ON THE RADIATIVE EQUILIBRIUM OF A STELLAR ATMOSPHERE. XXI

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ABSTRACT

In this paper the theory of diffuse reflection and transmission by a plane-parallel atmosphere of finite optical thickness is considered under conditions of (I) isotropic scattering with an albedo $\tilde{\omega}_0 \leqslant 1$, (II) scattering in accordance with Rayleigh's phase function, (III) scattering in accordance with the phase function $\lambda(1 + x \cos \Theta)$, and (IV) Rayleigh scattering with proper allowance for the polarization of the radiation field. In all cases considered, it has been possible to eliminate the constants of integration (which are twice as many as in the case of semi-infinite atmospheres) and express the solutions for the reflected and the transmitted radiations in closed forms in a general nth approximation. It is further shown how a pair of functions, $X(\mu)$ and $Y(\mu)$, depending only on the roots of a characteristic equation and the optical thickness of the atmosphere, play the same basic role in this theory as $H(\mu)$ does in the theory of semi-infinite atmospheres. The passage to the limit of infinite approximation and the determination of the exact laws of diffuse reflection and transmission are thus made possible.

1. *Introduction.*—In the earlier papers[1] of this series, the theory of the transfer of radiation in semi-infinite plane-parallel atmospheres has been developed to a point that it is possible to obtain by a definite algorism exact solutions for the various problems. But the corresponding theory for atmospheres of finite optical thicknesses is in a far less advanced stage. The difficulties confronting the development of this latter theory do not lie in the system of linear equations which replaces the equation of transfer in our scheme of approximation: they present, in fact, no problem which does not already require solution in the semi-infinite case.[2] The difficulties actually lie in the problem of eliminating the explicit appearance of the constants of integration in the solutions and expressing the angular distributions of the emergent radiations in terms of functions which involve only the roots of certain characteristic equations. This problem of the elimination of the constants and the reduction of the solution to the evaluation of a certain basic function (or a set of functions)[3] is of particular importance for passing to the limit of infinite approximation and obtaining the exact solutions. Thus, in analogy with the theory of semi-infinite atmospheres, we may expect that the angular distributions of the emergent radiations can be expressed in terms of certain functions which will be explicitly known in any finite approximation and which in the limit of infinite approximation will become solutions of functional equations of a standard form. We shall see that this reduction can be achieved in spite of the greater complexity of the problem arising from our present requirement of explicitly satisfying boundary conditions on both sides of the atmosphere and of obtaining solutions in closed forms for the expressions governing the radiation emergent from each of the two sides.

In this paper we shall take the first of the two principal steps required for the completion of the theory of radiative transfer in plane-parallel atmospheres of finite optical thicknesses. More particularly, in this paper, we shall carry out the elimination of the constants for the basic problem of diffuse reflection and transmission under a variety

[1] See particularly Papers XIV, XVI, XIX, and XX (*Ap. J.*, **105**, 164, 435, 1947; *ibid.*, **106**, 143, 145, 1947.

[2] Cf. Paper XII (eqs. [45] and [74]) (*ibid.*, **104**, 191, 1946).

[3] We shall see that, actually, a pair of functions defined in the interval (0, 1) is involved in the solutions for finite atmospheres.

of scattering conditions and show how, in each case considered, the solution can be expressed in terms of a single function defined in the interval $(-1, +1)$. The passage to the limit of infinite approximation and the exhibition of the relationship of these solutions[4] to the functional equations derived in Paper XVII[5] are postponed to a later paper.

I. ISOTROPIC SCATTERING WITH AN ALBEDO $\varpi_0 < 1$

2. *The expression of the boundary conditions and the emergent intensities in terms of two functions.*—As we shall see, the consideration of the problem of diffuse reflection and transmission by an atmosphere scattering radiation isotropically with an albedo $\varpi_0 < 1$ introduces us, in its simplest context, to a basic mathematical problem which is characteristic of this theory and which requires solution. Considering, then, the case of a plane-parallel atmosphere of a finite optical thickness, τ_1, on which is incident a parallel beam of radiation of net flux πF per unit area normal to itself, at an angle $\cos^{-1}\mu_0$ to the normal, we have[2]

$$I_i = \tfrac{1}{4}\varpi_0 F\left[\sum_{a=-n}^{+n}\frac{L_a e^{-k_a\tau}}{1+\mu_i k_a}+\frac{\gamma e^{-\tau/\mu_0}}{1+\mu_i/\mu_0}\right]\qquad (i=\pm 1,\ldots,\pm n), \quad (1)^{6}$$

for the solution of the intensities I_i in the nth approximation. In equation (1) the L_a's $(a = \pm 1, \ldots, \pm n)$ are the $2n$ constants of integration, and the k_a's $(a = \pm 1, \ldots, \pm n$ and $k_{+a} = -k_{-a})$ are the $2n$ roots of the characteristic equation

$$1 = \varpi_0 \sum_{j=1}^{n}\frac{a_j}{1-k^2\mu_j^2}, \qquad (2)$$

which occurs in pairs $(k_{+a} = -k_{-a})$, and

$$\gamma = H(\mu_0)\,H(-\mu_0) = \frac{(-1)^n}{\mu_1^2\cdots\mu_n^2}\,\frac{\prod_{i=1}^{n}(\mu_0^2-\mu_i^2)}{\prod_{a=1}^{n}(1-k_a^2\mu_0^2)}. \qquad (3)$$

The boundary conditions appropriate to our present problem are

$$I_{-i} = 0 \quad \text{at} \quad \tau = 0 \quad \text{and for} \quad i = 1, \ldots, n, \qquad (4)$$

and

$$I_{+i} = 0 \quad \text{at} \quad \tau = \tau_1 \quad \text{and for} \quad i = 1, \ldots, n. \qquad (5)$$

The equations which determine the $2n$ constants of integration are, therefore,

$$\sum_{a=-n}^{+n}\frac{L_a}{1-k_a\mu_i}+\frac{\gamma}{1-\mu_i/\mu_0} = 0 \qquad (i = 1, \ldots, n) \qquad (6)$$

and

$$\sum_{a=-n}^{+n}\frac{L_a e^{-k_a\tau_1}}{1+k_a\mu_i}+\frac{\gamma e^{-\tau_1/\mu_0}}{1+\mu_i/\mu_0} = 0 \qquad (i = 1, \ldots, n). \qquad (7)$$

[4] In the manner of Paper XIV.

[5] *Ap. J.*, **106**, 441, 1947.

[6] In the summation over a in this equation, the term $a = 0$ is omitted. We shall adopt this convention throughout. It is, therefore, always to be understood that in all summations (or products) extended over a from $-n$ to $+n$ the term with $a = 0$ does not occur.

In terms of the functions

$$S(\mu) = \sum_{a=-n}^{+n} \frac{L_a}{1-k_a\mu} + \frac{\gamma}{1-\mu/\mu_0} \tag{8}$$

and

$$T(\mu) = \sum_{a=-n}^{+n} \frac{L_a e^{-k_a\tau_1}}{1+k_a\mu} + \frac{\gamma e^{-\tau_1/\mu_0}}{1+\mu/\mu_0}, \tag{9}$$

the boundary conditions can be expressed in the form

$$S(\mu_i) = T(\mu_i) = 0 \qquad (i=1, \ldots, n). \tag{10}$$

In other words, the μ_i's ($i = 1, \ldots, n$) are zeros of both S and T.

In analogy with the procedure adopted in the case of semi-infinite atmospheres, we must now try to express the reflected and the transmitted intensities, $I(0, \mu)$ and $I(\tau_1, -\mu)$, ($0 \leqslant \mu \leqslant 1$), in terms of the same functions $S(\mu)$ and $T(\mu)$. It is, however, immediately apparent that $I(0, \mu)$ and $I(\tau_1, -\mu)$ cannot, simply, be proportional to $S(-\mu)$ and $T(-\mu)$, respectively, since these functions diverge for all those values of $\mu = k_a^{-1}$ for which $k_a^{-1} < 1$ ($a > 0$); in nonconservative cases (such as the present) divergence from this source will occur for ($n - 1$) values of μ in the interval $0 \leqslant \mu \leqslant 1$.[7] Consequently, a different procedure must be adopted for expressing the angular distributions of the reflected and the transmitted radiations. On consideration, it appears that the procedure which should be adopted is the following:

For the problem of diffuse reflection and transmission under consideration, the source function $\mathfrak{J}(\tau, \mu)$ is

$$\mathfrak{J}(\tau, \mu) = \tfrac{1}{2}\varpi_0 \int_{-1}^{+1} I(\tau, \mu')\, d\mu' + \tfrac{1}{4}\varpi_0 F e^{-\tau/\mu_0}, \tag{11}$$

or, in our scheme of approximation,

$$\mathfrak{J}(\tau, \mu) = \tfrac{1}{2}\varpi_0 \Sigma a_j I_j + \tfrac{1}{4}\varpi_0 F e^{-\tau/\mu_0}. \tag{12}$$

With the solution for I_i given by equation (1), the foregoing expression for $\mathfrak{J}(\tau, \mu)$ reduces to

$$\mathfrak{J}(\tau, \mu) = \tfrac{1}{4}\varpi_0 F\Big[\sum_{a=-n}^{+n} L_a e^{-k_a\tau} + \gamma e^{-\tau/\mu_0}\Big]. \tag{13}$$

Since, in general, the outward and the inward intensities, $I(\tau, +\mu)$ and $I(\tau, -\mu)$, ($0 < \mu < 1$), at any level τ, are derivable from the source function in accordance with the equations

$$I(\tau, +\mu) = \int_{\tau}^{\tau_1} \mathfrak{J}(t, \mu)\, e^{-(t-\tau)/\mu} \frac{dt}{\mu} \tag{14}$$

and

$$I(\tau, -\mu) = \int_{0}^{\tau} \mathfrak{J}(t, -\mu)\, e^{-(\tau-t)/\mu} \frac{dt}{\mu}, \tag{15}$$

we find that in our particular case

$$\left.\begin{aligned} I(\tau, +\mu) = \tfrac{1}{4}\varpi_0 F e^{+\tau/\mu}\Big\{ &\sum_{a=-n}^{+n} \frac{L_a}{1+k_a\mu}\left[e^{-\tau(1+k_a\mu)/\mu} - e^{-\tau_1(1+k_a\mu)/\mu}\right] \\ &+ \frac{\gamma}{1+\mu/\mu_0}\left[e^{-\tau(\mu_0+\mu)/\mu\mu_0} - e^{-\tau_1(\mu_0+\mu)/\mu\mu_0}\right]\Big\} \end{aligned}\right\} \tag{16}$$

[7] In addition, $T(-\mu)$ will also diverge for $\mu = \mu_0$.

and

$$I(\tau,-\mu)=\tfrac{1}{4}\varpi_0 F e^{-\tau/\mu}\left\{\sum_{a=-n}^{+n}\frac{L_a}{1-k_a\mu}\left[e^{\tau(1-k_a\mu)/\mu}-1\right]+\frac{\gamma}{1-\mu/\mu_0}\left[e^{\tau(\mu_0-\mu)/\mu\mu_0}-1\right]\right\}. \tag{17}$$

For the reflected and the transmitted intensities we therefore have

$$I(0,\mu)=\tfrac{1}{4}\varpi_0 F\left\{\sum_{a=-n}^{+n}\frac{L_a}{1+k_a\mu}\left[1-e^{-\tau_1(1+k_a\mu)/\mu}\right]+\frac{\gamma}{1+\mu/\mu_0}\left[1-e^{-\tau_1(\mu_0+\mu)/\mu\mu_0}\right]\right\} \tag{18}$$

and

$$I(\tau_1,-\mu)=\tfrac{1}{4}\varpi_0 F e^{-\tau_1/\mu}\left\{\sum_{a=-n}^{+n}\frac{L_a}{1-k_a\mu}\left[e^{\tau_1(1-k_a\mu)/\mu}-1\right]+\frac{\gamma}{1-\mu/\mu_0}\left[e^{\tau_1(\mu_0-\mu)/\mu\mu_0}-1\right]\right\}, \tag{19}$$

or, somewhat differently,

$$I(0,\mu)=\tfrac{1}{4}\varpi_0 F\left\{\sum_{a=-n}^{+n}\frac{L_a}{1+k_a\mu}+\frac{\gamma}{1+\mu/\mu_0}-e^{-\tau_1/\mu}\left[\sum_{a=-n}^{+n}\frac{L_a e^{-k_a\tau_1}}{1+k_a\mu}+\frac{\gamma e^{-\tau_1/\mu_0}}{1+\mu/\mu_0}\right]\right\}, \tag{20}$$

and

$$I(\tau_1,-\mu)=\tfrac{1}{4}\varpi_0 F\left\{\sum_{a=-n}^{+n}\frac{L_a e^{-k_a\tau_1}}{1-k_a\mu}+\frac{\gamma e^{-\tau_1/\mu_0}}{1-\mu/\mu_0}-e^{-\tau_1/\mu}\left[\sum_{a=-n}^{+n}\frac{L_a}{1-k_a\mu}+\frac{\gamma}{1-\mu/\mu_0}\right]\right\}. \tag{21}$$ [8]

According to our definitions of the functions $S(\mu)$ and $T(\mu)$, we can re-write the foregoing expressions for $I(0,+\mu)$ and $I(\tau_1,-\mu)$ in the forms

$$I(0,\mu)=\tfrac{1}{4}\varpi_0 F\left[S(-\mu)-e^{-\tau_1/\mu}T(\mu)\right] \tag{22}$$

and

$$I(\tau_1,-\mu)=\tfrac{1}{4}\varpi_0 F\left[T(-\mu)-e^{-\tau_1/\mu}S(\mu)\right]. \tag{23}$$ [9]

Thus, in the case of finite atmospheres, as in the case of semi-infinite atmospheres, there is a relationship of reciprocity between the equations which express the boundary conditions and the functions which describe the emergent radiations. The present relationship is naturally not so direct as the one encountered in the case of semi-infinite atmospheres. But it will appear that the relationship exemplified by equations (10), (22), and (23) is quite general and is precisely what is required to preserve the basic in-

[8] These expressions, do not, of course, diverge for any value of μ in the interval, $0 \leqslant \mu \leqslant 1$.

[9] Since $S(\mu_i) = T(\mu_i) = 0$,

$$I(0,\mu_i)=\tfrac{1}{4}\tilde{\omega}_0 FS(-\mu_i) \quad \text{and} \quad I(\tau_1,-\mu_i)=\tfrac{1}{4}\tilde{\omega}_0 FT(-\mu_i);$$

and this is in agreement with the solution (1) for the intensities at the points of the Gaussian division.

variances of the problem in all orders of approximation. And this last is, of course, an essential requirement for passing to the limit of infinite approximation on our method.

3. *The reduction to a problem in the theory of interpolation.*—In addition to the functions

$$P(\mu)=\prod_{i=1}^{n}(\mu-\mu_i) \quad \text{and} \quad R(\mu)=\prod_{\alpha=1}^{n}(1-k_\alpha\mu)\,, \tag{24}$$

which we have extensively used in the earlier papers, we shall now introduce the functions

$$W(\mu)=R(\mu)R(-\mu)=\prod_{\alpha=-n}^{+n}(1-k_\alpha\mu)=\prod_{\alpha=1}^{n}(1-k_\alpha^2\mu^2) \tag{25}$$

and

$$W_\alpha(\mu)=\prod_{\substack{\beta=-n\\ \beta\neq\alpha}}^{+n}(1-k_\beta\mu) \qquad (\alpha=\pm 1\,,\ldots\,,\pm n)\,. \tag{26}$$

Identities which follow from definitions (25) and (26) and which we shall find useful, are

$$W(\mu)=W(-\mu) \tag{27}$$

and

$$W_\alpha(\mu)=W_{-\alpha}(-\mu) \qquad (\alpha=1\,,\ldots\,,n)\,. \tag{28}$$

Now, from equations (8) and (9) it follows that

$$S(\mu)W(\mu)\left(1-\frac{\mu}{\mu_0}\right) \quad \text{and} \quad T(\mu)W(\mu)\left(1+\frac{\mu}{\mu_0}\right), \tag{29}$$

are polynomials of degree $2n$ in μ; and, according to equation (10), the μ_i's ($i=1, \ldots, n$) are zeros of both these polynomials. We may therefore write

$$S(\mu)=\frac{1}{\mu_1^2\ldots\mu_n^2}\,\frac{P(\mu)}{W(\mu)(1-\mu/\mu_0)}\,s(\mu) \tag{30}$$

and

$$T(\mu)=\frac{1}{\mu_1^2\ldots\mu_n^2}\,\frac{P(\mu)}{W(\mu)(1+\mu/\mu_0)}\,t(\mu)\,, \tag{31}$$ [10]

where $s(\mu)$ and $t(\mu)$ are polynomials of degree n in μ.

Two relations which follow immediately from equations (3), (8), (9), (30), and(31) are

$$\left.\begin{aligned}\gamma=\frac{1}{\mu_1^2\ldots\mu_n^2}\,\frac{P(\mu_0)P(-\mu_0)}{W(\mu_0)}&=\lim_{\mu\to\mu_0}\left(1-\frac{\mu}{\mu_0}\right)S(\mu)\\ &=\frac{1}{\mu_1^2\ldots\mu_n^2}\,\frac{P(\mu_0)}{W(\mu_0)}\,s(\mu_0)\,,\end{aligned}\right\} \tag{32}$$

and

$$\left.\begin{aligned}\gamma e^{-\tau_1/\mu_0}=\frac{e^{-\tau_1/\mu_0}}{\mu_1^2\ldots\mu_n^2}\,\frac{P(\mu_0)P(-\mu_0)}{W(\mu_0)}&=\lim_{\mu\to-\mu_0}\left(1+\frac{\mu}{\mu_0}\right)T(\mu)\\ &=\frac{1}{\mu_1^2\ldots\mu_n^2}\,\frac{P(-\mu_0)}{W(\mu_0)}\,t(-\mu_0)\,.\end{aligned}\right\} \tag{33}$$

[10] In eqs. (30) and (31) the factor $1/\mu_1^2\ldots\mu_n^2$ is introduced for reasons of convenience (see eqs. [32] and [33] below).

Hence

$$s(\mu_0) = P(-\mu_0) , \tag{34}$$

and

$$t(-\mu_0) = e^{-\tau_1/\mu_0} P(\mu_0) . \tag{35}$$

We next observe that, since (cf. eqs. [8] and [9])

$$L_a = \lim_{\mu \to 1/k_a} (1 - k_a \mu) S(\mu) \qquad (a = \pm 1 , \ldots , \pm n) , \tag{36}$$

and

$$L_a e^{-k_a \tau_1} = \lim_{\mu \to -1/k_a} (1 + k_a \mu) T(\mu) \qquad (a = \pm 1 , \ldots , \pm n) , \tag{37}$$

we must have

$$L_a = \frac{1}{\mu_1^2 \ldots \mu_n^2} \frac{P(1/k_a)}{W_a(1/k_a)(1 - 1/k_a\mu_0)} s(1/k_a) \qquad (a = \pm 1 , \ldots , \pm n) \tag{38}$$

and (cf. eq. [28])

$$L_a e^{-k_a \tau_1} = \frac{1}{\mu_1^2 \ldots \mu_n^2} \frac{P(-1/k_a)}{W_a(1/k_a)(1 - 1/k_a\mu_0)} t(-1/k_a) \qquad (a = \pm 1 , \ldots , \pm n) . \tag{39}$$

Comparing equations (38) and (39), we conclude that

$$s(1/k_a) = e^{k_a \tau_1} \frac{P(-1/k_a)}{P(+1/k_a)} t(-1/k_a) \qquad (a = \pm 1 , \ldots , \pm n) . \tag{40}$$

Re-writing this equation separately for the positive and the negative values of a, we have

$$s(1/k_a) = e^{k_a \tau_1} \frac{P(-1/k_a)}{P(+1/k_a)} t(-1/k_a) \qquad (a = 1 , \ldots , n) \tag{41}$$

and

$$t(1/k_a) = e^{k_a \tau_1} \frac{P(-1/k_a)}{P(+1/k_a)} s(-1/k_a) \qquad (a = 1 , \ldots , n) . \tag{42}$$

An immediate consequence of equations (41) and (42) is

$$s(1/k_a)\, s(-1/k_a) = t(1/k_a)\, t(-1/k_a) \qquad (a = \pm 1 , \ldots , \pm n) . \tag{43}$$

From this equation it follows that

$$s(\mu)\, s(-\mu) - t(\mu)\, t(-\mu) \equiv \text{constant } W(\mu) , \tag{44}$$

since the quantity on the left-hand side is a polynomial of degree $2n$ in μ and vanishes for $\mu = \pm 1/k_a (a = 1, \ldots , n)$.

Next, writing

$$F(\mu) = s(\mu) + t(\mu) \quad \text{and} \quad G(\mu) = s(\mu) - t(\mu) , \tag{45}$$

we find, from equations (41) and (42),

$$F(1/k_a) = + e^{k_a \tau_1} \frac{P(-1/k_a)}{P(+1/k_a)} F(-1/k_a) \qquad (a = 1 , \ldots , n) \tag{46}$$

and

$$G(1/k_a) = - e^{k_a \tau_1} \frac{P(-1/k_a)}{P(+1/k_a)} G(-1/k_a) \qquad (a = 1 , \ldots , n) . \tag{47}$$

In § 5 we shall show how equations (46) and (47), together with equations (34) and (35), just suffice to determine the polynomials $F(\mu)$ and $G(\mu)$ uniquely. The problem, as we shall see, is essentially one in the theory of interpolation.

A more symmetrical way of writing equations (46) and (47) is

$$F(1/k_a) = +\lambda_a F(-1/k_a) \qquad (a = 1, \ldots, n) \quad (48)$$

and

$$G(1/k_a) = -\lambda_a G(-1/k_a) \qquad (a = 1, \ldots, n), \quad (49)$$

where

$$\lambda_a = e^{k_a \tau_1} \frac{P(-1/k_a)}{P(+1/k_a)}. \quad (50)$$

4. *The solution of the basic mathematical problem.*—The mathematical problem to which we reduced the solution of $S(\mu)$ and $T(\mu)$ in § 3 can be formulated as follows:

To determine two polynomials $F(\mu)$ *and* $G(\mu)$ *of degree* n *in* μ *such that*

$$F(x_a) = +\lambda_a F(-x_a) \qquad (a = 1, \ldots, n), \quad (51)$$

and

$$G(x_a) = -\lambda_a G(-x_a) \qquad (a = 1, \ldots, n), \quad (52)$$

where x_a, $a = 1, \ldots, n$ *are* n *distinct values of the argument and* λ_a, $a = 1, \ldots, n$, *are* n *assigned numbers, all different from one another.*

As we have already remarked, this problem is essentially one in the theory of interpolation. It does not, however, seem to have been considered in literature before. But it will appear that the problem is closely associated with the method of solution, in a finite approximation, of a class of simultaneous pairs of functional equations of which equations (119) and (120) of Paper XVII are typical. The problem would therefore appear to merit a closer investigation than we can afford in this paper. We shall, however, give explicit solutions for $F(\mu)$ and $G(\mu)$ which satisfy the required conditions.

First, it may be noted that $F(\mu)$ and $G(\mu)$ are related by the identity (cf. eq. [44])

$$F(\mu)G(-\mu) + F(-\mu)G(\mu) \equiv \text{constant} \prod_{a=1}^{n} (x_a^2 - \mu^2), \quad (53)$$

since the quantity on the right-hand side is a polynomial of degree $2n$ in μ and vanishes for $\mu = \pm x_a (a = 1, \ldots, n)$.

Next it should be observed that in general *the conditions stated determine* $F(\mu)$ *and* $G(\mu)$ *uniquely, apart from a constant factor of proportionality.* That this is the case can be seen by writing $F(\mu)$ (for example) in the form

$$F(\mu) = \sum_{m=0}^{n} a_m \mu^m \quad (54)$$

and noting that the conditions of the problem (eq. [51]) require that

$$\sum_{m=0}^{n} a_m [1 + (-1)^{m+1} \lambda_a] x_a^m = 0 \qquad (a = 1, \ldots, n). \quad (55)$$

The $n + 1$ coefficients, $a_m (m = 0, \ldots, n)$, therefore satisfy a system of homogeneous linear equations of order n. Moreover, if the x_a's are all distinct and none of the λ_a's are equal to each other (as we have, indeed, assumed), the rank of the system (55) is also n Consequently, the coefficients a_m are all uniquely determined apart from a constant factor of proportionality. The polynomial $F(\mu)$ is therefore also determined apart from a constant factor of proportionality. Similar remarks clearly apply to $G(\mu)$ also.

The arguments of the preceding paragraph further establish that *n is the lowest degree of a polynomial (not identically zero) which will satisfy* n *conditions of the form* (51) *or* (52). On the other hand, polynomials of degree higher than n can be readily constructed in terms of $F(\mu)$ and $G(\mu)$ which will satisfy conditions (51) or (52). For example,

$$a_0 F(\mu) + a_1 \mu G(\mu) , \tag{56}$$

where a_0 and a_1 are two arbitrary constants, is the most general polynomial of degree $(n+1)$ in μ which will satisfy the n conditions (51); for, according to equations (51) and (52),

$$\left.\begin{aligned} a_0 F(x_a) + a_1 x_a G(x_a) &= a_0 \lambda_a F(-x_a) - a_1 x_a \lambda_a G(-x_a) \\ &= \lambda_a [a_0 F(\mu) + a_1 \mu G(\mu)]_{\mu=-x_a} \qquad (a = 1, \ldots, n) . \end{aligned}\right\} \tag{57}$$

Similarly,

$$b_0 G(\mu) + b_1 \mu F(\mu) , \tag{58}$$

where b_0 and b_1 are two arbitrary constants, is the most general polynomial of degree $(n+1)$ in μ which will satisfy conditions (52).

The foregoing observations can be readily extended to construct polynomials of any degree higher than n which will satisfy condition (51) or (52). We shall state the result in the form of the following theorem:

Theorem 1.—*The most general polynomials* $F^{(n+m)}(\mu)$ *and* $G^{(n+m)}(\mu)$ *of degree* $(n+m)$ *in* μ, $(m > 0)$, *which will satisfy the conditions*

$$F^{(n+m)}(x_a) = +\lambda_a F^{(n+m)}(-x_a) \qquad (a = 1, \ldots, n) \tag{59}$$

and

$$G^{(n+m)}(x_a) = -\lambda_a G^{(n+m)}(-x_a) \qquad (a = 1, \ldots, n) , \tag{60}$$

are

$$F^{(n+m)}(\mu) = \sum_{l=0}^{m} a_l \mu^l [\epsilon_{l,\,\text{even}} F(\mu) + \epsilon_{l,\,\text{odd}} G(\mu)] \tag{61}$$

and

$$G^{(n+m)}(\mu) = \sum_{l=0}^{m} b_l \mu^l [\epsilon_{l,\,\text{odd}} F(\mu) + \epsilon_{l,\,\text{even}} G(\mu)] , \tag{62}$$

where a_l *and* $b_l (l = 0, \ldots, n)$ *are arbitrary constants and*

$$\left.\begin{aligned} \epsilon_{l,\,\text{even}} &= 1 \text{ if } l \text{ is even} \\ &= 0 \text{ if } l \text{ is odd} \end{aligned}\right\} \text{ and } \left.\begin{aligned} \epsilon_{l,\,\text{odd}} &= 1 \text{ if } l \text{ is odd} \\ &= 0 \text{ if } l \text{ is even} \end{aligned}\right\} , \tag{63}$$

and $F(\mu)$ *and* $G(\mu)$ *are polynomials of degree* n *in* μ *which satisfy the same conditions as* $F^{(n+m)}(\mu)$ *and* $G^{(n+m)}(\mu)$, *respectively.*

This theorem suggests that *the polynomials* $F(\mu)$ *and* $G(\mu)$ *can be constructed by a process of induction;* for, if polynomials $F^{(n-1)}(\mu)$ and $G^{(n-1)}(\mu)$ of degree $(n-1)$ in μ which satisfy the $(n-1)$ conditions

$$F^{(n-1)}(x_a) = +\lambda_a F^{(n-1)}(-x_a) \qquad (a = 1, \ldots, n-1) \tag{64}$$

and

$$G^{(n-1)}(x_a) = -\lambda_a G^{(n-1)}(-x_a) \qquad (a = 1, \ldots, n-1) \tag{65}$$

are assumed known, then polynomials $F^{(n)}(\mu)$ and $G^{(n)}(\mu)$ of one higher degree satisfying conditions (51) and (52) appropriate to polynomials of degree n can be constructed. Thus, according to theorem 1,

$$F^{(n)}(\mu) = F^{(n-1)}(\mu) + a_1 \mu G^{(n-1)}(\mu) , \tag{66}$$

where a_1 is an arbitrary constant, will satisfy all the conditions (64) that are satisfied by $F^{(n-1)}(\mu)$. We therefore need to satisfy only the one additional condition,

$$F^{(n)}(x_n) = \lambda_n F^{(n)}(-x_n) . \tag{67}$$

This condition can be used to determine a_1. In this manner we find that, with the choice of

$$a_1 = -\frac{F^{(n-1)}(x_n) - \lambda_n F^{(n-1)}(-x_n)}{x_n [G^{(n-1)}(x_n) + \lambda_n G^{(n-1)}(-x_n)]} , \tag{68}$$

$F^{(n)}(\mu)$ defined as in equation (66) will satisfy all the required conditions. Similarly,

$$G^{(n)}(\mu) = G^{(n-1)}(\mu) + b_1 \mu F^{(n-1)}(\mu) , \tag{69}$$

where

$$b_1 = -\frac{G^{(n-1)}(x_n) + \lambda_n G^{(n-1)}(-x_n)}{x_n [F^{(n-1)}(x_n) - \lambda_n F^{(n-1)}(-x_n)]} , \tag{70}$$

will satisfy the n conditions (52). Thus polynomials of degree n satisfying the required number of conditions can be constructed if polynomials of one lower degree, each satisfying one less condition, are assumed known. On the other hand,

$$F^{(1)}(\mu) = (x_1 - \mu) + \lambda_1 (x_1 + \mu) \tag{71}$$

and

$$G^{(1)}(\mu) = (x_1 - \mu) - \lambda_1 (x_1 + \mu) \tag{72}$$

clearly satisfy the conditions appropriate to polynomials of degree 1. With this we have established that the solution to our problem can, in fact, be found by a process of induction.

While the construction by induction which we have outlined above solves our problem in principle, it is still unsatisfactory, in that the solution obtained by following the construction literally will be excessively complicated. This apparent complexity must, in part, be attributed to the fact that the manner of construction destroys the essential symmetry of the problem in the x_a's and the λ_a's. It would therefore seem that the construction of the polynomials $F(\mu)$ and $G(\mu)$ by a straightforward process of induction is not a significant approach to the problem. The method was therefore abandoned and the somewhat indirect method that we shall now describe has at least the merit of providing explicit formulae for the functions $F(\mu)$ and $G(\mu)$ which preserve the symmetries of the problem.

From the form of the solution for the case $n = 2$, it appeared that $F(\mu)$ and $G(\mu)$ must be expressible as linear combinations of the 2^n polynomials of the form

$$\prod_{a=1}^{n} (x_a \pm \mu) , \tag{73}$$

where in each of the n factors we have either a plus or a minus sign in the parentheses. Moreover, after some consideration, it also appeared that the coefficient of the term

$$\prod_{i=1}^{l} (x_{r_i} + \mu) \prod_{m=1}^{n-l} (x_{s_m} - \mu) , \tag{74}$$

where $r_1, \ldots, r_l$ and $s_1, \ldots, s_{n-l}$ are selections of l, respectively $n-l$, distinct integers from the set $(1, 2, \ldots, n)$, must be

$$\pm \lambda_{r_1} \ldots \lambda_{r_l} \frac{\prod_{m=1}^{n-l} \prod_{i=1}^{l} (x_{s_m} + x_{r_i})}{\prod_{m=1}^{n-l} \prod_{i=1}^{l} (x_{s_m} - x_{r_i})} . \tag{75}$$

But the decision regarding the sign turned out to be a rather more delicate matter. We shall, therefore, describe the method by which the decision was reached, as it will establish at the same time that the expression which we shall obtain does represent the solution to our problem.

According to our remarks in the preceding paragraph, we shall write

$$\left. \begin{aligned} F(\mu) = \sum_{2^n \text{ terms}} & \epsilon_{r_1 \ldots r_l} \lambda_{r_1} \ldots \lambda_{r_l} \\ & \times \frac{\prod_{m=1}^{n-l} \prod_{i=1}^{l} (x_{s_m} + x_{r_i})}{\prod_{m=1}^{n-l} \prod_{i=1}^{l} (x_{s_m} - x_{r_i})} \prod_{i=1}^{l} (x_{r_i} + \mu) \prod_{m=1}^{n-l} (x_{s_m} - \mu) , \end{aligned} \right\} \tag{76}$$

where

$$\epsilon^2_{r_1 \ldots r_l} = 1 \tag{77}$$

but is unspecified, otherwise, for the present.

It should be particularly noted that, in the summation on the right-hand side of equation (76), terms with the various factors $\lambda_{r_1} \ldots \lambda_{r_l}$ occur just exactly once.

Now if $F(\mu)$, as given by equation (76), represents a polynomial which satisfies the conditions of our problem, then we must have

$$F(x_{r_j}) = \lambda_{r_j} F(-x_{r_j}) . \tag{78}$$

According to equation (76), in $F(x_{r_j})$, there is *one and only one term* which occurs with the factor

$$\lambda_{r_1} \ldots \lambda_{r_j} \ldots \lambda_{r_l} , \tag{79}$$

and this arises, in fact, from the term

$$\left. \begin{aligned} \epsilon_{r_1 \ldots r_j \ldots r_l} \lambda_{r_1} \ldots \lambda_{r_j} \ldots \lambda_{r_l} & \frac{\prod_{m=1}^{n-l} \prod_{i=1}^{l} (x_{s_m} + x_{r_i})}{\prod_{m=1}^{n-l} \prod_{i=1}^{l} (x_{s_m} - x_{r_i})} \\ & \times \prod_{i=1}^{l} (x_{r_i} + \mu) \prod_{m=1}^{n-l} (x_{s_m} - \mu) , \end{aligned} \right\} \tag{80}$$

on the right-hand side of equation (76). The validity of equation (78) therefore requires that the term in $F(x_{r_j})$ arising from equation (80) must cancel the term in $F(-x_{r_j})$ which occurs with the factor

$$\lambda_{r_1} \dots \lambda_{r_{j-1}}\lambda_{r_{j+1}} \dots \lambda_{r_l} ; \tag{81}$$

and the only term on the right-hand side of equation (76) which occurs with this factor is

$$\left.\begin{aligned}&\epsilon_{r_1 \dots r_{j-1}r_{j+1} \dots r_l}\lambda_{r_1} \dots \lambda_{r_{j-1}}\lambda_{r_{j+1}} \dots \lambda_{r_l}\\ &\times\frac{\prod_{m=1}^{n-l}\prod_{i=1, i\neq j}^{l}(x_{s_m}+x_{r_i})\prod_{i=1, i\neq j}^{l}(x_{r_j}+x_{r_i})}{\prod_{m=1}^{n-l}\prod_{i=1, i\neq j}^{l}(x_{s_m}-x_{r_i})\prod_{i=1, i\neq j}^{l}(x_{r_j}-x_{r_i})}\prod_{i=1, i\neq j}^{l}(x_{r_i}+\mu)\prod_{m=1}^{n-l}(x_{s_m}-\mu)(x_{r_j}-\mu) .\end{aligned}\right\} \tag{82}$$

Putting $\mu = x_{r_j}$ in equation (80) and $\mu = -x_{r_j}$ in equation (82), we have, respectively,

$$\epsilon_{r_1 \dots r_j \dots r_l}\lambda_{r_1} \dots \lambda_{r_j} \dots \lambda_{r_l}\frac{\prod_{m=1}^{n-l}\prod_{i=1}^{l}(x_{s_m}+x_{r_i})}{\prod_{m=1}^{n-l}\prod_{i=1, i\neq j}^{l}(x_{s_m}-x_{r_i})}\prod_{i=1}^{l}(x_{r_i}+x_{r_j}) \tag{83}$$

and

$$\left.\begin{aligned}&\epsilon_{r_1 \dots r_{j-1}r_{j+1} \dots r_l}\lambda_{r_1} \dots \lambda_{r_{j-1}}\lambda_{r_{j+1}} \dots \lambda_{r_l}\\ &\times\frac{\prod_{m=1}^{n-l}\prod_{i=1, i\neq j}^{l}(x_{s_m}+x_{r_i})\prod_{i=1, i\neq j}^{l}(x_{r_j}+x_{r_i})}{\prod_{m=1}^{n-l}\prod_{i=1, i\neq j}^{l}(x_{s_m}-x_{r_i})\prod_{i=1, i\neq j}^{l}(x_{r_j}-x_{r_i})}\prod_{i=1, i\neq j}^{l}(x_{r_i}-x_{r_j})\prod_{m=1}^{n-l}(x_{s_m}+x_{r_j})(x_{r_j}+x_{r_j})\\ &\qquad=\epsilon_{r_1 \dots r_{j-1}r_{j+1} \dots r_l}\lambda_{r_1} \dots \lambda_{r_{j-1}}\lambda_{r_{j+1}} \dots \lambda_{r_l}(-1)^{l-1}\\ &\qquad\times\frac{\prod_{m=1}^{n-l}\prod_{i=1}^{l}(x_{s_m}+x_{r_i})}{\prod_{m=1}^{n-l}\prod_{i=1, i\neq j}^{l}(x_{s_m}-x_{r_i})}\prod_{i=1}^{l}(x_{r_i}+x_{r_j}) .\end{aligned}\right\} \tag{84}$$

Comparing (83) and (84), we observe that the validity of equation (78) requires only that

$$\epsilon_{r_1 \ldots r_l} = \epsilon_{r_1 \ldots r_{j-1} r_{j+1} \ldots r_l} (-1)^{l-1} . \quad (85)$$

Hence,

$$\epsilon_{r_1 \ldots r_l} = \epsilon_l = (-1)^{l-1} \epsilon_{l-1} . \quad (86)$$

Letting $\epsilon_n = 1$, we conclude from equation (86) that

$$\epsilon_n = +1 , \quad \epsilon_{n-1} = (-1)^{n-1} , \quad \epsilon_{n-2} = -1 , \quad \epsilon_{n-3} = (-1)^n , \quad \epsilon_{n-4} = +1 , \text{ etc.} \quad (87)$$

The ϵ's which occur in equation (76) can therefore be arranged in a sequence which we shall denote by $\epsilon_l^{(e)}$. With this choice of the ϵ's, equation (77) does, in fact, represent a solution for $F(\mu)$. (Any constant multiple of $F(\mu)$ will, of course, also be a solution.)

The solution for $G(\mu)$ can be constructed along similar lines. Thus $G(\mu)$ is also a linear combination of the 2^n polynomials (73), with, in fact, the same coefficients as $F(\mu)$ except for the ϵ-factor. And it can be verified that again ϵ depends only on the number of factors, l, in (73) which occurs with the positive sign in the parenthesis. However, in view of the minus sign in the conditions (52), we must now require that (cf. eq. [86])

$$\epsilon_l = \epsilon_{l-1} (-1)^l . \quad (88)$$

The ϵ's in the expansion for $G(\mu)$ therefore form the sequence

$$\epsilon_n = +1 . \quad \epsilon_{n-1} = (-1)^n , \quad \epsilon_{n-2} = -1 , \quad \epsilon_{n-3} = (-1)^{n-1} . \quad \epsilon_{n-4} = +1 , \text{ etc.,} \quad (89)$$

which we shall denote by $\epsilon_l^{(o)}$. We have thus established the following basic theorem:

Theorem 2.—The polynomials

$$F(\mu) = \sum_{2^n \text{ terms}} \epsilon_l^{(e)} \frac{\prod_{m=1}^{n-l} \prod_{i=1}^{l} (x_{s_m} + x_{r_i})}{\prod_{m=1}^{n-l} \prod_{i=1}^{l} (x_{s_m} - x_{r_i})} \prod_{i=1}^{l} \lambda_{r_i} (x_{r_i} + \mu) \prod_{m=1}^{n-l} (x_{s_m} - \mu) \quad (90)$$

and

$$G(\mu) = \sum_{2^n \text{ terms}} \epsilon_l^{(o)} \frac{\prod_{m=1}^{n-l} \prod_{i=1}^{l} (x_{s_m} + x_{r_i})}{\prod_{m=1}^{n-l} \prod_{i=1}^{l} (x_{s_m} - x_{r_i})} \prod_{i=1}^{l} \lambda_{r_i} (x_{r_i} + \mu) \prod_{m=1}^{n-l} (x_{s_m} - \mu) , \quad (91)$$

where $\epsilon_l^{(e)}$ *and* $\epsilon_l^{(o)}$ *denote the sequences*

$$\epsilon_l^{(e)} = +1 , \ (-1)^{n-1} , \ -1 , \ (-1)^n , \ +1 , \ (-1)^{n-1} , \ -1 , \ (-1)^n , \ldots , \quad (92)$$

and

$$\epsilon_l^{(o)} = +1 , \ (-1)^n , \ -1 , \ (-1)^{n-1} , \ +1 , \ (-1)^n , \ -1 , \ (-1)^{n-1} , \ldots , \quad (93)$$

satisfy the conditions

$$F(x_a) = \lambda_a F(-x_a) \quad \text{and} \quad G(x_a) = -\lambda_a G(-x_a) \quad (a = 1 , \ldots , n) , \quad (94)$$

where x_α, $\alpha = 1, \ldots, n$, *are* n *distinct values of the argument and* λ_α, $\alpha = 1, \ldots, n$, *are* n *assigned numbers all different from one another. Any other polynomial of degree n which satisfies either of these conditions must be a simple numerical multiple of* $F(\mu)$ *or* $G(\mu)$ *as defined.*

By writing

$$x_\alpha = \frac{1}{k_\alpha} \qquad (\alpha = 1, \ldots, n), \tag{95}$$

we can express the solutions (90) and (91) for $F(\mu)$ and $G(\mu)$ alternatively in the forms

$$F(\mu) = \sum_{2^n \text{ terms}} \epsilon_l^{(e)} \frac{\prod_{i=1}^{l}\prod_{m=1}^{n-l}(k_{r_i}+k_{s_m})}{\prod_{i=1}^{l}\prod_{m=1}^{n-l}(k_{r_i}-k_{s_m})} \prod_{i=1}^{l}(1+k_{r_i}\mu)\prod_{m=1}^{n-l}\frac{1}{\lambda_{s_m}}(1-k_{s_m}\mu) \tag{96}$$

and

$$G(\mu) = \sum_{2^n \text{ terms}} \epsilon_l^{(o)} \frac{\prod_{i=1}^{l}\prod_{m=1}^{n-l}(k_{r_i}+k_{s_m})}{\prod_{i=1}^{l}\prod_{m=1}^{n-l}(k_{r_i}-k_{s_m})} \prod_{i=1}^{l}(1+k_{r_i}\mu)\prod_{m=1}^{n-l}\frac{1}{\lambda_{s_m}}(1-k_{s_m}\mu)\,. \tag{97}$$

Now, examining the sequences (92) and (93), we observe that the terms n, $n-2$, etc., agree, while the terms $n-1$, $n-3$, etc., are of opposite signs. We can, therefore, express $F(\mu)$ and $G(\mu)$ in the forms

$$F(\mu) = C_0(\mu) + C_1(\mu) \tag{98}$$

and

$$G(\mu) = C_0(\mu) - C_1(\mu)\,, \tag{99}$$

where

$$C_0(\mu) = \sum_{2^{n-1} \text{ terms}}^{l=n,\, n-2,\, \ldots} \epsilon_l^{(0)} \frac{\prod_{i=1}^{l}\prod_{m=1}^{n-l}(k_{r_i}+k_{s_m})}{\prod_{i=1}^{l}\prod_{m=1}^{n-l}(k_{r_i}-k_{s_m})} \prod_{i=1}^{l}(1+k_{r_i}\mu)\prod_{m=1}^{n-l}\frac{1}{\lambda_{s_m}}(1-k_{s_m}\mu) \tag{100}$$

and

$$C_1(\mu) = (-1)^{n-1} \sum_{2^{n-1} \text{ term}}^{l=n-1;\, n-3,\, \ldots} \epsilon_l^{(1)} \frac{\prod_{i=1}^{l}\prod_{m=1}^{n-l}(k_{r_i}+k_{s_m})}{\prod_{i=1}^{l}\prod_{m=1}^{n-l}(k_{r_i}-k_{s_m})} \prod_{i=1}^{l}(1+k_{r_i}\mu)\prod_{m=1}^{n-l}\frac{1}{\lambda_{s_m}}(1-k_{s_m}\mu)\,, \quad (1C$$

where

$$\left.\begin{aligned} \epsilon_l^{(0)} &= +1 \text{ for integers of the form } n-4l \\ &= -1 \text{ for integers of the form } n-4l-2 \\ &= 0 \text{ otherwise} \end{aligned}\right\} \tag{102}$$

and

$$\left.\begin{aligned}\epsilon_l^{(1)} &= +1 \text{ for integers of the form } n-4l-1 \\ &= -1 \text{ for integers of the form } n-4l-3 \\ &= 0 \text{ otherwise}\end{aligned}\right\} . \tag{103}$$

In our future work we shall adopt equations (96)–(103) as our standard definitions of the various functions, and it will always have to be assumed that functions defined in this manner are meant unless something explicitly to the contrary is stated.

Finally, some properties of the functions $C_0(\mu)$ and $C_1(\mu)$ may be noted:

By virtue of equations (98) and (99), the identity (53) between $F(\mu)$ and $G(\mu)$ becomes, in our present notation (cf. eq. [25])

$$\left.\begin{aligned}C_0(\mu)C_0(-\mu) - C_1(\mu)C_1(-\mu) &= \text{constant} \prod_{a=1}^{n}(1-k_a^2\mu^2) \\ &= \text{constant } W(\mu) .\end{aligned}\right\} \tag{104}$$

Since $W(0) = 1$, we can re-write equation (104) in the form

$$C_0(\mu)C_0(-\mu) - C_1(\mu)C_1(-\mu) = [C_0^2(0) - C_1^2(0)]W(\mu) , \tag{105}$$

a relation which we shall find very useful in our further work.

Again, according to equations (51) and (52),

$$F(x_a) + G(x_a) = \lambda_a[F(-x_a) - G(-x_a)] , \tag{106}$$

and

$$F(x_a) - G(x_a) = \lambda_a[F(-x_a) + G(-x_a)] . \tag{107}$$

Expressing F and G as in equations (98) and (99), we find that the foregoing equations are equivalent, in our present notation, to

$$C_0(1/k_a) = \lambda_a C_1(-1/k_a) \qquad (a = 1, \ldots, n) \tag{108}$$

and

$$C_1(1/k_a) = \lambda_a C_0(-1/k_a) \qquad (a = 1, \ldots, n) . \tag{109}$$

Equations (108) and (109) will be formally equivalent to each other if (again formally!)

$$\lambda_a = \frac{1}{\lambda_{-a}} . \tag{110}$$

For λ_a defined as in equation (50), this is *actually* the case.

5. *Completion of the solutions for* s(μ) *and* t(μ).—Returning, now, to the solution for $s(\mu)$ and $t(\mu)$ at the point where we left it in § 3, we conclude that $s(\mu) + t(\mu)$ and $s(\mu) - t(\mu)$ must be proportional, respectively, to $F(\mu)$ and $G(\mu)$ as we have defined them in equations (96) and (97), with λ_a having the particular value given by equation (50). Expressing $F(\mu)$ and $G(\mu)$ in terms of $C_0(\mu)$ and $C_1(\mu)$, as in equations (100) and (101), we can therefore write

$$s(\mu) = q_0 C_0(\mu) + q_1 C_1(\mu) \tag{111}$$

and

$$t(\mu) = q_1 C_0(\mu) + q_0 C_1(\mu) , \tag{112}$$

where q_0 and q_1 are two constants. To determine these constants we make use of equations (34) and (35), which require that

$$s(\mu_0) = q_0 C_0(\mu_0) + q_1 C_1(\mu_0) = P(-\mu_0) \tag{113}$$

and

$$t(-\mu_0) = q_0 C_1(-\mu_0) + q_1 C_0(-\mu_0) = P(\mu_0)\, e^{-\tau_1/\mu_0}\,. \tag{114}$$

Solving these equations for q_0 and q_1, we find

$$q_0 = \frac{P(-\mu_0) C_0(-\mu_0) - e^{-\tau_1/\mu_0} P(\mu_0) C_1(\mu_0)}{C_0(\mu_0) C_0(-\mu_0) - C_1(\mu_0) C_1(-\mu_0)} \tag{115}$$

and

$$q_1 = \frac{e^{-\tau_1/\mu_0} P(\mu_0) C_0(\mu_0) - P(-\mu_0) C_1(-\mu_0)}{C_0(\mu_0) C_0(-\mu_0) - C_1(\mu_0) C_1(-\mu_0)}\,. \tag{116}$$

The denominator in equations (115) and (116) can be simplified by using equation (105). We thus find that

$$q_0 = \frac{1}{[C_0^2(0) - C_1^2(0)]\, W(\mu_0)} [P(-\mu_0) C_0(-\mu_0) - e^{-\tau_1/\mu_0} P(\mu_0) C_1(\mu_0)] \tag{117}$$

and

$$q_1 = \frac{1}{[C_0^2(0) - C_1^2(0)]\, W(\mu_0)} [e^{-\tau_1/\mu_0} P(\mu_0) C_0(\mu_0) - P(-\mu_0) C_1(-\mu_0)]\,. \tag{118}$$

With this determination of the constants q_0 and q_1, we have completed the formal solution of our problem.

6. *The solution for the reflected and the transmitted radiations.*—With $s(\mu)$ and $t(\mu)$ given by equations (111) and (112), equations (30) and (31) for $S(\mu)$ and $T(\mu)$ take the forms

$$S(\mu) = \frac{1}{\mu_1^2 \ldots \mu_n^2} \frac{P(\mu)}{W(\mu)} \frac{\mu_0}{\mu_0 - \mu} [q_0 C_0(\mu) + q_1 C_1(\mu)] \tag{119}$$

and

$$T(\mu) = \frac{1}{\mu_1^2 \ldots \mu_n^2} \frac{P(\mu)}{W(\mu)} \frac{\mu_0}{\mu_0 + \mu} [q_1 C_0(\mu) + q_0 C_1(\mu)]\,. \tag{120}$$

Substituting for $S(\mu)$ and $T(\mu)$ from the foregoing equations in equations (22) and (23), we obtain, after some minor rearranging of the terms, the following expressions for the reflected and the transmitted intensities:

$$\left.\begin{aligned} I(0, \mu) = \frac{1}{4} \frac{\varpi_0 F}{\mu_1^2 \ldots \mu_n^2} \frac{1}{W(\mu)} [q_0 \{P(-\mu) C_0(-\mu) - e^{-\tau_1/\mu} P(\mu) C_1(\mu)\} \\ - q_1 \{e^{-\tau_1/\mu} P(\mu) C_0(\mu) - P(-\mu) C_1(-\mu)\}] \frac{\mu_0}{\mu_0 + \mu} \end{aligned}\right\} \tag{121}$$

and

$$\left.\begin{aligned} I(\tau_1, -\mu) = \frac{1}{4} \frac{\varpi_0 F}{\mu_1^2 \ldots \mu_n^2} \frac{1}{W(\mu)} [q_1 \{P(-\mu) C_0(-\mu) - e^{-\tau_1/\mu} P(\mu) C_1(\mu)\} \\ - q_0 \{e^{-\tau_1/\mu} P(\mu) C_0(\mu) - P(-\mu) C_1(-\mu)\}] \frac{\mu_0}{\mu_0 - \mu}\,. \end{aligned}\right\} \tag{122}$$

Substituting, next, for q_0 and q_1 according to equations (117) and (118), we have

$$\left.\begin{aligned} I(0,\mu) = & \frac{1}{4}\frac{\varpi_0 F}{\mu_1^2\ldots\mu_n^2}\frac{1}{[C_0^2(0)-C_1^2(0)]}\frac{1}{W(\mu)W(\mu_0)}\frac{\mu_0}{\mu_0+\mu} \\ & \times [\{P(-\mu)C_0(-\mu)-e^{-\tau_1/\mu}P(\mu)C_1(\mu)\}\{P(-\mu_0)C_0(-\mu_0)-e^{-\tau_1/\mu_0}P(\mu_0)C_1(\mu_0)\} \\ & -\{e^{-\tau_1/\mu}P(\mu)C_0(\mu)-P(-\mu)C_1(-\mu)\}\{e^{-\tau_1/\mu_0}P(\mu_0)C_0(\mu_0)-P(-\mu_0)C_1(-\mu_0)\}] \end{aligned}\right\} \quad (123)$$

and

$$\left.\begin{aligned} I(\tau_1,-\mu) = & \frac{1}{4}\frac{\varpi_0 F}{\mu_1^2\ldots\mu_n^2}\frac{1}{[C_0^2(0)-C_1^2(0)]}\frac{1}{W(\mu)W(\mu_0)}\frac{\mu_0}{\mu_0-\mu} \\ & \times [\{P(-\mu)C_0(-\mu)-e^{-\tau_1/\mu}P(\mu)C_1(\mu)\}\{e^{-\tau_1/\mu_0}P(\mu_0)C_0(\mu_0)-P(-\mu_0)C_1(-\mu_0)\} \\ & -\{e^{-\tau_1/\mu}P(\mu)C_0(\mu)-P(-\mu)C_1(-\mu)\}\{P(-\mu_0)C_0(-\mu_0)-e^{-\tau_1/\mu_0}P(\mu_0)C_1(\mu_0)\}]. \end{aligned}\right\} \quad (124)$$

Now let

$$X(\mu) = \frac{(-1)^n}{\mu_1\ldots\mu_n}\frac{1}{[C_0^2(0)-C_1^2(0)]^{\frac{1}{2}}}\frac{1}{W(\mu)}[P(-\mu)C_0(-\mu)-e^{-\tau_1/\mu}P(\mu)C_1(\mu)] \quad (125)$$

and

$$Y(\mu) = \frac{(-1)^n}{\mu_1\ldots\mu_n}\frac{1}{[C_0^2(0)-C_1^2(0)]^{\frac{1}{2}}}\frac{1}{W(\mu)}[e^{-\tau_1/\mu}P(\mu)C_0(\mu)-P(-\mu)C_1(-\mu)]. \quad (126)$$

It will appear that functions $X(\mu)$ and $Y(\mu)$, defined in this manner, play the same fundamental role in the theory of atmospheres of finite optical thicknesses as the function $H(\mu)$ did in the theory of semi-infinite atmospheres.

In terms of the functions $X(\mu)$ and $Y(\mu)$, equations (123) and (124) for the reflected and the transmitted intensities take the following simple forms:

$$I(0,\mu) = \tfrac{1}{4}\varpi_0 F\frac{\mu_0}{\mu_0+\mu}[X(\mu)X(\mu_0)-Y(\mu)Y(\mu_0)] \quad (127)$$

and

$$I(\tau_1,-\mu) = \tfrac{1}{4}\varpi_0 F\frac{\mu_0}{\mu_0-\mu}[X(\mu)Y(\mu_0)-Y(\mu)X(\mu_0)]. \quad (128)$$

It will be seen that the solutions for $I(0, \mu)$ and $I(\tau_1, -\mu)$ given by equations (127) and (128) are of exactly the forms required by the functional equations satisfied by the scattering and the transmission functions (Paper XVII, § 6); they further bring into evidence Helmholtz' principle of reciprocity.

Finally, attention should be drawn to the fact that there is nothing in the analysis of the preceding sections which has depended on the k_a's being the roots of the particular characteristic equation (2) except that it has $2n$ roots which occur in pairs (i.e., $k_a = -k_{-a}$). The method of solution and the reduction to the basic problem considered in § 4 has depended only on the single circumstance of the solution for the intensities being of the form given by equations (1) and (3). Conversely, it follows that the expressions for the reflected and the transmitted radiations can always be brought to the forms (127) and (128), provided only that the intensities I_i at the points of the Gaussian division are given by equations of the general form of (1) and (3) and the k_a's are the roots of an equation of the form

$$1 = 2\sum_{j=1}^{n}\frac{a_j\Psi(\mu_j)}{1-k^2\mu_j^2}, \quad (129)$$

where the characteristic function, $\Psi(\mu)$, is an even polynomial, satisfying the condition

$$\int_0^1 \Psi(\mu)\, d\mu < \tfrac{1}{2}\,. \tag{130}$$ [11]

An obvious corollary of this observation is that for all equations of transfer of the form considered in Paper IX, § 4, the solution for the reflected and the transmitted intensities can be reduced to the forms given by equations (127) and (128), in which the functions $X(\mu)$ and $Y(\mu)$ are defined according to equations (24), (25), (100), (101), (125), and (126).

II. ISOTROPIC SCATTERING WITH UNIT ALBEDO

7. *The reduction for the case* $\varpi_0 = 1$.—The solution for the case of isotropic scattering with unit albedo cannot be obtained by simply letting $\varpi_0 = 1$ in the equations of the preceding sections (§§ 5 and 6); for, in this case, the various functions become indeterminate because two of the characteristic roots become zero. While there can, of course, be no difficulty of principle in properly passing to the limit $\varpi_0 = 1$ with due regard to the indeterminateness we have mentioned, it appears simpler to treat this case separately.

When $\varpi_0 = 1$, the solution for the intensities I_i at the points of the Gaussian division is (cf. Paper VIII)

$$\left.\begin{aligned} I_i = \tfrac{1}{4}F\left[\sum_{a=-n+1}^{+n-1} \frac{L_a e^{-k_a\tau}}{1+k_a\mu_i} + L_0(\tau+\mu_i) + L_n + \frac{\gamma e^{-\tau/\mu_0}}{1+\mu_i/\mu_0}\right] \\ (i = \pm 1, \ldots, \pm n)\,, \end{aligned}\right\} \tag{131}$$

where the k_a's ($a = \pm 1, \ldots, \pm n \mp 1$, and $k_{+a} = -k_{-a}$) are the $(2n-2)$ distinct nonvanishing roots of the characteristic equation

$$1 = \sum_{j=1}^{n} \frac{a_j}{1-k^2\mu_j^2} \tag{132}$$

and the L_a's ($a = 0, \pm 1, \ldots, \pm n \mp 1, n$) are the $2n$ constants of integration and

$$\gamma = H(\mu_0)\,H(-\mu_0) = \frac{1}{\mu_1^2 \cdots \mu_n^2}\,\frac{P(\mu_0)\,P(-\mu_0)}{W(\mu_0)}\,. \tag{133}$$

In terms of the functions

$$S(\mu) = \sum_{a=-n+1}^{+n-1} \frac{L_a}{1-k_a\mu} - L_0\mu + L_n + \frac{\gamma}{1-\mu/\mu_0} \tag{134}$$

and

$$T(\mu) = \sum_{a=-n+1}^{+n-1} \frac{L_a e^{-k_a\tau_1}}{1+k_a\mu} + L_0(\tau_1+\mu) + L_n + \frac{\gamma e^{-\tau_1/\mu_0}}{1+\mu/\mu_0}\,, \tag{135}$$

the boundary conditions requiring

$$I_{-i} = 0 \text{ at } \tau = 0 \quad \text{and} \quad I_{+i} = 0 \text{ at } \tau = \tau_1 \quad \text{for} \quad i = 1, \ldots, n\,, \tag{136}$$

can be written as

$$S(\mu_i) = T(\mu_i) = 0 \qquad (i = 1, \ldots, n)\,. \tag{137}$$

[11] Notice the exclusion of the equality sign here. This means that we exclude conservative cases from this discussion (see Sec. II below).

And, as in the case $\varpi_0 < 1$ (§ 2), it can be shown that, in this case also, the reflected and the transmitted intensities can be expressed in the forms (cf. eqs. [22] and [23])

$$I(0, \mu) = \tfrac{1}{4}F\,[S(-\mu) - e^{-\tau_1/\mu}T(\mu)] \qquad (138)$$

and

$$I(\tau_1, -\mu) = \tfrac{1}{4}F\,[T(-\mu) - e^{-\tau_1/\mu}S(\mu)]\,. \qquad (139)$$

Returning to equations (134) and (135), we observe that, since the μ_i's ($i = 1, \ldots, n$) are zeros of $S(\mu)$ and $T(\mu)$, we can write (cf. eqs. [30], [31], and n. 10)

$$S(\mu) = \frac{1}{\mu_1^2 \ldots \mu_n^2}\frac{P(\mu)}{W(\mu)}\frac{1}{1-\mu/\mu_0}\,s(\mu) \qquad (140)$$

and

$$T(\mu) = \frac{1}{\mu_1^2 \ldots \mu_n^2}\frac{P(\mu)}{W(\mu)}\frac{1}{1+\mu/\mu_0}\,t(\mu)\,, \qquad (141)$$

where $s(\mu)$ and $t(\mu)$ are polynomials of degree n in μ.

From equations (133)–(135), (140), and (141), it readily follows that (cf. eqs. [32]–[35])

$$s(\mu_0) = P(-\mu_0) \qquad \text{and} \qquad t(-\mu_0) = e^{-\tau_1/\mu_0}P(\mu_0)\,. \qquad (142)$$

But we now have only $(2n - 2)$ relations of the form (cf. eq. [40])

$$s(1/k_a) = e^{k_a\tau_1}\frac{P(-1/k_a)}{P(+1/k_a)}\,t(-1/k_a) \qquad (a = \pm 1, \ldots, \pm n \mp 1)\,. \qquad (143)$$

Consequently, $s(\mu) + t(\mu)$ and $s(\mu) - t(\mu)$ satisfy only $(n - 1)$ (instead of n) relations of the forms (46) and (47), respectively. In accordance with theorems 1 and 2 (§ 4), we therefore conclude that, in the present case, $s(\mu) + t(\mu)$ must be a linear combination of $F(\mu)$ and $\mu G(\mu)$ with numerical coefficients, where $F(\mu)$ and $G(\mu)$ are polynomials of degree $n - 1$, defined in the manner of equations (50), (96), and (97) in terms of the $n - 1$ positive nonvanishing roots of equation (132).[12] Similarly, $s(\mu) - t(\mu)$ must be a linear combination of $G(\mu)$ and $\mu F(\mu)$ with numerical coefficients. We can therefore write (cf. eqs. [98] and [99])

$$s(\mu) = (p_0 + q_0\mu)\,C_0(\mu) + (p_1 - q_1\mu)\,C_1(\mu) \qquad (144)$$

and

$$t(\mu) = (p_1 + q_1\mu)\,C_0(\mu) + (p_0 - q_0\mu)\,C_1(\mu)\,, \qquad (145)$$

where p_0, q_0, p_1, and q_1 are certain constants. Equations (142) provide two relations between these four constants. Two further relations can be obtained in the following manner:

According to equations (134), (135), (140), and (141), we have

$$\left.\begin{aligned} \frac{1}{\mu_1^2 \ldots \mu_n^2}P(\mu)\,s(\mu) &= \left(1-\frac{\mu}{\mu_0}\right)W(\mu)\,S(\mu) \\ &= (-1)^{n-1}k_1^2 \ldots k_{n-1}^2\left[\frac{L_0}{\mu_0}\mu^{2n} - \left(L_0 + \frac{L_n}{\mu_0}\right)\mu^{2n-1} + \ldots\right] \end{aligned}\right\} \qquad (146)$$

[12] We shall adopt this convention throughout. It is therefore always to be understood that W, W_a, F, G, C_0, C_1, X, and Y signify the functions defined as in eqs. (25), (26), (50), (96)–(103), (125), and (126) in terms of the positive nonvanishing roots of the particular characteristic equation which is appropriate in the context.

and

$$\left.\begin{aligned}\frac{1}{\mu_1 \cdots \mu_n^2} P(\mu)\, t(\mu) &= \left(1+\frac{\mu}{\mu_0}\right) W(\mu)\, T(\mu) \\ &= (-1)^{n-1} k_1^2 \cdots k_{n-1}^2 \left[\frac{L_0}{\mu_0}\mu^{2n} + \left(L_0 + \frac{L_n}{\mu_0} + \frac{L_0 \tau_1}{\mu_0}\right)\mu^{2n-1} + \cdots\right].\end{aligned}\right\} \tag{147}$$

From equations (146) and (147) it is apparent that the coefficients of μ^n in $s(\mu)$ and $t(\mu)$ are the same. This requires that in equations (144) and (145)

$$q_0 = q_1 = a \text{ (say)}. \tag{148}$$

We can therefore write

$$s(\mu) = (p_0 + a\mu)\, C_0(\mu) + (p_1 - a\mu)\, C_1(\mu) \tag{149}$$

and

$$t(\mu) = (p_1 + a\mu)\, C_0(\mu) + (p_0 - a\mu)\, C_1(\mu). \tag{150}$$

With the foregoing forms for $s(\mu)$ and $t(\mu)$, it is readily verified that

$$\left.\begin{aligned}P(\mu)\, s(\mu) = \mu^{2n}\{a[c_0^{(n-1)} - c_1^{(n-1)}]\} + \mu^{2n-1}\Big\{ a[c_0^{(n-2)} - c_1^{(n-2)}] \\ + p_0 c_0^{(n-1)} + p_1 c_1^{(n-1)} - \left(\sum_{i=1}^{n}\mu_i\right) a[c_0^{(n-1)} - c_1^{(n-1)}]\Big\} + \cdots\end{aligned}\right\} \tag{151}$$

and

$$\left.\begin{aligned}P(\mu)\, t(\mu) = \mu^{2n}\{a[c_0^{(n-1)} - c_1^{(n-1)}]\} + \mu^{2n-1}\Big\{ a[c_0^{(n-2)} - c_1^{(n-2)}] \\ + p_1 c_0^{(n-1)} + p_0 c_1^{(n-1)} - \left(\sum_{i=1}^{n}\mu_i\right) a[c_0^{(n-1)} - c_1^{(n-1)}]\Big\} + \cdots,\end{aligned}\right\} \tag{152}$$

where $c_0^{(n-1)}$, $c_0^{(n-2)}$ and $c_1^{(n-1}$, $c_1^{(n-2)}$ are the coefficients of the highest and the next highest powers of μ in $C_0(\mu)$ and $C_1(\mu)$, respectively.

Comparing equations (146) and (147) with (151) and (152), we conclude that

$$\left.\begin{aligned}\frac{1}{c_0^{(n-1)} - c_1^{(n-1)}}\Big\{ c_0^{(n-2)} - c_1^{(n-2)} + \frac{1}{a}[p_0 c_0^{(n-1)} + p_1 c_1^{(n-1)}]\Big\} - \sum_{i-1}^{n}\mu_i \\ = -\left(\mu_0 + \frac{L_n}{L_0}\right)\end{aligned}\right\} \tag{153}$$

and

$$\left.\begin{aligned}\frac{1}{c_0^{(n-1)} - c_1^{(n-1)}}\Big\{ c_0^{(n-2)} - c_1^{(n-2)} + \frac{1}{a}[p_1 c_0^{(n-1)} + p_0 c_1^{(n-1)}]\Big\} - \sum_{i=1}^{n}\mu_i \\ = +\left(\mu_0 + \frac{L_n}{L_0} + \tau_1\right).\end{aligned}\right\} \tag{154}$$

Adding equations (153) and (154), we obtain

$$\frac{1}{c_0^{(n-1)} - c_1^{(n-1)}}\Big\{ 2[c_0^{(n-2)} - c_1^{(n-2)}] + \frac{p_0 + p_1}{a}[c_0^{(n-1)} + c_1^{(n-1)}]\Big\} - 2\sum_{i=1}^{n}\mu_i = \tau_1. \tag{155}$$

Hence

$$a = Q(p_0 + p_1) , \tag{156}$$

where

$$Q = \frac{c_0^{(n-1)} + c_1^{(n-1)}}{\left(\tau_1 + 2\sum_{i-1}^{n} \mu_i\right)[c_0^{(n-1)} - c_1^{(n-1)}] - 2[c_0^{(n-2)} - c_1^{(n-2)}]} . \tag{157}$$

Now, from equations (142), (149), and (150) we have

$$(p_0 + a\mu_0) C_0(\mu_0) + (p_1 - a\mu_0) C_1(\mu_0) = P(-\mu_0) \tag{158}$$

and

$$(p_0 + a\mu_0) C_1(-\mu_0) + (p_1 - a\mu_0) C_0(-\mu_0) = e^{-\tau_1/\mu_0} P(\mu_0) . \tag{159}$$

Solving these equations for $(p_0 + a\mu_0)$ and $(p_1 - a\mu_0)$, we have (cf. eqs. [113]–[118])

$$p_0 + a\mu_0 = \frac{1}{[C_0^2(0) - C_1^2(0)]W(\mu_0)} [P(-\mu_0)C_0(-\mu_0) - e^{-\tau_1/\mu_0}P(\mu_0)C_1(\mu_0)] \tag{160}$$

and

$$p_1 - a\mu_0 = \frac{1}{[C_0^2(0) - C_1^2(0)]W(\mu_0)} [e^{-\tau_1/\mu_0}P(\mu_0)C_0(\mu_0) - P(-\mu_0)C_1(-\mu_0)] . \tag{161}$$

And, finally, according to equations (156), (160), and (161)

$$\left.\begin{aligned} a = \frac{Q}{[C_0^2(0) - C_1^2(0)]W(\mu_0)} & [P(-\mu_0)C_0(-\mu_0) - e^{-\tau_1/\mu_0}P(\mu_0)C_1(\mu_0) \\ & + e^{-\tau_1/\mu_0}P(\mu_0)C_0(\mu_0) - P(-\mu_0)C_1(-\mu_0)] . \end{aligned}\right\} \tag{162}$$

With this determination of the various constants, the solution of the formal problem is completed.

8. *The solution for the reflected and the transmitted radiations.*—The angular distributions of the reflected and the transmitted radiations are given by

$$I(0, \mu) = \frac{1}{4}\frac{F}{\mu_1^2 \ldots \mu_n^2}\frac{1}{W(\mu)} [P(-\mu)s(-\mu) - e^{-\tau_1/\mu}P(\mu)t(\mu)] \frac{\mu_0}{\mu_0 + \mu} \tag{163}$$

and

$$I(\tau_1, -\mu) = \frac{1}{4}\frac{F}{\mu_1^2 \ldots \mu_n^2 W(\mu)} [P(-\mu)t(-\mu) - e^{-\tau_1/\mu}P(\mu)s(\mu)] \frac{\mu_0}{\mu_0 - \mu} . \tag{164}$$

On the other hand, according to equations (149) and (150),

$$\left.\begin{aligned} & P(-\mu)s(-\mu) - e^{-\tau_1/\mu}P(\mu)t(\mu) \\ & \quad = (p_0 - a\mu)[P(-\mu)C_0(-\mu) - e^{-\tau_1/\mu}P(\mu)C_1(\mu)] \\ & \qquad - (p_1 + a\mu)[e^{-\tau_1/\mu}P(\mu)C_0(\mu) - P(-\mu)C_1(-\mu)] \\ & \quad = (p_0 + a\mu_0)[P(-\mu)C_0(-\mu) - e^{-\tau_1/\mu}P(\mu)C_1(\mu)] \\ & \qquad - (p_1 - a\mu_0)[e^{-\tau_1/\mu}P(\mu)C_0(\mu) - P(-\mu)C_1(-\mu)] \\ & \qquad - a(\mu_0 + \mu)[P(-\mu)C_0(-\mu) - e^{-\tau_1/\mu}P(\mu)C_1(\mu) \\ & \qquad\qquad + e^{-\tau_1/\mu}P(\mu)C_0(\mu) - P(-\mu)C_1(-\mu)] . \end{aligned}\right\} \tag{165}$$

Similarly,

$$\left.\begin{aligned} P(-\mu)\,t(-\mu) - e^{-\tau_1/\mu}P(\mu)\,s(\mu) & \\ = (p_1 - a\mu_0)\,[P(-\mu)\,C_0(-\mu) - e^{-\tau_1/\mu}P(\mu)\,C_1(\mu)] & \\ - (p_0 + a\mu_0)\,[e^{-\tau_1/\mu}P(\mu)\,C_0(\mu) - P(-\mu)\,C_1(-\mu)] & \\ + a(\mu_0 - \mu)\,[P(-\mu)\,C_0(-\mu) - e^{-\tau_1/\mu}P(\mu)\,C_1(\mu) & \\ + e^{-\tau_1/\mu}P(\mu)\,C_0(\mu) - P(-\mu)\,C_1(-\mu)]\,. & \end{aligned}\right\} \quad (166)$$

In the foregoing equations we can now substitute for $(p_0 + a\mu_0)$, $(p_1 - a\mu_0)$, and a in accordance with equations (160)–(162). In this manner we find

$$\left.\begin{aligned} I(0, \mu) = \tfrac{1}{4}F\frac{\mu_0}{\mu_0+\mu}\{X(\mu)\,X(\mu_0) - Y(\mu)\,Y(\mu_0) & \\ - Q(\mu_0+\mu)\,[X(\mu)+Y(\mu)]\,[X(\mu_0)+Y(\mu_0)]\} & \end{aligned}\right\} \quad (167)$$

and

$$\left.\begin{aligned} I(\tau_1, -\mu) = \tfrac{1}{4}F\frac{\mu_0}{\mu_0-\mu}\{X(\mu)\,Y(\mu_0) - Y(\mu)\,X(\mu_0) & \\ + Q(\mu_0-\mu)\,[X(\mu)+Y(\mu)]\,[X(\mu_0)+Y(\mu_0)]\}\,. & \end{aligned}\right\} \quad (168)$$

Equations (167) and (168) can be brought into forms analogous to solutions (127) and (128) for the case $\varpi_0 < 1$, if we introduce the functions

$$\psi(\mu) = X(\mu) - Q\mu\,[X(\mu) + Y(\mu)] \quad (169)$$

and

$$\phi(\mu) = Y(\mu) + Q\mu\,[X(\mu) + Y(\mu)]\,. \quad (170)$$

In terms of these functions, the angular distributions of the reflected and the transmitted radiations take the required forms:

$$I(0, \mu) = \tfrac{1}{4}F\frac{\mu_0}{\mu_0+\mu}\,[\psi(\mu)\,\psi(\mu_0) - \phi(\mu)\,\phi(\mu_0)] \quad (171)$$

and

$$I(\tau_1, -\mu) = \tfrac{1}{4}F\frac{\mu_0}{\mu_0-\mu}\,[\psi(\mu)\,\phi(\mu_0) - \phi(\mu)\,\psi(\mu_0)]\,. \quad (172)$$

III. SCATTERING IN ACCORDANCE WITH RAYLEIGH'S PHASE FUNCTION

9. *The azimuth independent term. The reduction of the solution.*—In solving the equation of transfer appropriately for the problem of diffuse reflection and transmission by a plane-parallel atmosphere scattering radiation in accordance with Rayleigh's phase function, we must distinguish three terms in the radiation field: an azimuth independent term, a term proportional to $\cos(\varphi - \varphi_0)$, and a term proportional to $\cos 2(\varphi - \varphi_0)$ (cf. Paper IX, eq. [152]). Considering, first, the azimuth independent term, we have the solution (cf. Papers III and IX, Sec. II)

$$\left.\begin{aligned} I_i = \tfrac{3}{32}F\Big[(3-\mu_i^2)\sum_{a=-n+1}^{+n-1}\frac{L_a e^{-k_a\tau}}{1+k_a\mu_i} + L_0(\tau+\mu_i) + L_n & \\ + (3-\mu_i^2)\frac{\gamma e^{-\tau/\mu_0}}{1+\mu_i/\mu_0}\Big] \qquad (i = \pm 1, \ldots, \pm n)\,, & \end{aligned}\right\} \quad (173)$$

where the k_a's ($a = \pm 1, \ldots, \pm n \mp 1$ and $k_{+a} = -k_{-a}$) are the $(2n-2)$ distinct nonvanishing roots of the characteristic equation

$$1 = \frac{3}{8}\sum_{j=1}^{n} \frac{a_j (3-\mu_j^2)}{1-k^2\mu_j^2}, \tag{174}$$

$$\gamma = H(\mu_0)\, H(-\mu_0), \tag{175}$$

and the rest of the symbols have their usual meanings.

In terms of the functions

$$S(\mu) = (3-\mu^2)\sum_{a=-n+1}^{+n-1} \frac{L_a}{1-k_a\mu} - L_0\mu + L_n + (3-\mu^2)\frac{\gamma}{1-\mu/\mu_0} \tag{176}$$

and

$$T(\mu) = (3-\mu^2)\sum_{a=-n+1}^{+n-1} \frac{L_a e^{-k_a\tau_1}}{1+k_a\mu} + L_0(\tau_1+\mu) + L_n + (3-\mu^2)\frac{\gamma e^{-\tau_1/\mu_0}}{1+\mu/\mu_0}, \tag{177}$$

the boundary conditions determining the constants of integration and the equations governing the reflected and the transmitted radiations can be written in the following forms:

$$S(\mu_i) = T(\mu_i) = 0 \qquad (i = 1, \ldots, n), \tag{178}$$

$$I(0, \mu) = \tfrac{3}{32}F\,[S(-\mu) - e^{-\tau_1/\mu}T(\mu)], \tag{179}$$

and

$$I(\tau_1, -\mu) = \tfrac{3}{32}F\,[T(-\mu) - e^{-\tau_1/\mu}S(\mu)]. \tag{180}$$

By virtue of the boundary conditions (eq. [178]), we can write

$$S(\mu) = \frac{1}{\mu_1^2 \ldots \mu_n^2}\frac{P(\mu)}{W(\mu)}\frac{1}{1-\mu/\mu_0}\, s(\mu) \tag{181}$$

an

$$T(\mu) = \frac{1}{\mu_1^2 \ldots \mu_n^2}\frac{P(\mu)}{W(\mu)}\frac{1}{1+\mu/\mu_0}\, t(\mu), \tag{182}$$

where $s(\mu)$ and $t(\mu)$ are polynomials of degree n in μ.

From equations (175)–(177), (181), and (182), it now follows that

$$s(\mu_0) = (3-\mu_0^2)\, P(-\mu_0) \tag{183}$$

and

$$t(-\mu_0) = (3-\mu_0^2)\, e^{-\tau_1/\mu_0}P(\mu_0). \tag{184}$$

And again from equations (176), (177), (181), and (182) we have

$$\left.\begin{aligned}\left(3-\frac{1}{k_a^2}\right)L_a = \frac{1}{\mu_1^2 \ldots \mu_n^2}\frac{P(+1/k_a)}{W_a(1/k_a)}\frac{1}{1-1/k_a\mu_0}\, s(1/k_a)\\ (a = \pm 1, \ldots, \pm n \mp 1)\end{aligned}\right\} \tag{185}$$

and

$$\left(3-\frac{1}{k_a^2}\right)L_a e^{-k_a\tau_1}=\frac{1}{\mu_1^2\ldots\mu_n^2}\frac{P(-1/k_a)}{W_a(1/k_a)}\frac{1}{1-1/k_a\mu_0}t(-1/k_a) \quad (a=\pm 1,\ldots,\pm n\mp 1). \qquad (186)$$

We therefore have the $2n-2$ relations

$$s(1/k_a)=e^{k_a\tau_1}\frac{P(-1/k_a)}{P(+1/k_a)}t(-1/k_a) \qquad (a=\pm 1,\ldots,\pm n\mp 1). \qquad (187)$$

Since, however, $s(\mu)$ and $t(\mu)$ are polynomials of degree n in μ, we conclude, in accordance with theorems 1 and 2 (§ 4), that $s(\mu)$ and $t(\mu)$ must be expressible in the forms (cf. eqs. [144] and [145])

$$s(\mu)=(p_0+q_0\mu)C_0(\mu)+(p_1-q_1\mu)C_1(\mu) \qquad (188)$$

and

$$t(\mu)=(p_1+q_1\mu)C_0(\mu)+(p_0-q_0\mu)C_1(\mu), \qquad (189)$$

where p_0, q_0, p_1, and q_1 are certain constants to be determined and $C_0(\mu)$ and $C_1(\mu)$ are polynomials of degree $n-1$ in μ defined in the manner of equations (100) and (101) in terms of the $(n-1)$ positive nonvanishing roots of the equation (174) (see n. 12).

From equations (183), (184), (188), and (189) we now have

$$(p_0+q_0\mu_0)C_0(\mu_0)+(p_1-q_1\mu_0)C_1(\mu_0)=(3-\mu_0^2)P(-\mu_0) \qquad (190)$$

and

$$(p_0+q_0\mu_0)C_1(-\mu_0)+(p_1-q_1\mu_0)C_0(-\mu_0)=(3-\mu_0^2)\,e^{-\tau_1/\mu_0}P(\mu_0). \qquad (191)$$

Solving these equations for $p_0+q_0\mu_0$ and $p_1-q_1\mu_0$, we have

$$p_0+q_0\mu_0=\frac{3-\mu_0^2}{[C_0^2(0)-C_1^2(0)]W(\mu_0)}[P(-\mu_0)C_0(-\mu_0)-e^{-\tau_1/\mu_0}P(\mu_0)C_1(\mu_0)] \qquad (192)$$

and

$$p_1-q_1\mu_0=\frac{3-\mu_0^2}{[C_0^2(0)-C_1^2(0)]W(\mu_0)}[e^{-\tau_1/\mu_0}P(\mu_0)C_0(\mu_0)-P(-\mu_0)C_1(-\mu_0)]. \qquad (193)$$

Next, combining equations (179)–(182), (188), and (189), we find, after some rearranging of the terms, that the reflected and the transmitted intensities can be expressed in the forms

$$\begin{aligned} I(0,\mu)=\tfrac{3}{32}\frac{F}{\mu_1^2\ldots\mu_n^2}\frac{1}{W(\mu)}\{&(p_0+q_0\mu_0)[P(-\mu)C_0(-\mu)-e^{-\tau_1/\mu}P(\mu)C_1(\mu)]\\ &-(p_1-q_1\mu_0)[e^{-\tau_1/\mu}P(\mu)C_0(\mu)-P(-\mu)C_1(-\mu)]\\ &-q_0(\mu_0+\mu)[P(-\mu)C_0(-\mu)-e^{-\tau_1/\mu}P(\mu)C_1(\mu)]\\ &-q_1(\mu_0+\mu)[e^{-\tau_1/\mu}P(\mu)C_0(\mu)-P(-\mu)C_1(-\mu)]\}\frac{\mu_0}{\mu_0+\mu} \end{aligned} \qquad (1\ldots$$

and

$$\left.\begin{aligned} I(\tau_1, -\mu) = \tfrac{3}{32}\frac{F}{\mu_1^2 \cdots \mu_n^2}\frac{1}{W(\mu)}\{(p_1 - q_1\mu_0)[P(-\mu)C_0(-\mu) - e^{-\tau_1/\mu}P(\mu)C_1(\mu)] \\ -(p_0 + q_0\mu_0)[e^{-\tau_1/\mu}P(\mu)C_0(\mu) - P(-\mu)C_1(-\mu)] \\ +q_1(\mu_0 - \mu)[P(-\mu)C_0(-\mu) - e^{-\tau_1/\mu}P(\mu)C_1(\mu)] \\ +q_0(\mu_0 - \mu)[e^{-\tau_1/\mu}P(\mu)C_0(\mu) - P(-\mu)C_1(-\mu)]\}\frac{\mu_0}{\mu_0 - \mu}. \end{aligned}\right\} \quad (195)$$

It remains to determine the constants q_0 and q_1.

10. *The determination of the constants* q_0 *and* q_1.—Putting $\mu = +\sqrt{3}$, respectively, $-\sqrt{3}$, in equations (176), (177), (181), and (182) we have

$$\left.\begin{aligned} \frac{1}{\mu_1^2 \cdots \mu_n^2}\frac{P(\pm\sqrt{3})}{W(\sqrt{3})}\frac{s(\pm\sqrt{3})}{1 \mp \frac{\sqrt{3}}{\mu_0}} &= S(\pm\sqrt{3}) \\ &= \mp\sqrt{3}L_0 + L_n \end{aligned}\right\} \quad (196)$$

and

$$\left.\begin{aligned} \frac{1}{\mu_1^2 \cdots \mu_n^2}\frac{P(\pm\sqrt{3})}{W(\sqrt{3})}\frac{t(\pm\sqrt{3})}{1 \pm \frac{\sqrt{3}}{\mu_0}} &= T(\pm\sqrt{3}) \\ &= (\tau_1 \pm \sqrt{3})L_0 + L_n\,. \end{aligned}\right\} \quad (197)$$

The foregoing equations can be simplified by using the relation (Paper XIV, eq. [267])

$$\frac{1}{\mu_1^2 \cdots \mu_n^2}\frac{P(+\sqrt{3})P(-\sqrt{3})}{W(\sqrt{3})} = -8\,. \quad (198)$$

We find

$$-\frac{8}{1 \mp \frac{\sqrt{3}}{\mu_0}}\frac{s(\pm\sqrt{3})}{P(\mp\sqrt{3})} = \mp\sqrt{3}L_0 + L_n \quad (199)$$

and

$$-\frac{8}{1 \pm \frac{\sqrt{3}}{\mu_0}}\frac{t(\pm\sqrt{3})}{P(\mp\sqrt{3})} = (\tau_1 \pm \sqrt{3})L_0 + L_n\,. \quad (200)$$

From equations (199) and (200) we obtain the following set of equations:

$$\frac{8}{1 \mp \frac{\sqrt{3}}{\mu_0}}\left[\frac{s(\pm\sqrt{3})}{P(\mp\sqrt{3})} - \frac{t(\mp\sqrt{3})}{P(\pm\sqrt{3})}\right] = L_0\tau_1\,, \quad (201)$$

$$-\frac{8}{1 + \frac{\sqrt{3}}{\mu_0}}\frac{t(+\sqrt{3})}{P(-\sqrt{3})} + \frac{8}{1 - \frac{\sqrt{3}}{\mu_0}}\frac{t(-\sqrt{3})}{P(+\sqrt{3})} = 2\sqrt{3}L_0\,, \quad (202)$$

and

$$-\frac{8}{1 + \frac{\sqrt{3}}{\mu_0}}\frac{s(-\sqrt{3})}{P(+\sqrt{3})} + \frac{8}{1 - \frac{\sqrt{3}}{\mu_0}}\frac{s(+\sqrt{3})}{P(-\sqrt{3})} = 2\sqrt{3}L_0\,. \quad (203)$$

Now, eliminating L_0 from equations (201)–(203), we obtain the equations

$$\left(1+\frac{\sqrt{3}}{\mu_0}\right)\left[\frac{s(+\sqrt{3})}{P(-\sqrt{3})}-\frac{(t-\sqrt{3})}{(P+\sqrt{3})}\right]=\left(1-\frac{\sqrt{3}}{\mu_0}\right)\left[\frac{s(-\sqrt{3})}{P(+\sqrt{3})}-\frac{t(+\sqrt{3})}{P(-\sqrt{3})}\right] \tag{204}$$

and

$$\left.\begin{aligned} 2\sqrt{3}\left\{\left(1+\frac{\sqrt{3}}{\mu_0}\right)\left[\frac{s(+\sqrt{3})}{P(-\sqrt{3})}-\frac{t(-\sqrt{3})}{P(+\sqrt{3})}\right]+\left(1-\frac{\sqrt{3}}{\mu_0}\right)\left[\frac{s(-\sqrt{3})}{P(+\sqrt{3})}-\frac{t(+\sqrt{3})}{P(-\sqrt{3})}\right]\right\} \\ =\tau_1\left\{\left(1+\frac{\sqrt{3}}{\mu_0}\right)\left[\frac{s(+\sqrt{3})}{P(-\sqrt{3})}+\frac{t(-\sqrt{3})}{P(+\sqrt{3})}\right]-\left(1-\frac{\sqrt{3}}{\mu_0}\right)\left[\frac{s(-\sqrt{3})}{P(+\sqrt{3})}+\frac{t(+\sqrt{3})}{P(-\sqrt{3})}\right]\right\} \\ =\frac{\sqrt{3}}{2}\left(1-\frac{3}{\mu_0^2}\right)L_0\tau_1 . \end{aligned}\right\} \quad (2$$

These equations can be further reduced to the forms

$$\left.\begin{aligned} (\sqrt{3}+\mu_0)[s(+\sqrt{3})P(+\sqrt{3})-t(-\sqrt{3})P(-\sqrt{3})] \\ +(\sqrt{3}-\mu_0)[s(-\sqrt{3})P(-\sqrt{3})-t(+\sqrt{3})P(+\sqrt{3})]=0 \end{aligned}\right\} \tag{206}$$

and

$$\left.\begin{aligned} 2\sqrt{3}\{(\sqrt{3}+\mu_0)[s(+\sqrt{3})P(+\sqrt{3})-t(-\sqrt{3})P(-\sqrt{3})]-(\sqrt{3}-\mu_0) \\ \times[s(-\sqrt{3})P(-\sqrt{3})-t(+\sqrt{3})P(+\sqrt{3})]\}=\tau_1\{(\sqrt{3}+\mu_0)[s(+\sqrt{3})P(+\sqrt{3}) \\ +t(-\sqrt{3})P(-\sqrt{3})]+(\sqrt{3}-\mu_0)[s(-\sqrt{3})P(-\sqrt{3})+t(+\sqrt{3})P(+\sqrt{3})]\}. \end{aligned}\right\} \quad (2$$

We now have to substitute for $s(\pm\sqrt{3})$ and $t(\pm\sqrt{3})$ in equations (206) and (207) according to equations (188) and (189). For this purpose, it is convenient to write $s(\pm\sqrt{3})$ and $t(\pm\sqrt{3})$ in the forms

$$\left.\begin{aligned} s(\pm\sqrt{3}) = (p_0+q_0\mu_0)C_0(\pm\sqrt{3})+(p_1-q_1\mu_0)C_1(\pm\sqrt{3}) \\ +(\sqrt{3}\mp\mu_0)[\pm q_0C_0(\pm\sqrt{3})\mp q_1C_1(\pm\sqrt{3})] \end{aligned}\right\} \tag{208}$$

and

$$\left.\begin{aligned} t(\pm\sqrt{3}) = (p_0+q_0\mu_0)C_1(\pm\sqrt{3})+(p_1-q_1\mu_0)C_0(\pm\sqrt{3}) \\ +(\sqrt{3}\pm\mu_0)[\mp q_0C_1(\pm\sqrt{3})\pm q_1C_0(\pm\sqrt{3})]. \end{aligned}\right\} \tag{209}$$

Substituting for $s(\pm\sqrt{3})$ and $t(\pm\sqrt{3})$ from equations (208) and (209) in equations (206) and (207), we find, after some lengthy but straightforward reductions, that

$$\left.\begin{aligned} (p_0+q_0\mu_0)(\sqrt{3}w_1+w_2\mu_0)+(p_1-q_1\mu_0)(-\sqrt{3}w_1+w_2\mu_0) \\ +(3-\mu_0^2)w_2(q_0-q_1)=0 \end{aligned}\right\} \tag{210}$$

and

$$\left.\begin{aligned} 2\sqrt{3}\{(p_0+q_0\mu_0)(\sqrt{3}w_2+w_1\mu_0)+(p_1-q_1\mu_0)(\sqrt{3}w_2-w_1\mu_0) \\ +(3-\mu_0^2)w_1(q_0+q_1)\}=\tau_1\{(p_0+q_0\mu_0)(\sqrt{3}w_4+w_3\mu_0) \\ +(p_1-q_1\mu_0)(\sqrt{3}w_4-w_3\mu_0)+(3-\mu_0^2)w_3(q_0+q_1)\}, \end{aligned}\right\} \tag{211}$$

where, for the sake of brevity, we have written

$$\left.\begin{aligned}\gamma_1&=C_0(\sqrt{3})P(\sqrt{3})-C_1(\sqrt{3})P(\sqrt{3})+C_0(-\sqrt{3})P(-\sqrt{3})-C_1(-\sqrt{3})P(-\sqrt{3}),\\ \gamma_2&=C_0(\sqrt{3})P(\sqrt{3})+C_1(\sqrt{3})P(\sqrt{3})-C_0(-\sqrt{3})P(-\sqrt{3})-C_1(-\sqrt{3})P(-\sqrt{3}),\\ \gamma_3&=C_0(\sqrt{3})P(\sqrt{3})-C_1(\sqrt{3})P(\sqrt{3})-C_0(-\sqrt{3})P(-\sqrt{3})+C_1(-\sqrt{3})P(-\sqrt{3}),\\ \gamma_4&=C_0(\sqrt{3})P(\sqrt{3})+C_1(\sqrt{3})P(\sqrt{3})+C_0(-\sqrt{3})P(-\sqrt{3})+C_1(-\sqrt{3})P(-\sqrt{3}).\end{aligned}\right\}\quad(212)$$

Equations (210) and (211) can be re-written in the forms

$$\left.\begin{aligned}(3-\mu_0^2)(q_0-q_1)+(p_0+q_0\mu_0)\left(\mu_0+\sqrt{3}\,\frac{w_1}{w_2}\right)&\\ +(p_1-q_1\mu_0)\left(\mu_0-\sqrt{3}\,\frac{w_1}{w_2}\right)&=0\end{aligned}\right\}\quad(213)$$

and

$$\left.\begin{aligned}(3-\mu_0^2)(q_0+q_1)+(p_0+q_0\mu_0)\left[\sqrt{3}\,\frac{w_4\tau_1-2\sqrt{3}w_2}{w_3\tau_1-2\sqrt{3}w_1}+\mu_0\right]&\\ +(p_1-q_1\mu_0)\left[\sqrt{3}\,\frac{w_4\tau_1-2\sqrt{3}w_2}{w_3\tau_1-2\sqrt{3}w_1}-\mu_0\right]&=0.\end{aligned}\right\}\quad(214)$$

Solving these equations for q_0 and q_1, we have

$$(3-\mu_0^2)\,q_0=-[(c_1+\mu_0)(p_0+q_0\mu_0)+c_2(p_1-q_1\mu_0)]\qquad(215)$$

and

$$(3-\mu_0^2)\,q_1=-[c_2(p_0+q_0\mu_0)+(c_1-\mu_0)(p_1-q_1\mu_0)],\qquad(216)$$

where

$$c_1=\frac{\sqrt{3}}{2}\left[\frac{w_4\tau_1-2\sqrt{3}w_2}{w_3\tau_1-2\sqrt{3}w_1}+\frac{w_1}{w_2}\right]\qquad(217)$$

and

$$c_2=\frac{\sqrt{3}}{2}\left[\frac{w_4\tau_1-2\sqrt{3}w_2}{w_3\tau_1-2\sqrt{3}w_1}-\frac{w_1}{w_2}\right].\qquad(218)$$

Since $(p_0+q_0\mu_0)$ and $(p_1-q_1\mu_0)$ have already been determined (eqs. [192] and [193]), the foregoing equations complete the formal solution for $S(\mu)$ and $T(\mu)$.

11. *The solution for the reflected and the transmitted radiations.*—Now, substituting for $(p_0+q_0\mu_0)$, $(p_1-q_1\mu_0)$, q_0, and q_1 according to equations (192), (193), (215), and (216) in equations (194) and (195) and introducing the functions $X(\mu)$ and $Y(\mu)$, defined as in equations (125) and (126), we have

$$\left.\begin{aligned}I(0,\mu)=\tfrac{3}{32}F\frac{\mu_0}{\mu_0+\mu}\{&(3-\mu_0^2)[X(\mu)X(\mu_0)-Y(\mu)Y(\mu_0)]\\ &+(\mu_0+\mu)X(\mu)[(c_1+\mu_0)X(\mu_0)+c_2Y(\mu_0)]\\ &+(\mu_0+\mu)Y(\mu)[c_2X(\mu_0)+(c_1-\mu_0)Y(\mu_0)]\}\end{aligned}\right\}\quad(219)$$

and

$$\left.\begin{aligned}I(\tau_1,-\mu)=\tfrac{3}{32}F\frac{\mu_0}{\mu_0-\mu}\{&(3-\mu_0^2)[X(\mu)Y(\mu_0)-Y(\mu)X(\mu_0)]\\ &-(\mu_0-\mu)X(\mu)[c_2X(\mu_0)+(c_1-\mu_0)Y(\mu_0)]\\ &-(\mu_0-\mu)Y(\mu)[(c_1+\mu_0)X(\mu_0)+c_2Y(\mu_0)]\}.\end{aligned}\right\}\quad(220)$$

After some rearranging, equations (219) and (220) can be brought to the forms

$$\left.\begin{aligned} I^{(0)}(0,\mu) = \tfrac{3}{32}F\frac{\mu_0}{\mu_0+\mu}\{&X^{(0)}(\mu)X^{(0)}(\mu_0)[3+c_1(\mu_0+\mu)+\mu\mu_0] \\ &-Y^{(0)}(\mu)Y^{(0)}(\mu_0)[3-c_1(\mu_0+\mu)+\mu\mu_0] \\ &+c_2(\mu_0+\mu)[X^{(0)}(\mu)Y^{(0)}(\mu_0)+Y^{(0)}(\mu)X^{(0)}(\mu_0)]\} \end{aligned}\right\} \quad (221)$$

and

$$\left.\begin{aligned} I^{(0)}(\tau_1,-\mu) = \tfrac{3}{32}F\frac{\mu_0}{\mu_0-\mu}\{&X^{(0)}(\mu)Y^{(0)}(\mu_0)[3-c_1(\mu_0-\mu)-\mu\mu_0] \\ &-Y^{(0)}(\mu)X^{(0)}(\mu_0)[3+c_1(\mu_0-\mu)-\mu\mu_0] \\ &-c_2(\mu_0-\mu)[X^{(0)}(\mu)X^{(0)}(\mu_0)+Y^{(0)}(\mu)Y^{(0)}(\mu_0)]\}. \end{aligned}\right\} \quad (222)$$

In equations (221) and (222) we have inserted a superscript "0" with the various functions to emphasize the fact that these equations represent only the solutions for the azimuth independent terms in the reflected and the transmitted intensities.

To complete the solution we need to find the remaining terms in the reflected and the transmitted intensities which are proportional to $\cos(\varphi-\varphi_0)$ and $\cos 2(\varphi-\varphi_0)$. The determination of these terms presents no difficulty, since the equations satisfied by $I^{(1)}(\tau,\mu)$ and $I^{(2)}(\tau,\mu)$ (cf. Paper IX, eqs. [154] and [155]) are of the standard form considered in Paper IX, § 4, and the analysis of Section I applies without any modifications. We can, therefore, write (cf. Paper IX, eqs. [190] and [194])

$$\left.\begin{aligned} I^{(1)}(0,\mu) &= -\tfrac{3}{8}F\mu\mu_0(1-\mu^2)^{\frac{1}{2}}(1-\mu_0^2)^{\frac{1}{2}}\frac{\mu_0}{\mu_0+\mu} \\ &\qquad\times[X^{(1)}(\mu)X^{(1)}(\mu_0)-Y^{(1)}(\mu)Y^{(1)}(\mu_0)], \\ I^{(1)}(\tau_1,-\mu) &= +\tfrac{3}{8}F\mu\mu_0(1-\mu^2)^{\frac{1}{2}}(1-\mu_0^2)^{\frac{1}{2}}\frac{\mu_0}{\mu_0-\mu} \\ &\qquad\times[X^{(1)}(\mu)Y^{(1)}(\mu_0)-Y^{(1)}(\mu)X^{(1)}(\mu_0)], \end{aligned}\right\} \quad (223)$$

and

$$\left.\begin{aligned} I^{(2)}(0,\mu) &= \tfrac{3}{32}F(1-\mu^2)(1-\mu_0^2)\frac{\mu_0}{\mu_0+\mu} \\ &\qquad\times[X^{(2)}(\mu)X^{(2)}(\mu_0)-Y^{(2)}(\mu)Y^{(2)}(\mu_0)], \\ I^{(2)}(\tau_1,-\mu) &= \tfrac{3}{32}F(1-\mu^2)(1-\mu_0^2)\frac{\mu_0}{\mu_0-\mu} \\ &\qquad\times[X^{(2)}(\mu)Y^{(2)}(\mu_0)-Y^{(2)}(\mu)X^{(2)}(\mu_0)], \end{aligned}\right\} \quad (224)$$

where $X^{(1)}$, $Y^{(1)}$ and $X^{(2)}$, $Y^{(2)}$ are defined in terms of the characteristic functions

$$\Psi^{(1)}(\mu) = \tfrac{3}{8}\mu^2(1-\mu^2) \quad \text{and} \quad \Psi^{(2)}(\mu) = \tfrac{3}{32}(1-\mu^2)^2. \quad (225)$$

In terms of the foregoing solutions (eqs. [221]–[224]), the reflected and the transmitted intensities are given by

$$\left.\begin{aligned} I(0,\mu,\varphi;\mu_0,\varphi_0) = I^{(0)}(0,\mu) &+ I^{(1)}(0,\mu)\cos(\varphi-\varphi_0) \\ &+ I^{(2)}(0,\mu)\cos 2(\varphi-\varphi_0) \end{aligned}\right\} \quad (226)$$

and

$$\left.\begin{aligned} I(\tau_1,-\mu,\varphi;\mu_0,\varphi_0)=I^{(0)}(\tau_1,-\mu)+I^{(1)}(\tau_1,-\mu)\cos(\varphi-\varphi_0) \\ +I^{(2)}(\tau_1,-\mu)\cos 2(\varphi-\varphi_0)\,. \end{aligned}\right\} \tag{227}$$

IV. SCATTERING IN ACCORDANCE WITH THE PHASE FUNCTION $\lambda(1 + x \cos \Theta)$

12. *The azimuth independent term.*—In the problem of diffuse reflection and transmission by a plane-parallel atmosphere scattering radiation in accordance with the phase function $\lambda(1 + x \cos \Theta)$ ($\lambda < 1$, $1 \geqslant x \geqslant -1$), we must distinguish between two terms in the radiation field: an azimuth independent term and a term proportional to cos $(\varphi - \varphi_0)$ (cf. Paper IX, eq. [6]). Considering, first, the azimuth independent term, we can express the reflected and the transmitted radiations in terms of the functions

$$S(\mu) = \sum_{a=-n}^{+n} \frac{L_a[1+x(1-\lambda)\mu/k_a]}{1-k_a\mu} + \gamma\frac{[1+x(1-\lambda)\mu\mu_0]}{1-\mu/\mu_0} \tag{228}$$

and

$$\left.\begin{aligned} T(\mu) = \sum_{a=-n}^{+n} \frac{L_a[1-x(1-\lambda)\mu/k_a]}{1+k_a\mu} e^{-k_a\tau_1} \\ +\gamma e^{-\tau_1/\mu_0}\frac{[1-x(1-\lambda)\mu\mu_0]}{1+\mu/\mu_0}, \end{aligned}\right\} \tag{229}$$

where the k_a's ($a = \pm 1, \ldots, \pm n$ and $k_{+a} = -k_{-a}$) are the $2n$ roots of the characteristic equation

$$1 = \lambda \sum_{j=1}^{n} \frac{a_j[1+x(1-\lambda)\mu_j^2]}{1-k^2\mu_j^2}, \tag{230}$$

$$\gamma = H(\mu_0)H(-\mu_0), \tag{231}$$

and the rest of the symbols have their usual meanings. We have

$$I(0,\mu) = \tfrac{1}{4}\lambda F[S(-\mu) - e^{-\tau_1/\mu}T(\mu)] \tag{232}$$

and

$$I(\tau_1,-\mu) = \tfrac{1}{4}\lambda F[T(-\mu) - e^{-\tau_1/\mu}S(\mu)]\,. \tag{233}$$

Moreover, the boundary conditions at $\tau = 0$ and $\tau = \tau_1$ require that

$$S(\mu_i) = T(\mu_i) = 0 \qquad (i = 1, \ldots, n)\,. \tag{234}$$

In view of these boundary conditions, we can write

$$S(\mu) = \frac{1}{\mu_1^2 \ldots \mu_n^2}\frac{P(\mu)}{W(\mu)}\frac{1}{1-\mu/\mu_0}s(\mu) \tag{235}$$

and

$$T(\mu) = \frac{1}{\mu_1^2 \ldots \mu_n^2}\frac{P(\mu)}{W(\mu)}\frac{1}{1+\mu/\mu_0}t(\mu), \tag{236}$$

where $s(\mu)$ and $t(\mu)$ are polynomials of degree $n + 1$ in μ.

From equations (228), (229), (231), (235), and (236) it follows that

$$s(\mu_0) = [1 + x(1-\lambda)\mu_0^2] P(-\mu_0) \tag{237}$$

and

$$t(-\mu_0) = [1 + x(1-\lambda)\mu_0^2] e^{-\tau_1/\mu_0} P(\mu_0) . \tag{238}$$

And again, from equations (228), (229), (235), and (236), we have

$$\left. L_a = \frac{1}{\mu_1^2 \ldots \mu_n^2} \frac{P(1/k_a)}{W_a(1/k_a)} \frac{1}{1 - 1/k_a\mu_0} \frac{s(1/k_a)}{1 + x(1-\lambda)/k_a^2} \quad (a = \pm 1, \ldots, \pm n) \right\} \tag{239}$$

and

$$\left. L_a e^{-k_a\tau_1} = \frac{1}{\mu_1^2 \ldots \mu_n^2} \frac{P(-1/k_a)}{W_a(1/k_a)} \frac{1}{1 - 1/k_a\mu_0} \frac{t(-1/k_a)}{1 + x(1-\lambda)/k_a^2} \quad (a = \pm 1, \ldots, \pm n) . \right\} \tag{240}$$

We therefore have the $2n$ relations

$$s(1/k_a) = e^{k_a\tau_1} \frac{P(-1/k_a)}{P(+1/k_a)} t(-1/k_a) \qquad (a = \pm 1, \ldots, \pm n) . \tag{241}$$

However, since $s(\mu)$ and $t(\mu)$ are polynomials of degree $(n+1)$ in μ, we conclude, in accordance with theorems 1 and 2 (§ 4), that $s(\mu)$ and $t(\mu)$ must be expressible in the forms

$$s(\mu) = (p_0 + q_0\mu) C_0(\mu) + (p_1 - q_1\mu) C_1(\mu) \tag{242}$$

and

$$t(\mu) = (p_1 + q_1\mu) C_0(\mu) + (p_0 - q_0\mu) C_1(\mu) , \tag{243}$$

where p_0, q_0, p_1, and q_1 are certain constants and $C_0(\mu)$ and $C_1(\mu)$ are defined as in equations (50), (100), and (101) in terms of the n positive roots of equation (230).

From equations (237), (238), (242), and (243) it now follows that

$$p_0 + q_0\mu_0 = \frac{1 + x(1-\lambda)\mu_0^2}{[C_0^2(0) - C_1^2(0)] W(\mu_0)} [P(-\mu_0) C_0(-\mu_0) - e^{-\tau_1/\mu_0} P(\mu_0) C_1(\mu_0)] \tag{244}$$

and

$$p_1 - q_1\mu_0 = \frac{1 + x(1-\lambda)\mu_0^2}{[C_0^2(0) - C_1^2(0)] W(\mu_0)} [e^{-\tau_1/\mu_0} P(\mu_0) C_0(\mu_0) - P(-\mu_0) C_1(-\mu_0)] . \tag{245}$$

It remains to determine the constants q_0 and q_1.

13. *The determination of the constants* q_0 *and* q_1.—Putting $\mu = 0$ in equations (228), (229), (235), and (236), we obtain

$$S(0) = \sum_{a=-n}^{+n} L_a + \gamma = \frac{(-1)^n}{\mu_1 \ldots \mu_n} s(0) \tag{246}$$

and

$$T(0) = \sum_{a=-n}^{+n} L_a e^{-k_a\tau_1} + \gamma e^{-\tau_1/\mu_0} = \frac{(-1)^n}{\mu_1 \ldots \mu_n} t(0) . \tag{247}$$

Now, according to equation (239),

$$\left.\begin{aligned}\sum_{a=-n}^{+n} L_a &= \frac{1}{\mu_1^2 \ldots \mu_n^2} \sum_{a=-n}^{+n} \frac{P(1/k_a)\, s(1/k_a)}{W_a(1/k_a)(1-1/k_a\mu_0)[1+x(1-\lambda)/k_a^2]} \\ &= -\frac{\mu_0}{\mu_1^2 \ldots \mu_n^2 W(\mu_0)} \sum_{a=-n}^{+n} \frac{k_a P(1/k_a)\, s(1/k_a)}{W_a(1/k_a)[1+x(1-\lambda)/k_a^2]} W_a(\mu_0) .\end{aligned}\right\} \quad (248)$$

To evaluate the sum on the right-hand side, we introduce the function

$$f(z) = \sum_{a=-n}^{+n} \frac{k_a P(1/k_a)\, s(1/k_a)}{W_a(1/k_a)[1+x(1-\lambda)/k_a^2]} W_a(z) \quad (249)$$

and express ΣL_a in terms of it. Thus

$$\sum_{a=-n}^{+n} L_a = -\frac{\mu_0}{\mu_1^2 \ldots \mu_n^2} \frac{f(\mu_0)}{W(\mu_0)} . \quad (250)$$

As defined in equation (249), $f(z)$ is a polynomial of degree $(2n-1)$ in z and takes the values

$$\frac{k_a P(1/k_a)\, s(1/k_a)}{1+x(1-\lambda)/k_a^2} \quad (251)$$

for $z = 1/k_a$ $(a = \pm 1, \ldots, \pm n)$. In other words,

$$\left.\begin{aligned} z[1+x(1-\lambda)z^2] f(z) - P(z)\, s(z) = 0 \quad &\text{for} \quad z = 1/k_a \\ &\text{and} \quad a = \pm 1, \ldots, \pm n . \end{aligned}\right\} \quad (252)$$

Hence there must exist a relation of the form

$$z[1+x(1-\lambda)z^2] f(z) = P(z)\, s(z) + W(z)[Az^2 + Bz + D] \quad (253)$$

where A, B, and D are certain constants to be determined.

Putting $z = 0$ in equation (253), we determine at once that

$$D = -(-1)^n \mu_1 \ldots \mu_n s(0) . \quad (254)$$

Next, putting $z = \pm i/\sqrt{x(1-\lambda)}$ in equation (253), we find

$$P(\pm i\zeta)\, s(\pm i\zeta) + W(i\zeta)[-A\zeta^2 \pm iB\zeta + D] = 0 , \quad (255)$$

where, for the sake of brevity, we have written

$$\zeta = \frac{1}{\sqrt{x(1-\lambda)}} , \quad (256)$$

Solving equations (255) for A and B, we find

$$A = \frac{D}{\zeta^2} + \frac{1}{2\zeta^2 W(i\zeta)} [P(i\zeta)\, s(i\zeta) + P(-i\zeta)\, s(-i\zeta)] \quad (257)$$

and

$$B = \frac{i}{2\zeta W(i\zeta)} [P(i\zeta)\, s(i\zeta) - P(-i\zeta)\, s(-i\zeta)] . \quad (258)$$

Returning to equation (253) and setting $z = \mu_0$ and remembering that

$$s(\mu_0) = \left(1 + \frac{\mu_0^2}{\zeta^2}\right) P(-\mu_0) \tag{259}$$

we obtain

$$f(\mu_0) = \frac{1}{\mu_0} P(\mu_0) P(-\mu_0) + \frac{W(\mu_0)}{\mu_0 [1 + x(1-\lambda)\mu_0^2]} [A\mu_0^2 + B\mu_0 + D) . \tag{260}$$

Using the foregoing expression for $f(\mu_0)$ in equation (250) and inserting for A, B, and D their values given by equations (254), (257), and (258), we find, after some reductions, that

$$\left. \begin{aligned} \sum_{a=-n}^{+n} L_a = -\gamma + \frac{(-1)^n}{\mu_1 \dots \mu_n} s(0) \\ - \frac{x(1-\lambda)\mu_0}{2\mu_1^2 \dots \mu_n^2 [1 + x(1-\lambda)\mu_0^2] W(i\zeta)} [\mu_0 \{P(i\zeta)\, s(i\zeta) + P(-i\zeta)\, s(-i\zeta)\} \\ + i\zeta \{P(i\zeta)\, s(i\zeta) - P(-i\zeta)\, s(-i\zeta)\}] . \end{aligned} \right\} \tag{261}$$

Similarly,

$$\left. \begin{aligned} \sum_{a=-n}^{+n} L_a e^{-k_a \tau_1} = -\gamma e^{-\tau_1/\mu_0} + \frac{(-1)^n}{\mu_1 \dots \mu_n} t(0) \\ + \frac{x(1-\lambda)\mu_0}{2\mu_1^2 \dots \mu_n^2 [1 + x(1-\lambda)\mu_0^2] W(i\zeta)} [-\mu_0 \{P(i\zeta)\, t(i\zeta) + P(-i\zeta)\, t(-i\zeta)\} \\ + i\zeta \{P(i\zeta)\, t(i\zeta) - P(-i\zeta)\, t(-i\zeta)\}] . \end{aligned} \right\} \tag{262}$$

Equations (261) and (262) evaluate the required sums.

According to equations (246), (247), (261), and (262), we clearly have

$$(\mu_0 + i\zeta) P(i\zeta)\, s(i\zeta) + (\mu_0 - i\zeta) P(-i\zeta)\, s(-i\zeta) = 0 \tag{263}$$

and

$$(\mu_0 - i\zeta) P(i\zeta)\, t(i\zeta) + (\mu_0 + i\zeta) P(-i\zeta)\, t(-i\zeta) = 0 . \tag{264}$$

We now have to substitute for $s(\pm i\zeta)$ and $t(\pm i\zeta)$ in equations (263) and (264) according to equations (242) and (243). For this purpose it is convenient to write $s(\pm i\zeta)$ and $t(\pm i\zeta)$ in the forms

$$\left. \begin{aligned} s(\pm i\zeta) = (p_0 + q_0\mu_0) C_0(\pm i\zeta) + (p_1 - q_1\mu_0) C_1(\pm i\zeta) \\ + (\mu_0 \mp i\zeta)[-q_0 C_0(\pm i\zeta) + q_1 C_1(\pm i\zeta)] \end{aligned} \right\} \tag{265}$$

and

$$\left. \begin{aligned} t(\pm i\zeta) = (p_0 + q_0\mu_0) C_1(\pm i\zeta) + (p_1 - q_1\mu_0) C_0(\pm i\zeta) \\ + (\mu_0 \pm i\zeta)[-q_0 C_1(\pm i\zeta) + q_1 C_0(\pm i\zeta)] . \end{aligned} \right\} \tag{266}$$

Substituting from equations (265) and (266) in equations (263) and (264), we find, after some minor reductions, that

$$\left. \begin{aligned} (\mu_0^2 + \zeta^2)(q_0\alpha_0 - q_1\alpha_1) = (p_0 + q_0\mu_0)(\alpha_0\mu_0 + \beta_0\zeta) \\ + (p_1 - q_1\mu_0)(\alpha_1\mu_0 + \beta_1\zeta) \end{aligned} \right\} \tag{267}$$

and

$$\left.\begin{aligned}(\mu_0^2+\zeta^2)(q_0\alpha_1-q_1\alpha_0) &= (p_0+q_0\mu_0)(\alpha_1\mu_0-\beta_1\zeta)\\ &\quad+(p_1-q_1\mu_0)(\alpha_0\mu_0-\beta_0\zeta)\,,\end{aligned}\right\} \qquad (268)$$

where

$$\left.\begin{aligned}\alpha_0 &= P(i\zeta)C_0(i\zeta)+P(-i\zeta)C_0(-i\zeta)\,,\\ \alpha_1 &= P(i\zeta)C_1(i\zeta)+P(-i\zeta)C_1(-i\zeta)\,,\\ \beta_0 &= i\,[P(i\zeta)C_0(i\zeta)-P(-i\zeta)C_0(-i\zeta)]\,,\\ \beta_1 &= i\,[P(i\zeta)C_1(i\zeta)-P(-i\zeta)C_1(-i\zeta)]\,.\end{aligned}\right\} \qquad (269)$$

Defined in this manner, α_0, α_1, β_0, and β_1 are all real constants.

Solving equations (267) and (268) for q_0 and q_1, we find

$$q_0=\frac{x(1-\lambda)}{1+x(1-\lambda)\mu_0^2}[(c_1+\mu_0)(p_0+q_0\mu_0)+c_2(p_1-q_1\mu_0)] \qquad (270)$$

and

$$q_1=\frac{x(1-\lambda)}{1+x(1-\lambda)\mu_0^2}[c_2(p_0+q_0\mu_0)+(c_1-\mu_0)(p_1-q_1\mu_0)]\,, \qquad (271)$$

where

$$c_1=\frac{1}{\sqrt{x(1-\lambda)}}\,\frac{\alpha_0\beta_0+\alpha_1\beta_1}{\alpha_0^2-\alpha_1^2} \qquad (272)$$

and

$$c_2=\frac{1}{\sqrt{x(1-\lambda)}}\,\frac{\alpha_0\beta_1+\alpha_1\beta_0}{\alpha_0^2-\alpha_1^2}\,. \qquad (273)$$

With this, the determination of the constants is completed.

14. *The solutions for the reflected and the transmitted radiations.*—It is apparent that for $s(\mu)$ and $t(\mu)$ given by equations (242) and (243), the equations governing the angular distributions of the reflected and the transmitted radiations can be reduced to the forms (194) and (195) with $\lambda/4$ replacing "3/32." With the expressions for $(p_0+q_0\mu_0)$, $(p_1-q_1\mu_0)$, q_0, and q_1 given by equations (244), (245), (270), and (271), we therefore have

$$\left.\begin{aligned}I(0,\mu) &= \tfrac{1}{4}\lambda F\{[1+x(1-\lambda)\mu_0^2][X(\mu)X(\mu_0)-Y(\mu)Y(\mu_0)]\\ &\quad-x(1-\lambda)(\mu_0+\mu)X(\mu)[(c_1+\mu_0)X(\mu_0)+c_2Y(\mu_0)]\\ &\quad-x(1-\lambda)(\mu_0+\mu)Y(\mu)[c_2X(\mu_0)+(c_1-\mu_0)Y(\mu_0)]\}\frac{\mu_0}{\mu_0+\mu}\end{aligned}\right\} \qquad (274)$$

and

$$\left.\begin{aligned}I(\tau_1,-\mu) &= \tfrac{1}{4}\lambda F\{[1+x(1-\lambda)\mu_0^2][X(\mu)Y(\mu_0)-Y(\mu)X(\mu_0)]\\ &\quad+x(1-\lambda)(\mu_0-\mu)X(\mu)[c_2X(\mu_0)+(c_1-\mu_0)Y(\mu_0)]\\ &\quad+x(1-\lambda)(\mu_0-\mu)Y(\mu)[(c_1+\mu_0)X(\mu_0)+c_2Y(\mu_0)]\}\frac{\mu_0}{\mu_0-\mu}\,.\end{aligned}\right\} \qquad (275)$$

Now, Helmholtz' principle of reciprocity as reformulated in Paper XIII (§ 4) requires that the scattering and the transmission matrices have the following property for transposition:

$$\left.\begin{aligned} S(\mu, \varphi; \mu_0, \varphi_0) &= \tilde{S}(\mu_0, \varphi; \mu, \varphi_0) \\ \text{and} \qquad T(\mu, \varphi; \mu_0, \varphi_0) &= \tilde{T}(\mu_0, \varphi_0; \mu, \varphi)\,. \end{aligned}\right\} \tag{552}$$

From equations (548)–(551) it is evident that the matrices $S^{(1)}$, $S^{(2)}$, $T^{(1)}$, and $T^{(2)}$ have the required symmetries. But it is not at once apparent that the matrices $S^{(0)}$ and $T^{(0)}$ are in conformity with the reciprocity principle; for, though the diagonal elements of $S^{(0)}$ and $T^{(0)}$ clearly satisfy the necessary conditions, for the nondiagonal elements the validity of the principle requires that (cf. eqs. [540] and [541] or [543] and [544])

$$\left[\frac{C_{0,r}^2(0) - C_{1,r}^2(0)}{C_{0,l}^2(0) - C_{1,l}^2(0)}\right]^{\frac{1}{2}} \nu_i = -\left[\frac{C_{0,l}^2(0) - C_{1,l}^2(0)}{C_{0,r}^2(0) - C_{1,r}^2(0)}\right]^{\frac{1}{2}} u_i \qquad (i = 1, 2)\,. \tag{553}$$

From equations (405), (406), (510), and (511) defining the constants ν_i amd u_i, it is seen that the condition (553) is equivalent to

$$\frac{1}{2}\left[\frac{C_{0,r}^2(0) - C_{1,r}^2(0)}{C_{0,l}^2(0) - C_{1,l}^2(0)}\right]^{\frac{1}{2}} (\gamma_1\gamma_4 + \gamma_2\gamma_3) = -\left[\frac{C_{0,l}^2(0) - C_{1,l}^2(0)}{C_{0,r}^2(0) - C_{1,r}^2(0)}\right]^{\frac{1}{2}} (c_1 c_4 + c_2 c_3) \tag{554}$$

or

$$\frac{\gamma_1\gamma_4 + \gamma_2\gamma_3}{c_1 c_4 + c_2 c_3} = -2\,\frac{C_{0,l}^2(0) - C_{1,l}^2(0)}{C_{0,r}^2(0) - C_{1,r}^2(0)}\,. \tag{555}$$

On the other hand, from the definitions of the various constants γ_i (eq. [396]), it readily follows that

$$\gamma_1\gamma_4 + \gamma_2\gamma_3 = 2\,[C_{0,l}(+1)\,C_{0,l}(-1) - C_{1,l}(+1)\,C_{1,l}(-1)]\,; \tag{556}$$

or, using the identity (105) satisfied by the C-functions in general, we have

$$\gamma_1\gamma_4 + \gamma_2\gamma_3 = 2\,[C_{0,l}^2(0) - C_{1,l}^2(0)]\,\Omega(1)\,. \tag{557}$$

Similarly,

$$c_1 c_4 + c_2 c_3 = 2\,[C_{0,r}^2(0) - C_{1,r}^2(0)]\,W(1)\,. \tag{558}$$

Hence (cf. Paper XI, eq. [133]),

$$\frac{\gamma_1\gamma_4 + \gamma_2\gamma_3}{c_1 c_4 + c_2 c_3} = \frac{C_{0,l}^2(0) - C_{1,l}^2(0)}{C_{0,r}^2(0) - C_{1,r}^2(0)}\,\frac{\Omega(1)}{W(1)} = -2\,\frac{C_{0,l}^2(0) - C_{1,l}^2(0)}{C_{0,r}^2(0) - C_{1,r}^2(0)}\,, \tag{559}$$

in agreement with equation (555). Thus our solution for S and T is in conformity with the requirements of the reciprocity principle.