int_{-1}^{+1} [I_{lr}(\tau, \mu) + I_{rr}(\tau, \mu)]\, \mu^2 d\mu = \tfrac{1}{4}\mu_0 F_r [-\mu_0 e^{-\tau/\mu_0} + \gamma_r^{(1)}\tau + \gamma_r^{(2)}] , \tag{452}$$

where $(I_{ll} + I_{rl})$ and $(I_{lr} + I_{rr})$ are the *total* intensities in the diffuse radiation field which are proportional, respectively, to F_l and F_r and where $\gamma_l^{(1)}$, $\gamma_l^{(2)}$, $\gamma_r^{(1)}$, and $\gamma_r^{(2)}$ are constants.

We shall now show how the integrals (449)–(452) enable us to eliminate the arbitrariness in the solution found in § 20 and determine Q explicitly in terms of the moments of $X_r(\mu)$ and $Y_r(\mu)$.

First, we may observe that, according to equations (321),

$$I_{ll}(0, \mu) + I_{rl}(0, \mu) = \frac{3}{16\mu}[S_{11}(\mu, \mu_0) + S_{21}(\mu, \mu_0)]\, F_l \tag{453}$$

$$I_{ll}(\tau_1, -\mu) + I_{rl}(\tau_1, -\mu) = \frac{3}{16\mu}[T_{11}(\mu, \mu_0) + T_{21}(\mu, \mu_0)]\, F_l , \tag{454}$$

$$I_{lr}(0, \mu) + I_{rr}(0, \mu) = \frac{3}{16\mu}[S_{12}(\mu, \mu_0) + S_{22}(\mu, \mu_0)]\, F_r , \tag{455}$$

and

$$I_{lr}(\tau_1, -\mu) + I_{rr}(\tau_1, -\mu) = \frac{3}{16\mu}[T_{12}(\mu, \mu_0) + T_{22}(\mu, \mu_0)]\, F_r . \tag{456}$$

Considering the part of the emergent intensities proportional to F_l and substituting for the relevant matrix elements of $\boldsymbol{S}$ and $\boldsymbol{T}$ from equations (353)–(360), we have

$$\begin{aligned} I_{ll}(0, \mu) + I_{rl}(0, \mu) = \tfrac{1}{2}\mu_0 \Big\{ & \frac{3}{4}\frac{1-\mu^2}{\mu_0+\mu}[X_l(\mu_0)\, X_l(\mu) - Y_l(\mu_0)\, Y_l(\mu)] \\ & + \tfrac{3}{4}X_l(\mu_0)[(\nu_4+\mu)\, X_l(\mu) - \nu_3 Y_l(\mu) + \tfrac{1}{2}\nu_1 Y_r(\mu) - \tfrac{1}{2}\nu_2 X_r(\mu) \\ & \qquad + \tfrac{1}{2}Q(\nu_2-\nu_1)\, \mu\{X_r(\mu) - Y_r(\mu)\}] \\ & + \tfrac{3}{4}Y_l(\mu_0)[(\nu_4-\mu)\, Y_l(\mu) - \nu_3 X_l(\mu) + \tfrac{1}{2}\nu_1 X_r(\mu) - \tfrac{1}{2}\nu_2 Y_r(\mu) \\ & \qquad + \tfrac{1}{2}Q(\nu_2-\nu_1)\, \mu\{X_r(\mu) - Y_r(\mu)\}]\Big\} F_l \end{aligned} \tag{457}$$

and

$$I_{ll}(\tau_1, -\mu) + I_{rl}(\tau_1, -\mu) = \tfrac{1}{2}\mu_0 \left\{ \frac{3}{4}\frac{1-\mu^2}{\mu_0 - \mu}[Y_l(\mu_0) X_l(\mu) - X_l(\mu_0) Y_l(\mu)] \right.$$
$$- \tfrac{3}{4} X_l(\mu_0)[(\nu_4 - \mu) Y_l(\mu) - \nu_3 X_l(\mu) + \tfrac{1}{2}\nu_1 X_r(\mu) - \tfrac{1}{2}\nu_2 Y_r(\mu)$$
$$+ \tfrac{1}{2}Q(\nu_2 - \nu_1)\mu\{X_r(\mu) - Y_r(\mu)\}] \tag{458}$$
$$- \tfrac{3}{4} Y_l(\mu_0)[(\nu_4 + \mu) X_l(\mu) - \nu_3 Y_l(\mu) + \tfrac{1}{2}\nu_1 Y_r(\mu) - \tfrac{1}{2}\nu_2 X_r(\mu)$$
$$\left. + \tfrac{1}{2}Q(\nu_2 - \nu_1)\mu\{X_r(\mu) - Y_r(\mu)\}] \right\} F_l .$$

Using equations (457) and (458), we can determine the emergent fluxes and the K's by evaluating the various integrals defining these quantities. The evaluations can all be carried out explicitly if proper use is made of the integral properties of the X- and Y-functions.[20] Comparing the resulting expressions for $F_l(0)$, $F_l(\tau_1)$, $K_l(0)$, and $K_l(\tau_1)$ with those given by equations (449) and (450) for $\tau = 0$ and $\tau = \tau_1$, we find, respectively,

$$\gamma_l^{(1)} = \tfrac{3}{4} X_l(\mu_0)[\nu_4 \alpha_1 - \nu_3 \beta_1 + \alpha_2 + \tfrac{1}{2}\nu_1 B_1 - \tfrac{1}{2}\nu_2 A_1 + \tfrac{1}{2}Q(\nu_2 - \nu_1)(A_2 - B_2)]$$
$$+ \tfrac{3}{4} Y_l(\mu_0)[\nu_4 \beta_1 - \nu_3 \alpha_1 - \beta_2 + \tfrac{1}{2}\nu_1 A_1 - \tfrac{1}{2}\nu_2 B_1 + \tfrac{1}{2}Q(\nu_2 - \nu_1)(A_2 - B_2)] , \tag{459}$$

$$\gamma_l^{(1)} = \tfrac{3}{4} X_l(\mu_0)[\nu_4 \beta_1 - \nu_3 \alpha_1 - \beta_2 + \tfrac{1}{2}\nu_1 A_1 - \tfrac{1}{2}\nu_2 B_1 + \tfrac{1}{2}Q(\nu_2 - \nu_1)(A_2 - B_2)]$$
$$+ \tfrac{3}{4} Y_l(\mu_0)[\nu_4 \alpha_1 - \nu_3 \beta_1 + \alpha_2 + \tfrac{1}{2}\nu_1 B_1 - \tfrac{1}{2}\nu_2 A_1 + \tfrac{1}{2}Q(\nu_2 - \nu_1)(A_2 - B_2)] , \tag{460}$$

$$\gamma_l^{(2)} = \tfrac{3}{4} X_l(\mu_0)[\nu_4 \alpha_2 - \nu_3 \beta_2 + \alpha_1 + \tfrac{1}{2}\nu_1 B_2 - \tfrac{1}{2}\nu_2 A_2 + \tfrac{1}{2}Q(\nu_2 - \nu_1)(A_3 - B_3)]$$
$$+ \tfrac{3}{4} Y_l(\mu_0)[\nu_4 \beta_2 - \nu_3 \alpha_2 - \beta_1 + \tfrac{1}{2}\nu_1 A_2 - \tfrac{1}{2}\nu_2 B_2 + \tfrac{1}{2}Q(\nu_2 - \nu_1)(A_3 - B_3)] , \tag{461}$$

and

$$\gamma_l^{(1)}\tau_1 + \gamma_l^{(2)}$$
$$= -\tfrac{3}{4} X_l(\mu_0)[\nu_4 \beta_2 - \nu_3 \alpha_2 - \beta_1 + \tfrac{1}{2}\nu_1 A_2 - \tfrac{1}{2}\nu_2 B_2 + \tfrac{1}{2}Q(\nu_2 - \nu_1)(A_3 - B_3)] \tag{462}$$
$$- \tfrac{3}{4} Y_l(\mu_0)[\nu_4 \alpha_2 - \nu_3 \beta_2 + \alpha_1 + \tfrac{1}{2}\nu_1 B_2 - \tfrac{1}{2}\nu_2 A_2 + \tfrac{1}{2}Q(\nu_2 - \nu_1)(A_3 - B_3)] .$$

It should, first, be observed that equations (459) and (460) are consistent with each other; for, to be consistent,

$$(\nu_4 + \nu_3)(\alpha_1 - \beta_1) + \alpha_2 + \beta_2 - \tfrac{1}{2}(\nu_2 + \nu_1)(A_1 - B_1) = 0 \tag{463}$$

must be true. With the abbreviations (421), equation (463) is equivalent to

$$a_2(\nu_4 + \nu_3) + 2\varpi_1 - b_2(\nu_2 + \nu_1) = 0 . \tag{464}$$

With the solutions (443) and (444) for $(\nu_2 + \nu_1)$ and $(\nu_4 + \nu_3)$, it is readily verified that equation (464) is indeed satisfied. We can accordingly combine equations (459) and (460) to give

$$\gamma_l^{(1)} = \tfrac{3}{8}[X_l(\mu_0) + Y_l(\mu_0)][(\nu_4 - \nu_3)(\alpha_1 + \beta_1) - \tfrac{1}{2}(\nu_2 - \nu_1)(A_1 + B_1)$$
$$+ \alpha_2 - \beta_2 + Q(\nu_2 - \nu_1)(A_2 - B_2)] . \tag{465}$$

[20] In the present context the relations of theorem 8 have to be used.

The factor of $X_l(\mu_0)+Y_l(\mu_0)$ on the right-hand side can be simplified considerably by using equations (443), (446), and (448). Thus

$$\begin{aligned}
&6(\nu_4-\nu_3)(\alpha_1+\beta_1)-3(\nu_2-\nu_1)(A_1+B_1)+6(\alpha_2-\beta_2)\\
&=\lambda\left[a_1(u_4+u_3-2Q)-b_1(\nu_2+\nu_1)+\frac{2\varpi_2}{\lambda}\right]\\
&=\frac{\lambda}{P_2a_2-2b_2\varpi_2}[a_1(a_2b_1-2\varpi_2P_1-2a_2P_2Q)\\
&\qquad -b_1(a_1a_2-4\varpi_1\varpi_2)+2\varpi_2(a_1P_1-2\varpi_1b_1+4\varpi_1P_2Q]\\
&=\frac{2\lambda P_2Q}{P_2a_2-2b_2\varpi_2}(-a_1a_2+4\varpi_1\varpi_2)\\
&=-2P_2Q(\nu_2-\nu_1)=-2Q(\nu_2-\nu_1)[3(A_0-B_0)-8].
\end{aligned}\tag{466}$$

Inserting this result in equation (465), we have

$$\gamma_l^{(1)}=-\tfrac{1}{8}Q(\nu_2-\nu_1)[3(A_0-B_0)-8-3(A_2-B_2)][X_l(\mu_0)+Y_l(\mu_0)].\tag{467}$$

Next, from equations (461), and (462) we obtain

$$\begin{aligned}\gamma_l^{(1)}\tau_1=-\tfrac{3}{4}[X_l(\mu_0)+Y_l(\mu_0)][(\nu_4-\nu_3)(\alpha_2+\beta_2)-\tfrac{1}{2}(\nu_2-\nu_1)(A_2+B_2)\\
+(\alpha_1-\beta_1)+Q(\nu_2-\nu_1)(A_3-B_3)].\end{aligned}\tag{468}$$

Again, the factor of $X_l(\mu_0)+Y_l(\mu_0)$ can be simplified by using equations (443) and (446); we find

$$\begin{aligned}\gamma_l^{(1)}\tau_1=-\tfrac{1}{8}(\nu_2-\nu_1)[3(A_0-A_2)+3(B_0-B_2)-8\\
-6Q\{(A_1-A_3)-(B_1-B_3)\}][X_l(\mu_0)+Y_l(\mu_0)].\end{aligned}\tag{469}$$

From equations (467) and (469) we now obtain

$$\tau_1=\frac{3(A_0-A_2)+3(B_0-B_2)-8-6Q[(A_1-A_3)-(B_1-B_3)]}{Q[3(A_0-A_2)-3(B_0-B_2)-8]}.\tag{470}$$

Solving this equation for Q, we have

$$Q=\frac{3(A_0-A_2)+3(B_0-B_2)-8}{[3(A_0-A_2)-3(B_0-B_2)-8]\tau_1+6(A_1-A_3)-6(B_1-B_3)}.\tag{471}$$

Introducing the notation (cf. eq. [10])

$$x_n^{(r)}=\frac{3}{8}\int_0^1(1-\mu^2)X_r(\mu)\mu^n d\mu\tag{472}$$

and

$$y_n^{(r)}=\frac{3}{8}\int_0^1(1-\mu^2)Y_r(\mu)\mu^n d\mu,\tag{473}$$

we can re-write equation (471) in the form

$$Q=\frac{x_0^{(r)}+y_0^{(r)}-1}{[x_0^{(r)}-y_0^{(r)}-1]\tau_1+2[x_1^{(r)}-y_1^{(r)}]}.\tag{474}$$

In this form we recognize the similarity of the present expression for Q with equation (202).

A similar consideration of the integrals (451) and (452) leads to the same value of Q, though the details of the calculation are somewhat more complicated.[21] However, it may be of interest to note that the constant $\gamma_r^{(1)}$ in equations (451) and (452) has the value (cf. eq. [467])

$$\gamma_r^{(1)} = -\tfrac{1}{8}Q\left[3\left(A_0 - A_2\right) - 3\left(B_0 - B_2\right) - 8\right]\left[\left(u_4 - u_3\right)\left\{X_r\left(\mu_0\right) + Y_r\left(\mu_0\right)\right\} - u_5\mu_0\left\{X_r\left(\mu_0\right) - Y_r\left(\mu_0\right)\right\}\right]. \tag{475}$$

With the foregoing determination of Q in terms of the moments of $X_r(\mu)$ and $Y_r(\mu)$, we have completed the solution of the problem.

22. *Concluding remarks.*—The analysis of the various problems of diffuse reflection and transmission presented in this paper has shown how problems of radiative transfer in plane-parallel atmospheres of finite optical thicknesses can be solved exactly; for, in every case considered, it was possible to reduce the complicated systems of functional equations representing the problem to pairs of equations of the standard form

$$X(\mu) = 1 + \mu\int_0^1 \frac{\Psi(\mu')}{\mu+\mu'}\left[X(\mu)X(\mu') - Y(\mu)Y(\mu')\right]d\mu' \tag{476}$$

and

$$Y(\mu) = e^{-\tau_1/\mu} + \mu\int_0^1 \frac{\Psi(\mu')}{\mu-\mu'}\left[Y(\mu)X(\mu') - X(\mu)Y(\mu')\right]d\mu'. \tag{477}$$

And, moreover, the expressions (155) and (156) for $X(\mu)$ and $Y(\mu)$, as rational functions involving the points of the Gaussian division and the roots of the characteristic equation

$$1 = \sum_{j=1}^{n} \frac{a_j\Psi(\mu_j)}{1-k^2\mu_j^2}, \tag{478}$$

provide approximate solutions of equations (476) and (477). Starting with these approximate solutions,[22] we can solve equations (476) and (477) by a process of iteration. Since the iteration will have to be performed for every required value of τ_1, the problem of tabulating the X- and Y-functions is much more elaborate than in the case of the H-functions. However, the existence of the *differential equations* (eqs. [18] and [19]),

$$\frac{\partial X(\mu,\tau_1)}{\partial\tau_1} = Y(\mu,\tau_1)\int_0^1 \frac{d\mu'}{\mu'}\Psi(\mu')Y(\mu',\tau_1) \tag{479}$$

and

$$\frac{\partial Y(\mu,\tau_1)}{\partial\tau_1} = -\frac{Y(\mu,\tau_1)}{\mu} + X(\mu,\tau_1)\int_0^1 \frac{d\mu'}{\mu'}\Psi(\mu')Y(\mu',\tau_1), \tag{480}$$

simplifies the tabulation problem considerably, since corrections for small changes in τ_1 can always be found with the aid of these equations.[23]

[21] In the reductions, use must be made of eqs. (13)–(16).

[22] For $\tau_1 \to 0$, a generalization of the method described by van de Hulst (*Ap. J.*, **107**, 220, 1948) in the context of the simpler equations (172) and (173) can also be used with considerable advantage. While the necessary generalizations of van de Hulst's method will be considered in a later paper of this series, it may be remarked here that the method is essentially one of solving equations (476) and (477) by an iteration scheme which is started with the "trial solutions" $X(\mu) = 1$ and $Y(\mu) = e^{-\tau_1/\mu}$.

[23] Expressions for the second- and higher-order derivatives can be easily derived from eqs. (479) and (480), so that Taylor expansions with as many terms as may be desired could be used.

In view of the importance of the problem and its long-standing nature, it is worthy of comment here that the basic problem underlying the theories relating to the illumination and polarization of the sunlit sky has now been solved exactly. The solution presented in Section V assumes that beyond $\tau = \tau_1$ there is a vacuum (or, equivalently, that there is a perfect absorber at $\tau = \tau_1$). However, the solution for the case in which there is a "ground" can be reduced to the "standard problem" considered in this paper.[24]

Again, while attention was concentrated in this paper on problems of diffuse reflection and transmission, it is evident that solutions for other problems in which there is a distribution of external sources through the medium can also be reduced to the X- and Y-functions of this paper.[25] We shall consider such problems in later papers of this series.

Finally, it should be remarked that, while the present paper solves the mathematical problem of the transfer equations, the practical use of the solutions must await the construction of tables of the X- and Y-functions appropriate for the various problems. The preparation of these tables is now being undertaken by Mrs. Frances Breen and the writer.

[24] Cf. van de Hulst, *op. cit.*

[25] E.g., a case in the theory of formation of stellar absorption lines leads to an equation of transfer with an external source function which increases linearly with the optical depth (cf. Paper XX, *Ap. J.*, **106,** 145, eq. [5], 1947). The exact solution for this problem can, nevertheless, be reduced to an H-function and its moments (Paper XX, eq. [47]).