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# ON THE RADIATIVE EQUILIBRIUM OF A STELLAR ATMOSPHERE XVII

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ABSTRACT

In this paper a systematic study is made of the various functional equations which characterize the transfer problems in plane-parallel atmospheres.

First, a functional equation relating the angular distribution of the emergent radiation from a semi-infinite atmosphere and the law of diffuse reflection by the same atmosphere is derived. This equation arises in consequence of the invariance of the emergent radiation from a semi-infinite atmosphere to the addition (or removal) of layers of arbitrary optical thickness to (or from) the atmosphere. Next, four functional equations governing the problem of transmission and diffuse reflection by an atmosphere of finite optical thickness are formulated. These equations have been derived under very general conditions; and they have been further reduced to a basic system of functional equations for the case in which scattering takes place in accordance with a phase function expressible as a series in Legendre polynomials. And, finally, further functional equations are formulated which determine the radiation field in the interior in terms of the scattering and transmission functions of atmospheres of finite optical thicknesses.

1. *Introduction.*—In the study of the transfer of radiation in stellar atmospheres the two basic problems are, first, the radiative equilibrium of a semi-infinite atmosphere with a constant net flux and, second, the law of diffuse reflection by the same atmosphere. It is, of course, apparent that the first of these problems is significant only in conservative cases; for in all other cases the equation of transfer will not admit the integral which insures the constancy of the net flux through the atmosphere.

In the problem of a semi-infinite atmosphere with a constant net flux, the radiation field will be axially symmetrical at each point about the direction normal to the plane of stratification, and greatest interest is attached to the *law of darkening*,

$$I(0, \mu) \qquad (0 \leqslant \mu \leqslant 1), \tag{1}$$

which expresses the angular distribution of the emergent radiation. On the other hand, in the problem of diffuse reflection, our principal interest is in the intensity,

$$I(\mu, \varphi; \mu_0, \varphi_0), \tag{2}$$

diffusely reflected in the direction $(\mu, \varphi)$ when a parallel beam of radiation of net flux $\pi F$ per unit area normal to itself is incident on the atmosphere in the direction $(-\mu_0, \varphi_0)$. This *reflected intensity* is generally expressed in terms of a *scattering function* $S(\mu, \varphi; \mu_0, \varphi_0)$ in the form

$$I(\mu, \varphi; \mu_0, \varphi_0) = \frac{1}{4\mu} S(\mu, \varphi; \mu_0, \varphi_0) F. \tag{3}$$

When the polarization of the scattered radiation has also to be allowed for, then, in the problem with a constant net flux, we must distinguish between the intensities $I_l$ and $I_r$ in the two states of polarization in which the electric vector vibrates in the meridian plane and at right angles to it, respectively. Correspondingly, in the problem of diffuse reflection we must consider a *scattering matrix*, $\mathbf{S}(\mu, \varphi; \mu_0, \varphi_0)$, defined suitably (cf. Papers XIII and XIV[1]).

Explicit solutions for the two basic problems have been found under a variety of con-

[1] *Ap. J.*, **105**, 151 (eq. [63]), and 164 (eq. [19]), 1947.



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ditions in the earlier papers of this series.[2] And an examination of these solutions reveals certain remarkable relationships between the angular distribution of the emergent radiation in the problem with a constant net flux and the law of diffuse reflection. The relationship is naturally the simplest in the case of isotropic scattering, when[3]

$$I(0, \mu) = \frac{\sqrt{3}}{4} F H(\mu) \tag{4}$$

and

$$\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right) S(\mu, \mu_0) = H(\mu) H(\mu_0), \tag{5}$$

where $H(\mu)$ satisfies the functional equation,

$$H(\mu) = 1 + \tfrac{1}{2}\mu H(\mu) \int_0^1 \frac{H(\mu')}{\mu+\mu'} d\mu'. \tag{6}$$

In the other cases the relationship is of a much more complex nature and has to be sought between the darkening function and the *azimuth independent* term,

$$S^{(0)}(\mu, \mu_0) = \frac{1}{2\pi} \int_0^{2\pi} S(\mu, \varphi; \mu_0, \varphi_0) \, d\varphi, \tag{7}$$

in the law of diffuse reflection.[4] The question now arises as to the origin and meaning of this relationship. While it should, in principle, be possible to go back to the original equations of transfer and derive the relationships in question as integrals of the problem, it would seem, in view of the complex nature of the relationships to be established, that such a procedure would not disclose the *physical* origin of the relationships. However, in this paper we shall show that there is an alternative way of looking at the problem which enables us to obtain in a simple way a functional equation relating $I(0, \mu)$ and $S^{(0)}(\mu, \mu_0)$. As we shall see, this functional equation arises essentially from the invariance of $I(0, \mu)$ to the addition (or removal) of layers of arbitrary thickness to (or from) the atmosphere. And it is to this invariance that we should attribute the relationship between the two basic problems.

The ideas leading to the establishment of the functional equation that we have just mentioned enable us to formulate further functional equations, which together appear to characterize the problems of radiative transfer in plane-parallel atmospheres (both finite and semi-infinite) in their entirety. In §§ 5–8 of this paper these other functional equations of the problem are derived.

## I. THE FUNCTIONAL EQUATION RELATING THE LAW OF DARKENING AND THE SCATTERING FUNCTION FOR SEMI-INFINITE PLANE-PARALLEL ATMOSPHERES

2. *The functional equation relating* $I(0, \mu)$ *and* $S^{(0)}(\mu, \mu_0)$.—Considering, first, the case in which the scattering of radiation is described simply in terms of a phase function $p(\cos\Theta)$, we shall suppose that

$$p(\cos\Theta) = \Sigma \varpi_l P_l(\cos\Theta), \tag{8}$$

where $\varpi_l$'s are constants and

$$\varpi_0 = 1, \tag{9}$$

[2] See esp. Paper XVI (*Ap. J.*, **105**, 435, 1947), where the solutions for the important cases of Rayleigh phase function and Rayleigh scattering are tabulated.

[3] Cf. Papers VIII (*Ap. J.*, **101**, 348, 1945) and XIV (*ibid.*, **105**, 164, eqs. [178]–[183]).

[4] When polarization is taken into account, we must consider the corresponding matrix, $\boldsymbol{S}^{(0)}(\mu, \mu_0)$ (see eq. [51], below).

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to insure the constancy of the net flux through the atmosphere. For this case the equation of transfer for the problem with a constant net flux (and no incident radiation) can be expressed in the form

$$\mu \frac{dI(\tau, \mu)}{d\tau} = I(\tau, \mu) - \frac{1}{2} \int_{-1}^{+1} j(\mu, \mu') I(\tau, \mu') d\mu', \tag{10}$$

where

$$j(\mu, \mu') = \Sigma \varpi_l P_l(\mu) P_l(\mu') \tag{11}$$

is a symmetrical function in the variables $\mu$ and $\mu'$.

We shall find it convenient to re-write equation (10) in the form

$$\mu \frac{dI(\tau, \mu)}{d\tau} = I(\tau, \mu) - B(\tau, \mu), \tag{12}$$

where the source function

$$B(\tau, \mu) = \frac{1}{2} \int_{-1}^{+1} j(\mu, \mu') I(\tau, \mu') d\mu'. \tag{13}$$

Now consider the radiation at a depth $d\tau$ below the boundary of the atmosphere at $\tau = 0$. The radiation field at this level, which is directed *inward*, can be inferred from the equation of transfer (12); for, since at $\tau = 0$ there is no incident intensity, at the level $d\tau$ we must have an inward intensity,

$$I(d\tau, -\mu') = \frac{1}{\mu'} B(0, -\mu') d\tau, \tag{14}$$

in the direction $-\mu'$. This inward-directed radiation will be reflected by the atmosphere below $d\tau$ by the law of diffuse reflection of a semi-infinite plane-parallel atmosphere. If this law is expressed in the form (3), the reflection of the radiation (14) will contribute to the *outward* intensity at the level $d\tau$, the amount

$$\frac{d\tau}{4\pi\mu} \int_0^1 \int_0^{2\pi} S(\mu, \varphi; \mu', \varphi') B(0, -\mu') \frac{d\mu'}{\mu'} d\varphi', \tag{15}$$

or, in view of the axial symmetry of $B(0, -\mu')$,

$$\frac{d\tau}{2\mu} \int_0^1 S^{(0)}(\mu, \mu') B(0, -\mu') \frac{d\mu'}{\mu'}, \tag{16}$$

where $S^{(0)}(\mu, \mu')$ is defined as in equation (7).

Now *the outward intensity* I$(d\tau, \mu)$ *at the level* $d\tau$ *can differ from* I$(0, \mu)$ *only by the amount (16)*; for the intensity and the angular distribution of the emergent radiation from a semi-infinite plane-parallel atmosphere cannot be altered by the removal (or addition) of layers of arbitrary optical thickness from (or to) the atmosphere. Consequently, the removal of the layers above the level $d\tau$ must restore $I(d\tau, \mu)$ to $I(0, \mu)$. We must therefore have

$$I(d\tau, \mu) = I(0, \mu) + \frac{d\tau}{2\mu} \int_0^1 S^{(0)}(\mu, \mu') B(0, -\mu') \frac{d\mu'}{\mu'}. \tag{17}$$

On the other hand, from the equation of transfer (12), we can directly conclude that

$$\left.\begin{aligned} I(d\tau, \mu) &= I(0, \mu) + d\tau \left(\frac{dI}{d\tau}\right)_{\tau=0} \\ &= I(0, \mu) + \frac{d\tau}{\mu} [I(0, \mu) - B(0, \mu)]. \end{aligned}\right\} \tag{18}$$

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Combining equations (17) and (18) and passing to the limit $d\tau = 0$, we obtain

$$I(0, \mu) = B(0, \mu) + \frac{1}{2}\int_0^1 S^{(0)}(\mu, \mu') B(0, -\mu') \frac{d\mu'}{\mu'}. \tag{19}$$

But, according to equation (13),

$$B(0, \mu) = \frac{1}{2}\int_0^1 j(\mu, \mu'') I(0, \mu'') d\mu''. \tag{20}$$

Using this expression for $B(0, \mu)$ in equation (19), we have

$$\left. \begin{aligned} I(0, \mu) = & \frac{1}{2}\int_0^1 j(\mu, \mu'') I(0, \mu'') d\mu'' \\ & + \frac{1}{4}\int_0^1 \int_0^1 S^{(0)}(\mu, \mu') j(-\mu', \mu'') I(0, \mu'') \frac{d\mu'}{\mu'} d\mu''. \end{aligned} \right\} \tag{21}$$

Finally, substituting for $j(\mu, \mu')$ according to equation (11), we find

$$\left. \begin{aligned} I(0, \mu) = & \frac{1}{2}\sum_l \varpi_l \left[\int_0^1 P_l(\mu') I(0, \mu') d\mu'\right] \\ & \times \left[P_l(\mu) + (-1)^l \frac{1}{2}\int_0^1 S^{(0)}(\mu, \mu') P_l(\mu') \frac{d\mu'}{\mu'}\right]. \end{aligned} \right\} \tag{22}$$

Equation (22) is a functional equation relating $I(0, \mu)$ and $S^{(0)}(\mu, \mu')$; and it is apparent how this equation will determine $I(0, \mu)$ uniquely in terms of $S^{(0)}(\mu, \mu')$, and conversely

3. *Illustrations of the functional equation* (22).—We shall illustrate equation (22) by considering, first, the case of isotropic scattering. In this case

$$\varpi_0 = 1, \quad \text{and} \quad \varpi_l = 0 \quad (l \neq 0), \tag{23}$$

and (cf. eq. [5])

$$\left(\frac{1}{\mu} + \frac{1}{\mu'}\right) S^{(0)}(\mu, \mu') = H(\mu) H(\mu'), \tag{24}$$

and equation (22) reduces to

$$I(0, \mu) = \left[\frac{1}{2}\int_0^1 I(0, \mu') d\mu'\right]\left[1 + \tfrac{1}{2}\mu H(\mu)\int_0^1 \frac{H(\mu')}{\mu + \mu'} d\mu'\right]. \tag{25}$$

Using equation (6), satisfied by $H(\mu)$, we find the foregoing equation to be

$$I(0, \mu) = \left[\frac{1}{2}\int_0^1 I(0, \mu') d\mu'\right] H(\mu). \tag{26}$$

It is seen that equation (26) is consistent with itself, since, according to Paper XIV, equation (180),

$$\int_0^1 H(\mu) d\mu = 2. \tag{27}$$

We may therefore write

$$I(0, \mu) = \text{constant } H(\mu) \tag{28}$$

and determine the constant of proportionality from the condition

$$2\int_0^1 I(0, \mu) \mu d\mu = F. \tag{29}$$

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We find in this manner that (cf. Paper XIV, eqs. [182]–[183])

$$I(0,\mu)=\frac{\sqrt{3}}{4}FH(\mu)\,, \tag{30}$$

in agreement with equation (4).

As a second illustration of equation (22), we shall consider the case of scattering in accordance with Rayleigh's phase function. In this case

$$\varpi_0=1\,,\qquad \varpi_2=\tfrac{1}{2}\,,\qquad \text{and}\qquad \varpi_l=0\qquad (l\neq 0,\,2)\,, \tag{31}$$

and equation (22) reduces to

$$\left.\begin{aligned}I(0,\mu)=\frac{1}{2}\Big[\int_0^1 I(0,\mu')\,d\mu'\Big]\Big[1+\frac{1}{2}\int_0^1 S^{(0)}(\mu,\mu')\frac{d\mu'}{\mu'}\Big]\\+\frac{1}{16}\Big[\int_0^1 I(0,\mu')(3\mu'^2-1)\,d\mu'\Big]\\\times\Big[3\mu^2-1+\frac{1}{2}\int_0^1 S^{(0)}(\mu,\mu')(3\mu'^2-1)\frac{d\mu'}{\mu'}\Big]\,.\end{aligned}\right\} \tag{32}$$

Also, for this case (Paper XIV, eqs. [231] and [246] or Paper XVI, Sec. II),

$$\left(\frac{1}{\mu}+\frac{1}{\mu'}\right)S^{(0)}(\mu,\mu')=\tfrac{3}{8}H^{(0)}(\mu)\,H^{(0)}(\mu')\,[3-c(\mu+\mu')+\mu\mu']\,, \tag{33}$$

where $H^{(0)}(\mu)$ is defined in terms of the functional equation (cf. Paper XIV, eq. [253]),

$$H^{(0)}(\mu)=1+\tfrac{3}{16}\mu H^{(0)}(\mu)\int_0^1\frac{H^{(0)}(\mu')}{\mu+\mu'}(3-\mu'^2)\,d\mu'\,, \tag{34}$$

and

$$c=\frac{\alpha_2}{\alpha_1}\,, \tag{35}$$ [5]

$\alpha_2$ and $\alpha_1$ being the moments of order 1 and 2 of $H^{(0)}(\mu)$.

Inserting the reflection function (33) in equation (32), we find, after some reductions in which repeated use is made of Paper XIV, equations (254)–(256), that

$$\left.\begin{aligned}I(0,\mu)=\tfrac{3}{16}(3A_0-A_2)\,H^{(0)}(\mu)\\+\tfrac{3}{32}[-2A_0\alpha_2+3A_2\alpha_1^2+3A_2\alpha_2(2-\alpha_0)]\frac{1}{\alpha_1}\mu H^{(0)}(\mu)\,,\end{aligned}\right\} \tag{36}$$

where $A_0$ and $A_2$ are the (unknown!) moments of order 0 and 2 of $I(0,\mu)$.

The constants $A_0$ and $A_2$ in equation (36) can be determined by taking the zero and second moments of this equation. We find that

$$A_0=\tfrac{3}{16}(3A_0-A_2)\,\alpha_0+\tfrac{3}{32}[-2A_0\alpha_2+3A_2\alpha_1^2+3A_2\alpha_2(2-\alpha_0)] \tag{37}$$

and

$$A_2=\tfrac{3}{16}(3A_0-A_2)\,\alpha_2+\tfrac{3}{32}[-2A_0\alpha_2+3A_2\alpha_1^2+3A_2\alpha_2(2-\alpha_0)]\frac{\alpha_3}{\alpha_1}\,. \tag{38}$$

[5] In Paper XIV the constant $c$ was defined somewhat differently (cf. eq. [264]). But, since the discriminant of eq. (262) vanishes, we could equally well have written

$$c=\frac{9\alpha_0-16}{3\alpha_1}\,.$$

Using the relation $9\alpha_0-3\alpha_2=16$ (Paper XIV, eq. [254]) in this equation for $c$, we obtain formula (35).

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These equations determine $A_0$ and $A_2$. However, a simple inspection shows that

$$\frac{A_0}{a_0}=\frac{A_2}{a_2}=\text{constant} \tag{39}$$

satisfies equations (37) and (38); for (cf. Paper XIV, eqs. [254] and [255])

$$3a_0-a_2=\tfrac{16}{3}, \tag{40}$$

and

$$\left.\begin{aligned}-2a_0a_2+3a_2a_1^2+3a_2^2(2-a_0)&=a_2[-2a_0+3a_1^2+(9a_0-16)(2-a_0)]\\&=a_2[3a_1^2-9a_0^2+32(a_0-1)]=0.\end{aligned}\right\} \tag{41}$$

Hence,

$$I(0,\mu)=\text{constant } H^{(0)}(\mu); \tag{42}$$

and the constant of proportionality can again be determined from the flux condition (29). In this manner we find that

$$I(0,\mu)=\frac{1}{2a_1}FH^{(0)}(\mu), \tag{43}$$

in agreement with Paper XIV, equation (274).

**4.** *The functional equations relating the laws of darkening and diffuse reflection in cases in which the polarization of the scattered radiation is taken into account.*—As sufficiently illustrative of the general problem, we shall consider the case of Rayleigh scattering for which the general equations of transfer have already been formulated in earlier papers.[6] However, in our present context we require the equations of transfer for the case of axial symmetry when it is sufficient to distinguish only between the intensities $I_l$ and $I_r$ in the two states of polarization in which the electric vector vibrates in the meridian plane and at right angles to it, respectively.[7] And the equations of transfer relevant to this problem have been derived in Paper X (eqs. [37] and [38]). For our present purposes it is convenient to combine these equations into a single vector equation, considering $I_l(\tau,\mu)$ and $I_r(\tau,\mu)$ as the components of a *two*-dimensional vector,

$$\boldsymbol{I}=(I_l,I_r). \tag{44}$$

The equation of transfer (Paper X, eqs. [37] and [38]) for $I_l$ and $I_r$ can then be expressed in the form (cf. Paper XIV, eqs. [10] and [15])

$$\mu\frac{d\boldsymbol{I}}{d\tau}=\boldsymbol{I}(\tau,\mu)-\frac{3}{8}\int_{-1}^{+1}\boldsymbol{J}(\mu,\mu')\boldsymbol{I}(\tau,\mu')\,d\mu', \tag{45}$$

where $\boldsymbol{J}(\mu,\mu')$ denotes the matrix,

$$\boldsymbol{J}(\mu,\mu')=\begin{pmatrix}2(1-\mu^2)(1-\mu'^2)+\mu^2\mu'^2 & \mu^2\\ \mu'^2 & 1\end{pmatrix}. \tag{46}$$

Introducing the source function,

$$\boldsymbol{B}(\tau,\mu)=\frac{3}{8}\int_{-1}^{+1}\boldsymbol{J}(\mu,\mu')\boldsymbol{I}(\tau,\mu')\,d\mu', \tag{47}$$

[6] Papers X (*Ap. J.*, **103**, 351, 1946), XI (*ibid.*, **104**, 110, 1946), XIII (*ibid.*, **105**, 151, 1947), XIV (*ibid.*, p. 164), and XV (*ibid.*, p. 424).

[7] The other Stokes parameters, $U=(I_l-I_r)\tan 2\chi$ and $V=(I_l-I_r)\sec 2\chi\tan 2\beta$ are identically zero for the problem with a constant net flux.

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we can re-write equation (45) in the form

$$\mu \frac{dI}{d\tau} = I(\tau, \mu) - B(\tau, \mu). \tag{48}$$

To derive the functional equation for $I(0, \mu)$, we proceed exactly as in § 2 by considering the radiation field at a level $d\tau$ below the boundary of the atmosphere at $\tau = 0$. At this level there will be an inward intensity, of amount

$$\frac{d\tau}{\mu'} B(0, \mu'), \tag{49}$$

in the direction $-\mu'$. On account of the axial symmetry of this field and the consequent vanishing of the component $U$, the radiation (49) will be reflected by the atmosphere below $d\tau$ in accordance with the two-dimensional scattering matrix (cf. Paper XVI, § 3),

$$\frac{3}{16\mu} S^{(0)}(\mu, \mu'), \tag{50}$$

where

$$+\frac{1}{\mu'}\Big) S^{(0)}(\mu, \mu') =$$
$$H_l(\mu) H_l(\mu')[1 - c(\mu + \mu') + \mu\mu'] \quad qH_l(\mu) H_r(\mu')(\mu + \mu')$$
$$H_r(\mu) H_l(\mu')(\mu + \mu') \quad H_r(\mu) H_r(\mu')[1 + c(\mu + \mu') + \mu\mu'] \Big). \tag{51}$$

(The definitions of the various quantities occurring on the right-hand side of the foregoing equation will be found in Paper XIV, Part II).

The reflection of the radiation (49) by the atmosphere below $d\tau$ will, accordingly, contribute to the outward intensity, $I(d\tau, \mu)$, at the level $d\tau$, the amount

$$\frac{3}{8\mu} d\tau \int_0^1 S^{(0)}(\mu, \mu') B(0, -\mu') \frac{d\mu'}{\mu'}. \tag{52}$$

Again, from the invariance of $I(0, \mu)$ to the removal (or addition) of layers of arbitrary thickness from (or to) the atmosphere, we conclude that $I(d\tau, \mu)$ can differ from $I(0, \mu)$ only by the amount (52). Hence,

$$I(d\tau, \mu) = I(0, \mu) + \frac{3}{8\mu} d\tau \int_0^1 S^{(0)}(\mu, \mu') B(0, -\mu') \frac{d\mu'}{\mu'}. \tag{53}$$

On the other hand, from the equation of transfer we directly find that

$$I(d\tau, \mu) = I(0, \mu) + \frac{d\tau}{\mu}[I(0, \mu) - B(0, \mu)]. \tag{54}$$

Combining equations (53) and (54) and passing to the limit $d\tau = 0$, we obtain

$$I(0, \mu) = B(0, \mu) + \frac{3}{8} \int_0^1 S^{(0)}(\mu, \mu') B(0, -\mu') \frac{d\mu'}{\mu'}. \tag{55}$$

Finally, substituting for $B(0, \mu)$ according to equation (47) in equation (55), we have

$$\left.\begin{aligned} I(0, \mu) = \frac{3}{8}\Big[ &\int_0^1 J(\mu, \mu') I(0, \mu') d\mu' \\ &+ \frac{3}{8} \int_0^1 \int_0^1 S^{(0)}(\mu, \mu') J(-\mu', \mu'') I(0, \mu'') \frac{d\mu'}{\mu'} d\mu'' \Big]. \end{aligned}\right\} \tag{56}$$

This is the required functional equation relating $I(0, \mu)$ and $S^{(0)}(\mu, \mu')$.

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It should, of course, be possible to deduce the solution for $I(0, \mu)$ from the known form of $S^{(0)}(\mu, \mu')$. The necessary calculations are likely to be somewhat tedious, since it is not too easy even to verify that the solution (Paper XIV, eqs. [119] and [120])

$$I(0, \mu) = \frac{3}{8\sqrt{2}} F \begin{pmatrix} qH_l(\mu) \\ H_r(\mu)(\mu + c) \end{pmatrix} \tag{57}$$

actually satisfies equation (56). Thus, for $I(0, \mu)$ given by equation (57),

$$B(0, \mu) = \frac{3}{8\sqrt{2}} F \begin{pmatrix} q(1 - \mu^2) + c\mu^2 \\ c \end{pmatrix}, \tag{58}$$

and the validity of equation (56) requires that the equation

$$\left.\begin{aligned} \begin{pmatrix} qH_l(\mu) \\ H_r(\mu)(\mu + c) \end{pmatrix} = \begin{pmatrix} q(1 - \mu^2) + c\mu^2 \\ c \end{pmatrix} + \tfrac{3}{8}\mu \int_0^1 \frac{d\mu'}{\mu + \mu'} \\ \times \begin{pmatrix} 2H_l(\mu)H_l(\mu')[1 - c(\mu + \mu') + \mu\mu'] & qH_l(\mu)H_r(\mu')(\mu + \mu') \\ qH_r(\mu)H_l(\mu')(\mu + \mu') & H_r(\mu)H_r(\mu')[1 - c(\mu + \mu') + \mu\mu'] \end{pmatrix} \\ \times \begin{pmatrix} q(1 - \mu'^2) + c\mu'^2 \\ c \end{pmatrix} \end{aligned}\right\} \quad (5$$

be true. By a series of reductions, in which the various integral properties[8] of the functions $H_l(\mu)$ and $H_r(\mu)$ are used repeatedly it can be shown that this is indeed the case. But we shall not go into the details of these reductions here.

## II. FUNCTIONAL EQUATIONS FOR TRANSFER PROBLEMS IN PLANE-PARALLEL ATMOSPHERES

5. *Functional equations for the problem of transmission and diffuse reflection by a plane-parallel atmosphere of finite optical thickness.*—In Paper XIV we formulated the functional equation satisfied by the scattering matrix in the problem of diffuse reflection, on Rayleigh's law, by a semi-infinite plane-parallel atmosphere. The corresponding functional equation for the case of scattering according to a general phase function (but not allowing for the polarization of the scattered radiation) had been derived earlier by V. A. Ambarzumian.[9] The principle underlying the derivation of these functional equations is the invariance, first pointed out byAmbarzumian, of the intensity and the angular distribution of the reflected light to the addition (or removal) of layers of arbitrary optical thickness to (or from) the atmosphere. It is apparent that similar functional equations must govern the problem of transmission and reflection by a plane-parallel atmosphere of finite optical thickness, $\tau_1$ (say). This possibility has, indeed, been envisaged by Ambarzumian,[10] who stated that the required functional equations can be obtained from the invariance of the reflected and the transmitted light to the addition of a layer of a certain optical thickness to the atmosphere above $\tau = 0$ and the *simultaneous* removal of a layer of equal optical thickness from the atmosphere, below, at $\tau = \tau_1$. This invariance will lead to two functional equations governing the intensities and the angular distributions of the reflected and the transmitted radiations. Ambarzumian has, in fact, written down these two equations for the case of isotropic scattering.[11] However, on consideration it

[8] Cf. Paper XIV, §§ 7 and 8.

[9] *J. Phys. Acad. Sc. U.S.S.R.*, **8**, 64, 1944.

[10] *C.R.* (Doklady) *Acad. d. Sc. U.R.S.S.*, **38**, 229, 1943.

[11] *Ibid.*, eqs. (8) and (9). The derivation of these equations is not given in this short note. However, from the details of the treatment of the semi-infinite case, which are given, it is not difficult to reconstruct the proof that the author probably had in mind. The present writer has been unable to trace any later publication in which Ambarzumian has returned to these matters.

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appears that the two functional equations which can be derived from the invariance referred to by Ambarzumian are not sufficient to characterize the problem completely. Actually, it would seem that four equations are necessary to make the problem determinate, and we shall show how these four equations arise by considering in detail the case of scattering according to a general phase function. The modifications required to include the polarization of the scattered radiation will be indicated later.

The equation of transfer for the problem of diffuse reflection by a plane-parallel atmosphere can be written in the form[12]

$$\mu \frac{dI(\tau, \mu, \varphi)}{d\tau} = I(\tau, \mu, \varphi) - B(\tau, \mu, \varphi), \tag{60}$$

where the source function $B(\tau, \mu, \varphi)$ is given by

$$\left. \begin{aligned} B(\tau, \mu, \varphi) = \frac{1}{4\pi} \int_{-1}^{+1} \int_0^{2\pi} p(\mu, \varphi; \mu', \varphi') I(\tau, \mu', \varphi') d\mu' d\varphi' \\ + \tfrac{1}{4} F e^{-\tau/\mu_0} p(\mu, \varphi; -\mu_0, \varphi_0), \end{aligned} \right\} \tag{61}$$

when a parallel beam of radiation of flux $\pi F$ per unit area normal to itself is incident on the atmosphere in the direction $(-\mu_0, \varphi_0)$. In equation (61), $p(\mu, \varphi; \mu', \varphi')$ is the phase function governing the probability that the radiation in the direction $(\mu', \varphi')$ will be scattered in the direction $(\mu, \varphi)$.

We shall suppose that the optical thickness of the atmosphere is $\tau_1$. We shall then be interested not only in the intensity $I(0, \mu, \varphi)$, $(0 \leqslant \mu \leqslant 1)$, reflected by the atmosphere in the direction $(\mu, \varphi)$, but also in the intensity $I(\tau_1, \mu, \varphi)$ transmitted in the direction $(-\mu, \varphi)$.[13] We shall express these intensities in the forms

$$I(0, \mu, \varphi) = \frac{1}{4\mu} F S(\tau_1; \mu, \varphi; \mu_0, \varphi_0) \tag{62}$$

and

$$I(\tau_1, \mu, \varphi) = \frac{1}{4\mu} F T(\tau_1; \mu, \varphi; \mu_0, \varphi_0), \tag{63}$$

to emphasize the fact that we are considering the reflection and transmission by an atmosphere of optical thickness $\tau_1$.

Now consider the radiation field present at a depth $d\tau$ below the boundary of the atmosphere at $\tau = 0$. At this level there will be the incident flux, reduced to the amount

$$\pi F \left(1 - \frac{d\tau}{\mu_0}\right), \tag{64}$$

and a diffuse radiation field. The intensity in this latter diffuse field, which is directed inward, can be inferred from the equation of transfer: for, since at $\tau = 0$ there is no inward intensity, at the level $d\tau$ we must have the intensity

$$\frac{d\tau}{\mu'} B(0, -\mu', \varphi') \tag{65}$$

in the direction $(-\mu', \varphi')$. Both the radiation fields (64) and (65) will be reflected by the atmosphere of optical thickness $\tau_1 - d\tau$ below $d\tau$ and will contribute

$$\left. \begin{aligned} \frac{F}{4\mu}\left(1 - \frac{d\tau}{\mu_0}\right) S(\tau_1 - d\tau; \mu, \varphi; \mu_0, \varphi_0) \\ + \frac{d\tau}{4\pi\mu} \int_0^1 \int_0^{2\pi} S(\tau_1 - d\tau; \mu, \varphi; \mu', \varphi') B(0, -\mu', \varphi') \frac{d\mu'}{\mu'} d\varphi' \end{aligned} \right\} \tag{66}$$

[12] Cf. Paper XII (*Ap. J.*, **104**, 191, 1946), eqs. (1)–(3).

[13] Note the minus sign here.

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to the intensity in the outward direction $(\mu, \varphi)$ at $d\tau$. Hence

$$\left. \begin{aligned} I(d\tau, \mu, \varphi) = \frac{F}{4\mu}\Big[\Big(1-\frac{d\tau}{\mu_0}\Big) S(\tau_1; \mu, \varphi; \mu_0, \varphi_0) - \frac{\partial S(\tau_1; \mu, \varphi; \mu_0, \varphi_0)}{\partial \tau_1} d\tau\Big] \\ + \frac{d\tau}{4\pi\mu}\int_0^1\int_0^{2\pi} S(\tau_1; \mu, \varphi; \mu', \varphi') B(0, -\mu', \varphi') \frac{d\mu'}{\mu'} d\varphi', \end{aligned} \right\} \quad (67)$$

where we have neglected all quantities of order higher than the first in $d\tau$. On the other hand, from the equation of transfer it directly follows that

$$I(d\tau, \mu, \varphi) = \frac{F}{4\mu}\Big(1+\frac{d\tau}{\mu}\Big) S(\tau_1; \mu, \varphi; \mu_0, \varphi_0) - \frac{d\tau}{\mu} B(0, \mu, \varphi). \quad (68)$$

Combining equations (67) and (68), we have

$$\left. \begin{aligned} \tfrac{1}{4}F\Big[\Big(\frac{1}{\mu}+\frac{1}{\mu_0}\Big) S(\tau_1; \mu, \varphi; \mu_0, \varphi_0) + \frac{\partial S(\tau_1; \mu, \varphi; \mu_0, \varphi_0)}{\partial \tau_1}\Big] \\ = B(0, \mu, \varphi) + \frac{1}{4\pi}\int_0^1\int_0^{2\pi} S(\tau_1; \mu, \varphi; \mu', \varphi') B(0, -\mu', \varphi') \frac{d\mu'}{\mu'} d\varphi'. \end{aligned} \right\} \quad (69)$$

In addition to equation (69), there is a further relation which can be derived from considerations relating to the radiation field present at the depth $d\tau$. The relation arises from the fact that the transmission of the flux $\pi F$ incident on $\tau = 0$ by the entire atmosphere of optical thickness $\tau_1$ must be the same as the transmission of the radiations (64) and (65) by the atmosphere of optical thickness $\tau_1 - d\tau$ below the level $d\tau$. Of the flux (64), the amount transmitted in the direction $(-\mu, \varphi)$ by the atmosphere below $d\tau$ is given by

$$\frac{F}{4\mu}\Big(1-\frac{d\tau}{\mu_0}\Big) T(\tau_1 - d\tau; \mu, \varphi; \mu_0, \varphi_0), \quad (70)$$

or, to the first order in $d\tau$,

$$\frac{F}{4\mu}\Big[\Big(1-\frac{d\tau}{\mu_0}\Big) T(\tau_1; \mu, \varphi; \mu_0, \varphi_0) - \frac{\partial T(\tau_1; \mu, \varphi; \mu_0, \varphi_0)}{\partial \tau_1} d\tau\Big]. \quad (71)$$

On the other hand, the transmission of the diffuse field (65) will contribute to the radiation in the direction $(-\mu, \varphi)$ the additional intensity

$$\left. \begin{aligned} \frac{d\tau}{4\pi\mu}\int_0^1\int_0^{2\pi} T(\tau_1; \mu, \varphi; \mu', \varphi') B(0, -\mu', \varphi') \frac{d\mu'}{\mu'} d\varphi' \\ + \frac{d\tau}{\mu} B(0, -\mu, \varphi)\, e^{-\tau_1/\mu}, \end{aligned} \right\} \quad (72)$$

where the second term arises from the intensity in the *diffuse* field (65), which is already in the direction $(-\mu, \varphi)$.

Adding contributions (71) and (72) and remembering that this must equal

$$\frac{F}{4\mu} T(\tau_1; \mu, \varphi; \mu_0, \varphi_0), \quad (73)$$

we obtain

$$\left. \begin{aligned} \tfrac{1}{4}F\Big[\frac{1}{\mu_0} T(\tau_1; \mu, \varphi; \mu_0, \varphi_0) + \frac{\partial T(\tau_1; \mu, \varphi; \mu_0, \varphi_0)}{\partial \tau_1}\Big] \\ = B(0, -\mu, \varphi)\, e^{-\tau_1/\mu} + \frac{1}{4\pi}\int_0^1\int_0^{2\pi} T(\tau_1; \mu, \varphi; \mu', \varphi') B(0, -\mu', \varphi') \frac{d\mu'}{\mu'} d\varphi'. \end{aligned} \right\} \quad (74)$$

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Consider, next, the radiation field present at the level $\tau_1 - d\tau$. Since there is no outward intensity at $\tau = \tau_1$, at the level $\tau_1 - d\tau$ there must be the intensity

$$\frac{d\tau}{\mu'} B(\tau_1, \mu', \varphi')\,, \tag{75}$$

in the direction $(+\mu', \varphi')$. The reflection of this radiation by the atmosphere *above* $\tau_1 - d\tau$ will contribute to the intensity in the direction $(-\mu, \varphi)$ the amount

$$\frac{d\tau}{4\pi\mu} \int_0^1 \int_0^{2\pi} S(\tau_1;\, \mu, \varphi;\, \mu', \varphi') B(\tau_1, \mu', \varphi') \frac{d\mu'}{\mu'}\, d\varphi'\,. \tag{76}$$

In addition, there will be the radiation directly transmitted in the direction $(-\mu, \varphi)$ by the atmosphere above the level $\tau_1 - d\tau$. The intensity arising from this account is given by

$$\frac{F}{4\mu}\left[T(\tau_1;\, \mu;\, \varphi;\, \mu_0, \varphi_0) - \frac{\partial T(\tau_1;\, \mu, \varphi;\, \mu_0, \varphi_0)}{\partial \tau_1}\, d\tau\right]. \tag{77}$$

Hence

$$\left.\begin{aligned} I(\tau_1 - d\tau, -\mu, \varphi) &= \frac{F}{4\mu}\left[T(\tau_1;\, \mu, \varphi;\, \mu_0, \varphi_0) - \frac{\partial T(\tau_1;\, \mu, \varphi;\, \mu_0, \varphi_0)}{\partial \tau_1}\, d\tau\right] \\ &\quad + \frac{d\tau}{4\pi\mu} \int_0^1 \int_0^{2\pi} S(\tau_1;\, \mu, \varphi;\, \mu', \varphi') B(\tau_1, \mu', \varphi') \frac{d\mu'}{\mu'}\, d\varphi'\,. \end{aligned}\right\} \tag{78}$$

But this must equal

$$I(\tau_1 - d\tau, -\mu, \varphi) = \frac{F}{4\mu}\left(1 + \frac{d\tau}{\mu}\right) T(\tau_1;\, \mu, \varphi;\, \mu_0, \varphi_0) - \frac{d\tau}{\mu} B(\tau_1, -\mu, \varphi)\,, \tag{79}$$

which follows from the equation of transfer. From equations (78) and (79) we now obtain

$$\left.\begin{aligned} &\tfrac{1}{4}F\left[\frac{1}{\mu} T(\tau_1;\, \mu, \varphi;\, \mu_0, \varphi_0) + \frac{\partial T(\tau_1;\, \mu, \varphi;\, \mu_0, \varphi_0)}{\partial \tau_1}\right] \\ &\quad = B(\tau_1, -\mu, \varphi) + \frac{1}{4\pi} \int_0^1 \int_0^{2\pi} S(\tau_1;\, \mu, \varphi;\, \mu', \varphi') B(\tau_1, \mu', \varphi') \frac{d\mu'}{\mu'}\, d\varphi'\,. \end{aligned}\right\} \tag{80}$$

Again, in addition to equation (80), there is a further relation which can be derived from considerations relating to the field present at the level $\tau_1 - d\tau$. The relation arises from the fact that the reflection of the flux $\pi F$ incident on $\tau = 0$ by the entire atmosphere of optical thickness $\tau_1$ must be the same as the reflection of the same flux by the atmosphere of optical thickness $\tau_1 - d\tau$ and the transmission of the radiation (75) incident on the level $\tau_1 - d\tau$ from below, by the atmosphere above it. In other words, we must have

$$\left.\begin{aligned} \frac{F}{4\mu} S(\tau_1, \mu, \varphi;\, \mu_0, \varphi_0) &= \frac{F}{4\mu}\left[S(\tau_1, \mu, \varphi;\, \mu_0, \varphi_0) - \frac{\partial S(\tau_1;\, \mu, \varphi;\, \mu_0, \varphi_0)}{\partial \tau_1}\, d\tau\right] \\ &\quad + \frac{d\tau}{4\pi\mu} \int_0^1 \int_0^{2\pi} T(\tau_1;\, \mu, \varphi;\, \mu', \varphi') B(\tau_1, \mu', \varphi') \frac{d\mu'}{\mu'}\, d\varphi' \\ &\quad + \frac{d\tau}{\mu} B(\tau_1, \mu, \varphi)\, e^{-\tau_1/\mu}\,, \end{aligned}\right\} \tag{81}$$

where the three terms on the right-hand side arise, respectively, from the reflection of the flux $\pi F$ by the atmosphere of optical thickness $\tau_1 - d\tau$, from the *diffuse* transmis-

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sion of the radiation (75) incident on the level $\tau_1 - d\tau$, and, finally, from the intensity in the field (75) already in the direction $(\mu, \varphi)$. From equation (81) it now follows that

$$\left.\begin{aligned}\tfrac{1}{4}F\frac{\partial S(\tau_1;\mu,\varphi;\mu_0,\varphi_0)}{\partial\tau_1} &= B(\tau_1,\mu,\varphi)\,e^{-\tau_1/\mu}\\ &\quad+\frac{1}{4\pi}\int_0^1\int_0^{2\pi}T(\tau_1;\mu,\varphi;\mu',\varphi')B(\tau_1,\mu',\varphi')\frac{d\mu'}{\mu'}\,d\varphi'.\end{aligned}\right\}\quad(82)$$

Now, according to equations (61)–(63)

$$\left.\begin{aligned}B(0,\mu,\varphi) &= \tfrac{1}{4}F\Big[p(\mu,\varphi;-\mu_0,\varphi_0)\\ &\quad+\frac{1}{4\pi}\int_0^1\int_0^{2\pi}p(\mu,\varphi;\mu'',\varphi'')\,S(\tau_1;\mu'',\varphi'';\mu_0,\varphi_0)\frac{d\mu''}{\mu''}\,d\varphi''\Big],\end{aligned}\right\}\quad(83)$$

and

$$\left.\begin{aligned}B(\tau_1,\mu,\varphi) &= \tfrac{1}{4}F\Big[p(\mu,\varphi;-\mu_0,\varphi_0)\,e^{-\tau_1/\mu_0}\\ &\quad+\frac{1}{4\pi}\int_0^1\int_0^{2\pi}p(\mu,\varphi;-\mu'',\varphi'')T(\tau_1;\mu'',\varphi'';\mu_0,\varphi_0)\frac{d\mu''}{\mu''}\,d\varphi''\Big].\end{aligned}\right\}\quad(84)$$

Using equations (83) and (84) in equations (69), (74), (80), and (82), we obtain

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$$\left.\begin{aligned}\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right) S(\tau_1;\mu,\varphi;\mu_0,\varphi_0)+\frac{\partial S(\tau_1;\mu,\varphi;\mu_0,\varphi_0)}{\partial\tau_1} &= p(\mu,\varphi;-\mu_0,\varphi_0)\\ &+\frac{1}{4\pi}\int_0^1\int_0^{2\pi} p(\mu,\varphi;\mu'',\varphi'')\, S(\tau_1;\mu'',\varphi'';\mu_0,\varphi_0)\frac{d\mu''}{\mu''}\,d\varphi''\\ &+\frac{1}{4\pi}\int_0^1\int_0^{2\pi} S(\tau_1;\mu,\varphi;\mu',\varphi')\, p(\mu',\varphi';\mu_0,\varphi_0)\frac{d\mu'}{\mu'}\,d\varphi'\\ +\frac{1}{16\pi^2}\int_0^1\int_0^{2\pi}\int_0^1\int_0^{2\pi} S(\tau_1;\mu,\varphi;\mu',\varphi')\, p(-\mu'\ \varphi';\mu'',\varphi'')\, & S(\tau_1;\mu'',\varphi'';\mu_0,\varphi_0)\frac{d\mu'}{\mu'}\,d\varphi'\,\frac{d\mu''}{\mu''}\,d\varphi'',\end{aligned}\right\} \quad (85)$$

$$\left.\begin{aligned}\frac{\partial S(\tau_1;\mu,\varphi;\mu_0,\varphi_0)}{\partial\tau_1} &= p(\mu,\varphi;-\mu_0,\varphi_0)\exp\left\{-\tau_1\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right)\right\}\\ &+\frac{e^{-\tau_1/\mu}}{4\pi}\int_0^1\int_0^{2\pi} p(\mu,\varphi;-\mu'',\varphi'')\, T(\tau_1;\mu'',\varphi'';\mu_0,\varphi_0)\frac{d\mu''}{\mu''}\,d\varphi''\\ &+\frac{e^{-\tau_1/\mu_0}}{4\pi}\int_0^1\int_0^{2\pi} T(\tau_1;\mu,\varphi;\mu',\varphi')\, p(\mu',\varphi';-\mu_0,\varphi_0)\frac{d\mu'}{\mu'}\,d\varphi'\\ +\frac{1}{16\pi^2}\int_0^1\int_0^{2\pi}\int_0^1\int_0^{2\pi} T(\tau_1;\mu,\varphi;\mu',\varphi')\, p(\mu',\varphi';-\mu'',\varphi'')\, & T(\tau_1;\mu'',\varphi'';\mu_0,\varphi_0)\frac{d\mu'}{\mu'}\,d\varphi'\,\frac{d\mu''}{\mu''}\,d\varphi'',\end{aligned}\right\} \quad (86)$$

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$$\left.\begin{aligned}\frac{1}{\mu_0}T(\tau_1;\,\mu,\varphi;\,\mu_0,\varphi_0)+\frac{\partial T(\tau_1;\,\mu,\varphi;\,\mu_0,\varphi_0)}{\partial\tau_1}&=e^{-\tau_1/\mu}p(\mu,\varphi;\,\mu_0,\varphi_0)\\&+\frac{e^{-\tau_1/\mu}}{4\pi}\int_0^1\int_0^{2\pi}p(-\mu,\varphi;\,\mu'',\varphi'')\,S(\tau_1;\,\mu'',\varphi'';\,\mu_0,\varphi_0)\frac{d\mu''}{\mu''}\,d\varphi''\\&+\frac{1}{4\pi}\int_0^1\int_0^{2\pi}T(\tau_1;\,\mu,\varphi;\,\mu',\varphi')\,p(\mu',\varphi';\,\mu_0,\varphi_0)\frac{d\mu'}{\mu'}\,d\varphi'\\+\frac{1}{16\pi^2}\int_0^1\int_0^{2\pi}\int_0^1\int_0^{2\pi}T(\tau_1;\,\mu,\varphi;\,\mu',\varphi')\,p(-\mu',\varphi';\,\mu'',\varphi'')&\,S(\tau_1;\,\mu'',\varphi'';\,\mu_0,\varphi_0)\frac{d\mu'}{\mu'}\,d\varphi'\frac{d\mu''}{\mu''}\,d\varphi'',\end{aligned}\right\}\quad(87)$$

and

$$\left.\begin{aligned}\frac{1}{\mu}T(\tau_1;\,\mu,\varphi;\,\mu_0,\varphi_0)+\frac{\partial T(\tau_1;\,\mu,\varphi;\,\mu_0,\varphi_0)}{\partial\tau_1}&=e^{-\tau_1/\mu_0}p(\mu,\varphi;\,\mu_0,\varphi_0)\\&+\frac{1}{4\pi}\int_0^1\int_0^{2\pi}p(\mu,\varphi;\,\mu'',\varphi'')\,T(\tau_1;\,\mu'',\varphi'';\,\mu_0,\varphi_0)\frac{d\mu''}{\mu''}\,d\varphi''\\&+\frac{e^{-\tau_1/\mu_0}}{4\pi}\int_0^1\int_0^{2\pi}S(\tau_1;\,\mu,\varphi;\,\mu',\varphi')\,p(\mu',\varphi';\,-\mu_0,\varphi_0)\frac{d\mu'}{\mu'}\,d\varphi'\\+\frac{1}{16\pi^2}\int_0^1\int_0^{2\pi}\int_0^1\int_0^{2\pi}S(\tau_1;\,\mu,\varphi;\,\mu',\varphi')\,p(\mu',\varphi';\,-\mu'',\varphi'')&\,T(\tau_1;\,\mu'',\varphi'';\,\mu_0,\varphi_0)\frac{d\mu'}{\mu'}\,d\varphi'\frac{d\mu''}{\mu''}\,d\varphi''.\end{aligned}\right\}\quad(88)$$

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Equations (85)–(88) represent the four functional equations governing the problem of transmission and reflection by a plane-parallel atmosphere of finite optical thickness. A simple inspection of these equations shows that

$$S(\tau_1; \mu, \varphi; \mu_0, \varphi_0) = S(\tau_1; \mu_0, \varphi_0; \mu, \varphi) \tag{89}$$

and

$$T(\tau_1; \mu, \varphi; \mu_0, \varphi_0) = T(\tau_1; \mu_0, \varphi_0; \mu, \varphi) . \tag{90}$$

Equations (89) and (90) are the expression of the reciprocity principle for the problem on hand.

Finally, we may remark that, when we allow for the polarization of the scattered radiation, we must consider a scattering matrix, $\mathbf{S}(\tau_1; \mu, \varphi; \mu_0, \varphi_0)$, and a *transmission matrix*, $\mathbf{T}(\tau_1; \mu, \varphi; \mu_0, \varphi_0)$, in place of the functions $S(\tau_1; \mu, \varphi; \mu_0, \varphi_0)$ and $T(\tau_1; \mu, \varphi; \mu_0, \varphi_0)$. It is, however, clear that the functional equations satisfied by $\mathbf{S}(\tau_1; \mu, \varphi; \mu_0, \varphi_0)$ and $\mathbf{T}(\tau_1; \mu, \varphi; \mu_0, \varphi_0)$ will be of the same forms as equations (85)–(88), only they will be matrix equations in which a *phase matrix* $\mathbf{P}(\mu, \varphi; \mu', \varphi')$ will play the same role as the phase function $p(\mu, \varphi; \mu', \varphi')$. Thus, in the case of Rayleigh scattering, the matrix $-\frac{3}{4}\mathbf{Q}\,\mathbf{J}(\mu, \varphi; -\mu', \varphi')$, defined as in Paper XIV, equations (10) and (17), will replace $p(\mu, \varphi; \mu', \varphi')$.

6. *The reduction of the functional equations (85)–(88) for the case in which the phase function is expressible as a series in Legendre polynomials.*—For the case in which the phase function $p(\cos\Theta)$ is expressible as a series in Legendre polynomials in the form

$$p(\cos\Theta) = \sum_l \varpi_l P_l(\cos\Theta) , \tag{91}$$ [14]

the functional equations (85)–(88) can be reduced in the following manner:

First, we may observe that, for a phase function of the form (91),

$$\left.\begin{aligned} p(\mu, \varphi; \mu', \varphi') = \sum_l \varpi_l \Big[ & P_l(\mu) P_l(\mu') \\ & + 2\sum_{m=1}^{l} \frac{(l-m)!}{(l+m)!} P_l^m(\mu) P_l^m(\mu') \cos m(\varphi - \varphi') \Big] . \end{aligned}\right\} \tag{92}$$

Rearranging the series on the right-hand side of this equation, we can write

$$p(\mu, \varphi; \mu', \varphi') = \sum_m \Big[ \sum_{l=m} \varpi_l^m P_l^m(\mu) P_l^m(\mu') \Big] \cos m(\varphi - \varphi') , \tag{93}$$

where

$$\varpi_l^m = (2 - \delta_{0,m}) \varpi_l \frac{(l-m)!}{(l+m)!} \qquad (l = m, m+1, \ldots) \tag{94}$$

and

$$\left.\begin{aligned} \delta_{0,m} &= 1 \quad \text{if} \quad m = 0 \\ &= 0 \quad \text{otherwise} . \end{aligned}\right\} \tag{95}$$

From the expansion (93) for the phase function, it follows that the scattering and the transmission functions, $S(\tau_1; \mu, \varphi; \mu_0, \varphi_0)$ and $T(\tau_1; \mu, \varphi; \mu_0, \varphi_0)$, must be expressible in the forms

$$S(\tau_1; \mu, \varphi; \mu_0, \varphi_0) = \sum_m S^{(m)}(\tau_1; \mu, \mu_0) \cos m(\varphi - \varphi_0) \tag{96}$$

[14] In all practical cases the series on the right-hand side will be a terminating one; but it is not necessary to make this restriction at this point.

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and

$$T(\tau_1;\mu,\varphi;\mu_0,\varphi_0)=\sum_m T^{(m)}(\tau_1;\mu,\mu_0)\cos m(\varphi-\varphi_0)\,, \tag{97}$$

where, as the notation implies, $S^{(m)}$ and $T^{(m)}$ are functions of $\tau_1$, $\mu$, and $\mu_0$ only.

Substituting the forms (96) and (97) for $S$ and $T$ in equations (85)–(88), we find that the equations for the various Fourier components separate and that they can be further reduced to the following forms:

$$\left.\begin{aligned}&\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right)S^{(m)}(\tau_1;\mu,\mu_0)+\frac{\partial S^{(m)}(\tau_1;\mu,\mu_0)}{\partial\tau_1}\\&=\sum_{l=m}\varpi_l^m(-1)^{m+l}\left[P_l^m(\mu)+\frac{(-1)^{m+l}}{2(2-\delta_{0,m})}\int_0^1 S^{(m)}(\tau_1;\mu,\mu')P_l^m(\mu')\frac{d\mu'}{\mu'}\right]\\&\qquad\times\left[P_l^{(m)}(\mu_0)+\frac{(-1)^{m+l}}{2(2-\delta_{0,m})}\int_0^1 P_l^m(\mu'')S^{(m)}(\tau_1;\mu'',\mu_0)\frac{d\mu''}{\mu''}\right],\end{aligned}\right\}\tag{98}$$

$$\left.\begin{aligned}&\frac{\partial S^{(m)}(\tau_1;\mu,\mu_0)}{\partial\tau_1}=\sum_{l=m}\varpi_l^m(-1)^{m+l}\\&\qquad\times\left[e^{-\tau_1/\mu}P_l^m(\mu)+\frac{1}{2(2-\delta_{0,m})}\int_0^1 T^{(m)}(\tau_1;\mu,\mu')P_l^m(\mu')\frac{d\mu'}{\mu'}\right]\\&\qquad\times\left[e^{-\tau_1/\mu_0}P_l^m(\mu_0)+\frac{1}{2(2-\delta_{0,m})}\int_0^1 P_l^m(\mu'')T^{(m)}(\tau_1;\mu'',\mu_0)\frac{d\mu''}{\mu''}\right],\end{aligned}\right\}\tag{99}$$

$$\left.\begin{aligned}&\frac{1}{\mu_0}T^{(m)}(\tau_1;\mu,\mu_0)+\frac{\partial T^{(m)}(\tau_1;\mu,\mu_0)}{\partial\tau_1}\\&=\sum_{l=m}\varpi_l^m\left[P_l^m(\mu_0)+\frac{(-1)^{m+l}}{2(2-\delta_{0,m})}\int_0^1 P_l^m(\mu'')S^{(m)}(\tau_1;\mu'',\mu_0)\frac{d\mu''}{\mu''}\right]\\&\qquad\times\left[e^{-\tau_1/\mu}P_l^m(\mu)+\frac{1}{2(2-\delta_{0,m})}\int_0^1 T^{(m)}(\tau_1;\mu,\mu')P_l^m(\mu')\frac{d\mu'}{\mu'}\right],\end{aligned}\right\}\tag{100}$$

and

$$\left.\begin{aligned}&\frac{1}{\mu}T^{(m)}(\tau_1;\mu,\mu_0)+\frac{\partial T^{(m)}(\tau_1;\mu,\mu_0)}{\partial\tau_1}\\&=\sum_{l=m}\varpi_l^m\left[P_l^m(\mu)+\frac{(-1)^{m+l}}{2(2-\delta_{0,m})}\int_0^1 S^{(m)}(\tau_1;\mu,\mu')P_l^m(\mu')\frac{d\mu'}{\mu'}\right]\\&\qquad\times\left[e^{-\tau_1/\mu_0}P_l^m(\mu_0)+\frac{1}{2(2-\delta_{0,m})}\int_0^1 P_l^m(\mu'')T^{(m)}(\tau_1;\mu'',\mu_0)\frac{d\mu''}{\mu''}\right].\end{aligned}\right\}\tag{101}$$

If we now let

$$\psi_l^m(\tau_1,\mu)=P_l^m(\mu)+\frac{(-1)^{m+l}}{2(2-\delta_{0,m})}\int_0^1 S^{(m)}(\tau_1;\mu,\mu')P_l^m(\mu')\frac{d\mu'}{\mu'}\tag{102}$$

and

$$\phi_l^m(\tau_1,\mu)=e^{-\tau_1/\mu}P_l^m(\mu)+\frac{1}{2(2-\delta_{0,m})}\int_0^1 T^{(m)}(\tau_1;\mu,\mu')P_l^m(\mu')\frac{d\mu'}{\mu'},\tag{103}$$

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then, in view of equations (89) and (90), we can re-write equations (98)–(101) in the forms

$$\left.\begin{aligned}&\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right)S^{(m)}(\tau_1;\mu,\mu_0)\\&\qquad+\frac{\partial S^{(m)}(\tau_1;\mu,\mu_0)}{\partial\tau_1}=\sum_{l=m}\varpi_l^m(-1)^{m+l}\psi_l^m(\tau_1,\mu)\psi_l^m(\tau_1,\mu_0),\end{aligned}\right\}\quad(104)$$

$$\frac{\partial S^{(m)}(\tau_1;\mu,\mu_0)}{\partial\tau_1}=\sum_{l=m}\varpi_l^m(-1)^{m+l}\phi_l^m(\tau_1,\mu)\phi_l^m(\tau_1,\mu_0),\quad(105)$$

$$\frac{1}{\mu}T^{(m)}(\tau_1;\mu,\mu_0)+\frac{\partial T^{(m)}(\tau_1;\mu,\mu_0)}{\partial\tau_1}=\sum_{l=m}\varpi_l^m\psi_l^m(\tau_1,\mu)\phi_l^m(\tau_1,\mu_0)\quad(106)$$

and

$$\frac{1}{\mu_0}T^{(m)}(\tau_1;\mu,\mu_0)+\frac{\partial T^{(m)}(\tau_1;\mu,\mu_0)}{\partial\tau_1}=\sum_{l=m}\varpi_l^m\psi_l^m(\tau_1,\mu_0)\phi_l^m(\tau_1,\mu).\quad(107)$$

Alternative forms of the foregoing equations are

$$\left.\begin{aligned}&\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right)S^{(m)}(\tau_1;\mu,\mu_0)\\&=\sum_{l=m}\varpi_l^m(-1)^{m+l}[\psi_l^m(\tau_1,\mu)\psi_l^m(\tau_1,\mu_0)-\phi_l^m(\tau_1,\mu)\phi_l^m(\tau_1,\mu_0)],\end{aligned}\right\}\quad(108)$$

$$\left.\begin{aligned}&\left(\frac{1}{\mu}-\frac{1}{\mu_0}\right)T^{(m)}(\tau_1;\mu,\mu_0)\\&=\sum_{l=m}\varpi_l^m[\psi_l^m(\tau_1,\mu)\phi_l^m(\tau_1,\mu_0)-\psi_l^m(\tau_1,\mu_0)\phi_l^m(\tau_1,\mu)],\end{aligned}\right\}\quad(109)$$

$$\frac{\partial S^{(m)}(\tau_1;\mu,\mu_0)}{\partial\tau_1}=\sum_{l=m}\varpi_l^m(-1)^{m+l}\phi_l^m(\tau_1,\mu)\phi_l^m(\tau_1,\mu_0),\quad(110)$$

and

$$\left.\begin{aligned}&\left(\frac{1}{\mu}-\frac{1}{\mu_0}\right)\frac{\partial T^{(m)}(\tau_1;\mu,\mu_0)}{\partial\tau_1}\\&=\sum_{l=m}\varpi_l^m\left[\frac{1}{\mu}\psi_l^m(\tau_1,\mu_0)\phi_l^m(\tau_1,\mu)-\frac{1}{\mu_0}\psi_l^m(\tau_1,\mu)\phi_l^m(\tau_1,\mu_0)\right].\end{aligned}\right\}\quad(111)$$

Finally, substituting for $S^{(m)}(\tau_1;\mu,\mu')$ and $T^{(m)}(\tau_1;\mu,\mu')$ from equations (108) and (109) in equations (102) and (103), we obtain

$$\left.\begin{aligned}\psi_l^m(\tau_1,\mu)&=P_l^m(\mu)+\frac{1}{2}\sum_{k=m}(-1)^{k+l}\varpi_k\frac{(k-m)!}{(k+m)!}\mu\\&\times\int_0^1\frac{d\mu'}{\mu+\mu'}[\psi_k^m(\tau_1,\mu)\psi_k^m(\tau_1,\mu')-\phi_k^m(\tau_1,\mu)\phi_k^m(\tau_1,\mu')]P_l^m(\mu')\end{aligned}\right\}\quad(112)$$

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and

$$\left.\begin{aligned}\phi_l^m(\tau_1,\mu) &= P_l^m(\mu)\, e^{-\tau_1/\mu} + \frac{1}{2}\sum_{k=m} \varpi_k \frac{(k-m)!}{(k+m)!}\mu \\ &\times \int_0^1 \frac{d\mu'}{\mu-\mu'}\left[\psi_k^m(\tau_1,\mu')\,\phi_k^m(\tau_1,\mu) - \psi_k^m(\tau_1,\mu)\,\phi_k^m(\tau_1,\mu')\right] P_l^m(\mu')\,,\end{aligned}\right\} \quad (113)$$

where we have further substituted for $\varpi_k^m$ in accordance with equation (94).

7. *The functional equations for the case of isotropic scattering.*—In the case of isotropic scattering,

$$\varpi_l = 0 \qquad (l \neq 0)\,, \quad (114)$$

and equations (108)–(113) reduce to

$$\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right) S(\tau_1;\,\mu,\,\mu_0) = \varpi_0\left[\psi(\tau_1,\mu)\,\psi(\tau_1,\mu_0) - \phi(\tau_1,\mu)\,\phi(\tau_1,\mu_0)\right], \quad (115)$$

$$\left(\frac{1}{\mu}-\frac{1}{\mu_0}\right) T(\tau_1;\,\mu,\,\mu_0) = \varpi_0\left[\psi(\tau_1,\mu)\,\phi(\tau_1,\mu_0) - \psi(\tau_1,\mu_0)\,\phi(\tau_1,\mu)\right], \quad (116)$$

$$\frac{\partial S(\tau_1;\,\mu,\,\mu_0)}{\partial \tau_1} = \varpi_0 \phi(\tau_1,\mu)\,\phi(\tau_1,\mu_0)\,, \quad (117)$$

$$\left(\frac{1}{\mu}-\frac{1}{\mu_0}\right)\frac{\partial T(\tau_1;\,\mu,\,\mu_0)}{\partial \tau_1} = \varpi_0\left[\frac{1}{\mu}\psi(\tau_1,\mu_0)\,\phi(\tau_1,\mu) - \frac{1}{\mu_0}\psi(\tau_1,\mu)\,\phi(\tau_1,\mu_0)\right], \quad (118)$$

$$\psi(\tau_1,\mu) = 1 + \tfrac{1}{2}\varpi_0\mu\int_0^1 \frac{d\mu'}{\mu+\mu'}\left[\psi(\tau_1,\mu)\,\psi(\tau_1,\mu') - \phi(\tau_1,\mu)\,\phi(\tau_1,\mu')\right], \quad (119)$$

and

$$\phi(\tau_1,\mu) = e^{-\tau_1/\mu} + \tfrac{1}{2}\varpi_0\mu\int_0^1 \frac{d\mu'}{\mu-\mu'}\left[\psi(\tau_1,\mu')\,\phi(\tau_1,\mu) - \psi(\tau_1,\mu)\,\phi(\tau_1,\mu')\right]. \quad (120)$$

Equations (115), (116), (119), and (120) agree with the equations given by Ambarzumian.[10] But equations (117) and (118) are new.

The question now arises as to whether the solution of the system of equations (115)–(120) can be obtained in closed forms when the integrals on the right-hand sides of equations (119) and (120) are replaced by Gauss sums in the $n$th approximation. This question is related to the elimination of the constants in the method of solution of the earlier papers of this series (Paper XII, for example) and to the still larger question of whether the systems of functional equations to which we were led in § 6 can be reduced to single (or, more possibly, a pair of) functional equations of standard forms. All these questions, as they arise in connection with transfer problems in semi-infinite atmospheres, were essentially answered in Paper XIV; but they remain to be investigated in the present more general context. (See note added at end of paper.)

8. *Functional equations governing the radiation field in the interior of plane-parallel atmospheres.*—We have concerned ourselves, so far, with only the radiations emerging from the boundaries of plane-parallel atmospheres. But it is clear that the ideas, particularly those leading to the functional equations derived in §§ 2, 4, and 5, can be applied equally to formulate functional equations for the radiation field in the interior. In this section we shall give some examples of such functional equations.

We shall consider, first, the radiation field in an atmosphere of optical thickness $\tau_1$, on which is incident a parallel beam of radiation of flux $\pi F$, in the direction $(-\mu_0, \varphi_0)$. At a depth $\tau$ in this atmosphere, there will be the incident flux reduced to the amount

$$\pi F e^{-\tau/\mu_0}\,, \quad (121)$$

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as well as a diffuse radiation field characterized by the intensity $I(\tau, \mu, \varphi)$. To distinguish between the radiation in the outward ($0 \leqslant \mu \leqslant 1$) and the inward ($0 > \mu \geqslant -1$) directions, we shall write

$$I(\tau, +\mu, \varphi) \qquad (0 \leqslant \mu \leqslant 1) \tag{122}$$

and

$$I(\tau, -\mu, \varphi) \qquad (0 < \mu \leqslant 1). \tag{123}$$

Now the atmosphere below $\tau$ will reflect the radiations (121) and (123) according to the reflective power of an atmosphere of optical thickness $(\tau_1 - \tau)$ and will contribute to an outward intensity in the direction $(+\mu, \varphi)$ which must equal $I(\tau, +\mu, \varphi)$. In other words,

$$\left.\begin{aligned} I(\tau, +\mu, \varphi) &= \frac{1}{4\mu} F e^{-\tau/\mu_0} S(\tau_1 - \tau; \mu, \varphi; \mu_0, \varphi_0) \\ &+ \frac{1}{4\pi\mu} \int_0^1 \int_0^{2\pi} S(\tau_1 - \tau; \mu, \varphi; \mu', \varphi') I(\tau; -\mu', \varphi') d\mu' d\varphi' . \end{aligned}\right\} \tag{124}$$

Similarly, we must have

$$\left.\begin{aligned} I(\tau, -\mu, \varphi) &= \frac{1}{4\mu} FT(\tau; \mu, \varphi; \mu_0, \varphi_0) \\ &+ \frac{1}{4\pi\mu} \int_0^1 \int_0^{2\pi} S(\tau; \mu, \varphi; \mu', \varphi') I(\tau, +\mu', \varphi') d\mu' d\varphi' , \end{aligned}\right\} \tag{125}$$

expressing the fact that the intensity in the direction $(-\mu, \varphi)$ may be regarded as resulting from the transmission of the incident flux by the part of the atmosphere above $\tau$ and the reflection of the radiation (122) incident on $\tau$, from below.

Functional equations of a different sort arise from considerations of the type which led to the functional equations (74) and (82) in § 5. Thus, from the equivalence of the transmission by the atmosphere of optical thickness $\tau_1$ and the transmission of the radiations (121) and (123) by the atmosphere of optical thickness $(\tau_1 - \tau)$ below the level $\tau$, we conclude that

$$\left.\begin{aligned} &\frac{1}{4\mu} FT(\tau_1; \mu, \varphi; \mu_0, \varphi_0) = \\ &\frac{1}{4\mu} F e^{-\tau/\mu_0} T(\tau_1 - \tau; \mu, \varphi; \mu_0, \varphi_0) + I(\tau, -\mu, \varphi)\, e^{-(\tau_1 - \tau)/\mu} \\ &\qquad + \frac{1}{4\pi\mu} \int_0^1 \int_0^{2\pi} T(\tau_1 - \tau; \mu, \varphi; \mu', \varphi') I(\tau; -\mu', \varphi') d\mu' d\varphi' . \end{aligned}\right\} \tag{126}$$

The three terms on the right-hand side of equation (126) represent, respectively, the contributions from the transmission of the flux (121) by the atmosphere below $\tau$, the intensity in the diffuse field (123) already in the direction $(-\mu, \varphi)$ and the diffuse transmission of the field (123) by the atmosphere of optical thickness $(\tau_1 - \tau)$.

Similarly, we must have

$$\left.\begin{aligned} \frac{1}{4\mu} FS(\tau_1; \mu, \varphi; \mu_0, \varphi_0) &= \frac{1}{4\mu} FS(\tau; \mu, \varphi; \mu_0, \varphi_0) + I(\tau, +\mu, \varphi)\, e^{-\tau/\mu} \\ &+ \frac{1}{4\pi\mu} \int_0^1 \int_0^{2\pi} T(\tau; \mu, \varphi; \mu', \varphi') I(\tau, +\mu', \varphi') d\mu' d\varphi' , \end{aligned}\right\} \tag{127}$$

which expresses the equivalence of the reflection by the atmosphere of optical thickness $\tau_1$ and the reflection by the part of the atmosphere above $\tau$ and the transmission of the radiation (122) incident on $\tau$ from below.

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Equations (126) and (127) can be re-written in the forms

$$\left.\begin{aligned} I(\tau,+\mu,\varphi)\,e^{-\tau/\mu} &= \frac{F}{4\mu}\left[S(\tau_1;\mu,\varphi;\mu_0,\varphi_0)-S(\tau;\mu,\varphi;\mu_0,\varphi_0)\right] \\ &\quad -\frac{1}{4\pi\mu}\int_0^1\int_0^{2\pi}T(\tau;\mu,\varphi;\mu'\ \varphi')\,I(\tau,+\mu',\varphi')\,d\mu' d\varphi' \end{aligned}\right\} \quad (128)$$

and

$$\left.\begin{aligned} & I(\tau,-\mu,\varphi)\,e^{-(\tau_1-\tau)/\mu} = \\ & \frac{F}{4\mu}\left[T(\tau_1;\mu,\varphi;\mu_0,\varphi_0)-e^{-\tau/\mu_0}T(\tau_1-\tau;\mu,\varphi;\mu_0,\varphi_0)\right] \\ & \qquad -\frac{1}{4\pi\mu}\int_0^1\int_0^{2\pi}T(\tau_1-\tau;\mu,\varphi;\mu',\varphi')\,I(\tau,-\mu',\varphi')\,d\mu' d\varphi'\,. \end{aligned}\right\} \quad (129)$$

It is clear that equations (124) and (125) or (128) and (129) will suffice to determine the radiation field in the interior uniquely, in terms of the scattering and transmission functions of atmospheres of finite optical thicknesses.

The functional equations analogous to equations (124)–(129) for the axially symmetric radiation field in the interior of a semi-infinite atmosphere with a constant net flux are

$$I(\tau,+\mu)=I(0,+\mu)+\frac{1}{2\mu}\int_0^1 S^{(0)}(\infty;\mu,\mu')\,I(\tau,-\mu')\,d\mu'\,, \quad (130)$$

$$I(0,+\mu)=I(\tau,+\mu)\,e^{-\tau/\mu}+\frac{1}{2\mu}\int_0^1 T^{(0)}(\tau;\mu,\mu')\,I(\tau,+\mu')\,d\mu'\,, \quad (131)$$

and

$$I(\tau,-\mu)=\frac{1}{2\mu}\int_0^1 S^{(0)}(\tau;\mu,\mu')\,I(\tau,+\mu')\,d\mu'\,, \quad (132)$$

where $S^{(0)}$ and $T^{(0)}$ are the azimuth independent terms in the scattering and the transmission functions defined as in equation (7). Equations (130), (131), and (132) express, respectively, the invariance of the emergent intensity to the removal of layers of arbitrary optical thickness from the atmosphere, the consideration that the emergent intensity may be regarded as the transmission of the radiation $I(\tau,+\mu)$ by the atmosphere above the level $\tau$, and the fact that the inward intensity prevailing at any level arises in consequence of the reflection of the outward radiation field by the atmosphere above $\tau$.

It is evident that the functional equations (124)–(132), together with the equations (85)–(88), satisfied by the scattering and the transmission functions, are entirely equivalent to the transfer problems formulated in terms of the equations of transfer and boundary conditions.

In a later paper we propose to undertake a detailed study of the various functional equations which we have formulated in this paper; but it is of interest to recall, meantime, that the basic ideas underlying the formulation of these functional equations resemble those which were introduced by Sir George Stokes and Lord Rayleigh in their treatment of the reflection and transmission of light by piles of plates.[15]

*Note added April 15.*—The questions raised at the end of § 7 have now been answered. In particular, solutions in closed forms for equations (119) and (120) have been found when the integrals on the right-hand sides are replaced by the corresponding Gauss sums. It is hoped to publish the results of these investigations in the near future.

[15] *Mathematical and Physical Papers of Sir George Stokes*, IV (Cambridge, England, 1904), 145; and *Scientific Papers of Lord Rayleigh*, VI (Cambridge, England, 1920), 492. I am indebted to Sir K. S. Krishnan for drawing my attention to these early investigations.

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