ON THE RADIATIVE EQUILIBRIUM OF A STELLAR ATMOSPHERE. XVI

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ABSTRACT

In this paper we tabulate the various H-functions which occur in the solutions of transfer problems involving Rayleigh's phase function and Rayleigh scattering (including the state of polarization of the scattered radiation). The H-functions have been determined numerically as the solutions of the exact functional equations which they satisfy.

The exact laws of darkening in the two states of polarization of the emergent radiation from an electron-scattering atmosphere are also tabulated in this paper.

1. *Introduction.*—In Paper XIV[1] of this series, it was shown how exact solutions for a large class of problems in the theory of radiative transfer can be obtained by simply letting the H-functions (defined in terms of the Gaussian division and characteristic roots), which appear in their solutions (in the nth approximation), become solutions of associated functional equations of a certain standard form, namely,

$$H(\mu) = 1 + \mu H(\mu) \int_0^1 \frac{H(\mu')\Psi(\mu')}{\mu + \mu'} d\mu' , \tag{1}$$

where the *characteristic function* $\Psi(\mu)$ is an even polynomial in μ, satisfying the condition

$$\int_0^1 \Psi(\mu)\, d\mu \leqslant \tfrac{1}{2} . \tag{2}$$

The practical importance of this result arises from the fact that, starting with a solution for $H(\mu)$ in the third approximation (for example), we can obtain by a process of iteration the exact H-functions which characterize the problem. And, as we have further seen in Paper XIV, once the exact H-functions have been determined in this (or similar fashion), such other constants as the solutions may involve can be evaluated directly in terms of the moments of these functions.

We have now completed the numerical solution of the various H-functions which occur in the problems considered in Papers III, IX (Part II), X, XI, XIII, and XV; and these solutions are tabulated in this paper. Reference may be made here to the exact laws of darkening and to the degree of polarization of the emergent radiation from an electron-scattering atmosphere, which are also tabulated (see Table 6).

I. TRANSFER PROBLEMS INVOLVING RAYLEIGH'S PHASE FUNCTION

The problem of a semi-infinite plane-parallel atmosphere with a constant net flux πF. *The law of darkening.*—

$$I(\mu) = \tfrac{3}{4} qFH^{(0)}(\mu) .$$

The law of diffuse reflection from a semi-infinite plane-parallel atmosphere.—

$$\begin{aligned} I(\mu, \varphi; \mu_0, \varphi_0) = \tfrac{3}{32} F \{ & H^{(0)}(\mu) H^{(0)}(\mu_0) [3 - c(\mu + \mu_0) + \mu\mu_0] \\ & - 4\mu\mu_0 (1-\mu^2)^{\frac{1}{2}} (1-\mu_0^2)^{\frac{1}{2}} H^{(1)}(\mu) H^{(1)}(\mu_0) \cos(\varphi - \varphi_0) \\ & + (1-\mu^2)(1-\mu_0^2) H^{(2)}(\mu) H^{(2)}(\mu_0) \cos 2(\varphi - \varphi_0) \} \frac{\mu_0}{\mu + \mu_0} . \end{aligned}$$

[1] *Ap. J.*, **105**, 164, 1947.

The characteristic functions in terms of which $H^{(0)}(\mu)$, $H^{(1)}(\mu)$, and $H^{(2)}(\mu)$ are defined are, respectively,

$$\Psi^{(0)}(\mu) = \tfrac{3}{16}(3-\mu^2)\,,$$

$$\Psi^{(1)}(\mu) = \tfrac{3}{8}\mu^2(1-\mu^2)\,,$$

and

$$\Psi^{(2)}(\mu) = \tfrac{3}{32}(1-\mu^2)^2\,.$$

The constants q and c are given by

$$q=\frac{2}{3\alpha_1} \quad \text{and} \quad c=\frac{\alpha_2}{\alpha_1}\,,$$

where α_1 and α_2 are the moments of order 1 and 2 of $H^{(0)}(\mu)$. Further, q and c are related according to

$$8q^2 = 3 - c^2\,.$$

TABLE 1

THE FUNCTIONS $H^{(0)}(\mu)$, $H^{(1)}(\mu)$, AND $H^{(2)}(\mu)$, OBTAINED AS SOLUTIONS OF THE EXACT FUNCTIONAL EQUATIONS

μ	$H^{(0)}(\mu)$	$H^{(1)}(\mu)$	$H^{(2)}(\mu)$	μ	$H^{(0)}(\mu)$	$H^{(1)}(\mu)$	$H^{(2)}(\mu)$
0.	1.00000	1.00000	1.00000	0.55	2.17098	1.02539	1.03586
0.05	1.14691	1.00430	1.01145	0.60	2.26650	1.02652	1.03679
0.10	1.26470	1.00786	1.01724	0.65	2.36162	1.02757	1.03761
0.15	1.37457	1.01089	1.02134	0.70	2.45639	1.02854	1.03836
0.20	1.48009	1.01352	1.02448	0.75	2.55085	1.02944	1.03904
0.25	1.58281	1.01582	1.02700	0.80	2.64503	1.03028	1.03966
0.30	1.68355	1.01785	1.02909	0.85	2.73899	1.03106	1.04024
0.35	1.78287	1.01968	1.03085	0.90	2.83274	1.03179	1.04076
0.40	1.88105	1.02132	1.03236	0.95	2.92631	1.03247	1.04125
0.45	1.97836	1.02280	1.03368	1.00	3.01973	1.03312	1.04170
0.50	2.07496	1.02415	1.03483				

TABLE 2

THE CONSTANTS DERIVED FROM THE EXACT FUNCTION, $H^{(0)}(\mu)$

$\alpha_0 = 2.06088$ $\quad q = 0.55835$

$\alpha_1 = 1.19400$ $\quad c = 0.71139$

$\alpha_2 = 0.84940$

TABLE 3

THE EXACT LAW OF DARKENING FOR AN ATMOSPHERE WITH A CONSTANT NET FLUX AND SCATTERING RADIATION IN ACCORDANCE WITH RAYLEIGH'S PHASE FUNCTION

μ	$I(0,\mu)/F$	$I(0,\mu)/I(0,1)$	μ	$I(0,\mu)/F$	$I(0,\mu)/I(0,1)$
0.	0.41876	0.33116	0.55	0.90912	0.71893
0.05	.48028	.37981	0.60	0.94912	0.75056
.10	.52961	.41881	0.65	0.98896	0.78206
.15	.57562	.45520	0.70	1.02864	0.81345
.20	.61981	.49014	0.75	1.06820	0.84473
.25	.66282	.52416	0.80	1.10764	0.87592
.30	.70501	.55752	0.85	1.14698	0.90703
.35	.74660	.59041	0.90	1.18624	0.93808
.40	.78771	.62292	0.95	1.22543	0.96906
.45	.82846	.65514	1.00	1.26455	1.00000
0.50	0.86891	0.68791			

II. TRANSFER PROBLEMS WHICH INVOLVE RAYLEIGH SCATTERING AND IN WHICH THE POLARIZATION OF THE SCATTERED RADIATION IS ACCURATELY ALLOWED FOR

The problem of a semi-infinite plane-parallel atmosphere with a constant net flux πF. *The law of darkening in the two states of polarization.*—

$$I_l(0,\mu)=\tfrac{3}{8}F\frac{q}{\sqrt{2}}H_l(\mu)$$

and

$$I_r(0,\mu)=\tfrac{3}{8}F\frac{1}{\sqrt{2}}H_r(\mu)(\mu+c)\,.$$

The law of diffuse reflection of a parallel beam of partially elliptically polarized light from a semi-infinite plane-parallel atmosphere.—

$$I(\mu,\varphi;\mu_0,\varphi_0)=\frac{3}{16\mu}Q\,[S^{(0)}(\mu,\mu_0)+(1-\mu^2)^{\frac{1}{2}}(1-\mu_0^2)^{\frac{1}{2}}S^{(1)}(\mu,\varphi;\mu_0,\varphi_0)+S^{(2)}(\mu,\varphi;\mu_0,\varphi_0)\,]\,F$$

and

$$V(\mu,\varphi;\mu_0,\varphi_0)=\tfrac{3}{8}V_0[-\mu\mu_0H_v(\mu)H_v(\mu_0)+(1-\mu^2)^{\frac{1}{2}}(1-\mu_0^2)^{\frac{1}{2}}H_r(\mu)H_r(\mu_0)\cos(\varphi-\varphi_0)\,]\frac{\mu_0}{\mu+\mu_0},$$

where

$$\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right)S^{(0)}(\mu,\mu_0)=$$

$$\begin{pmatrix}2H_l(\mu)H_l(\mu_0)[1-c(\mu+\mu_0)+\mu\mu_0] & qH_l(\mu)H_r(\mu_0)(\mu+\mu_0) & 0\\ qH_r(\mu)H_l(\mu_0)(\mu+\mu_0) & H_r(\mu)H_r(\mu_0)[1+c(\mu+\mu_0)+\mu\mu_0] & 0\\ 0 & 0 & 0\end{pmatrix}$$

$$\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right)S^{(1)}(\mu,\varphi;\mu_0,\varphi_0)=H^{(1)}(\mu)H^{(1)}(\mu_0)\times\begin{pmatrix}-4\mu\mu_0\cos(\varphi-\varphi_0) & 0 & -2\mu\sin(\varphi-\varphi_0)\\ 0 & 0 & 0\\ -2\mu_0\sin(\varphi-\varphi_0) & 0 & \cos(\varphi-\varphi_0)\end{pmatrix}$$

and

$$\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right)S^{(2)}(\mu,\varphi;\mu_0,\varphi_0)=H^{(2)}(\mu)H^{(2)}(\mu_0)\times\begin{pmatrix}\mu^2\mu_0^2\cos 2(\varphi-\varphi_0) & -\mu^2\cos 2(\varphi-\varphi_0) & \mu^2\mu_0\sin 2(\varphi-\varphi_0)\\ -\mu_0^2\cos 2(\varphi-\varphi_0) & \cos 2(\varphi-\varphi_0) & -\mu_0\sin 2(\varphi-\varphi_0)\\ \mu\mu_0^2\sin 2(\varphi-\varphi_0) & -\mu\sin 2(\varphi-\varphi_0) & -\mu\mu_0\cos 2(\varphi-\varphi_0)\end{pmatrix}.$$

The characteristic functions in terms of which $H_l(\mu)$, $H_r(\mu)$, $H_v(\mu)$, $H^{(1)}(\mu)$, and $H^{(2)}(\mu)$ are defined, are, respectively,

$$\Psi_l(\mu)=\tfrac{3}{4}(1-\mu^2)\qquad \Psi_r(\mu)=\tfrac{3}{8}(1-\mu^2)\,;\qquad \Psi_v(\mu)=\tfrac{3}{4}\mu^2$$

and

$$\Psi^{(1)}(\mu)=\tfrac{3}{8}(1-\mu^2)(1+2\mu^2)\,;\qquad \Psi^{(2)}(\mu)=\tfrac{3}{16}(1+\mu^2)^2\,.$$

The constants q and c are given by

$$q=2\,\frac{4\,(A_1+2\,a_1)\;-\;3\,(A_0a_1+A_1a_0)}{3\,(A_1^2+2\,a_1^2)}$$

and

$$c=\frac{8\,(A_1-a_1)\;+3\,(2\,a_1a_0-A_1A_0)}{3\,(A_1^2+2\,a_1^2)}\,,$$

where a_0, A_0, and a_1, A_1, are the moments of order 0 and 1 of $H_l(\mu)$ and $H_r(\mu)$, respectively. Further, the constants q and c are related according to

$$q^2=2\,(1-c^2)\,.$$

TABLE 4

THE FUNCTIONS $H_l(\mu)$, $H_r(\mu)$, $H_v(\mu)$, $H^{(1)}(\mu)$, AND $H^{(2)}(\mu)$ OBTAINED AS SOLUTIONS OF THE EXACT FUNCTIONAL EQUATIONS

μ	$H_l(\mu)$	$H_r(\mu)$	$H_v(\mu)$	$H^{(1)}(\mu)$	$H^{(2)}(\mu)$
0	1.0000	1.00000	1.000000	1.00000	1.00000
0.05	1.1814	1.05737	1.020307	1.07301	1.04967
0.10	1.3255	1.09113	1.038341	1.12164	1.08621
0.15	1.4596	1.11703	1.054580	1.16151	1.11762
0.20	1.5884	1.13816	1.069338	1.19571	1.14552
0.25	1.7137	1.15594	1.082844	1.22577	1.17075
0.30	1.8367	1.17128	1.095250	1.25256	1.19383
0.35	1.9579	1.18468	1.106714	1.27674	1.21508
0.40	2.0778	1.19654	1.117346	1.29872	1.23476
0.45	2.1966	1.20713	1.127237	1.31882	1.25308
0.50	2.3146	1.21668	1.136467	1.33733	1.27019
0.55	2.4319	1.22532	1.145102	1.35444	1.28624
0.60	2.5486	1.23320	1.153203	1.37030	1.30132
0.65	2.6649	1.24042	1.160816	1.38507	1.31554
0.70	2.7807	1.24705	1.167989	1.39886	1.32895
0.75	2.8962	1.25318	1.174758	1.41179	1.34166
0.80	3.0113	1.25886	1.181158	1.42392	1.35371
0.85	3.1262	1.26414	1.187217	1.43533	1.36515
0.90	3.2408	1.26906	1.192967	1.44608	1.37601
0.95	3.3552	1.27366	1.198427	1.45625	1.38638
1.00	3.4695	1.27797	1.203620	1.46586	1.39625

TABLE 5

THE CONSTANTS DERIVED FROM THE EXACT FUNCTIONS, $H_l(\mu)$ AND $H_r(\mu)$

$a_0=2.29767$	$A_1=0.61733$
$A_0=1.19736$	$q=0.68989$
$a_1=1.34864$	$c=0.87294$

2. *Remarks on the computation of the* H*-functions; the accuracy of the tabulated solutions* —We shall briefly indicate the procedure which was adopted in the solution of the vari ous functional equations satisfied by $H\,(\mu)$.

First, we may observe that, as a general rule, the functional equation in the forr (Paper XIV, eq. [171])

$$\frac{1}{H\,(\mu)}=\left\{\left[1-2\int_0^1\Psi\,(\mu)\,d\mu\right]^{\frac{1}{2}}+\int_0^1\frac{H\,(\mu')\,\Psi\,(\mu')}{\mu+\mu'}\,\mu'd\mu'\right\} \tag{3}$$

was found more suitable for purposes of iteration than the equation in its original forr

The solution was generally started with the third approximation for $H(\mu)$ in terms of the Gaussian division and characteristic roots. (In conservative cases [cf. Paper XIV, p. 190] the fourth approximation was available.) The first iteration was performed with this approximate solution for $H(\mu)$ in accordance with equation (3), evaluating the integral on the right-hand side for various values of μ. It was found that in most cases the first iterate had to be obtained for steps of 0.05 over the entire interval $0 \leqslant \mu \leqslant 1$. However, in the second and the following iterations, it was sufficient to evaluate the iterate for fewer and fewer values of the argument, since for the intermediate values the function could be predicted with accuracy by interpolating among the differences between the successive iterates. The process of iteration was continued until the function did not change within the limits of the accuracy of the calculations.

TABLE 6

THE EXACT LAWS OF DARKENING IN THE TWO STATES OF POLARIZATION FOR AN ELECTRON-SCATTERING ATMOSPHERE; DEGREE OF POLARIZATION OF THE EMERGENT RADIATION

μ	I_l/F	I_r/F	I/F	Law of Darkening in State of Polarization		Degree of Polarization
				l	r	
0.	0.18294	0.23147	0.41441	0.28823	0.36470	0.11713
0.05.	.21613	.25877	0.47490	0.34053	0.40771	.08979
0.10.	.24247	.28150	0.52397	0.38203	0.44352	.07448
0.15.	.26702	.30299	0.57001	0.42070	0.47739	.06311
0.20.	.29057	.32381	0.61439	0.45782	0.51019	.05410
0.25.	.31350	.34420	0.65770	0.49394	0.54231	.04667
0.30.	.33599	.36429	0.70029	0.52939	0.57397	.04041
0.35.	.35817	.38417	0.74234	0.56432	0.60529	.03502
0.40.	.38010	.40388	0.78398	0.59888	0.63634	.03033
0.45.	.40184	.42346	0.82530	0.63313	0.66719	.02619
0.50.	.42343	.44294	0.86637	0.66714	0.69788	.02252
0.55.	.44489	.46233	0.90722	0.70095	0.72844	.01923
0.60.	.46624	.48165	0.94789	0.73459	0.75888	.01627
0.65.	.48750	.50092	0.98842	0.76809	0.78924	.01358
0.70.	.50869	.52013	1.02882	0.80147	0.81950	.011123
0.75.	.52981	.53930	1.06911	0.83475	0.84971	.008880
0.80.	.55087	.55844	1.10931	0.86794	0.87986	.006818
0.85.	.57189	.57754	1.14943	0.90105	0.90996	.004919
0.90.	.59286	.59661	1.18947	0.93409	0.94001	.003155
0.95.	.61379	.61566	1.22945	0.96707	0.97002	0.001522
1.00.	0.63469	0.63469	1.26938	1.00000	1.00000	0

The numerical computations were carried out in all stages with at least six-figure accuracy. It is believed that in the tabulated solutions errors exceeding two (and in rare cases three) units in the last place retained are unlikely.

It may be further stated that, as a check on the accuracy attained at each stage of the iteration, the integral

$$\int_0^1 H(\mu) \Psi(\mu)\, d\mu) \tag{4}$$

was evaluated numerically and compared with its exact value (Paper XIV, eq. [165]),

$$1 - \left[1 - 2\int_0^1 \Psi(\mu)\, d\mu\right]^{\frac{1}{2}}. \tag{5}$$

In conservative cases the further integral (Paper XIV, eq. [177]),

$$\int_0^1 H(\mu)\Psi(\mu)\,\mu d\mu = \left[2\int_0^1 \Psi(\mu)\,\mu^2 d\mu\right]^{\frac{1}{2}}, \tag{6}$$

was also used. In Table 7 we exhibit this comparision between the various integrals evaluated with the aid of the tabulated functions and their exact values. It is seen that the agreement between the two sets of values confirms our estimate of the accuracy of the tabulated solutions.

TABLE 7

COMPARISON OF THE VALUES OF CERTAIN INTEGRALS EVALUATED WITH THE AID OF THE TABULATED FUNCTIONS WITH THEIR EXACT VALUES

Case	Integral	Exact Value	Value Found Numerically
I.....	$\frac{3}{16}\int_0^1 H^{(0)}(\mu)(3-\mu^2)\,d\mu$	1.00000	0.99998
	$\frac{3}{16}\int_0^1 H^{(0)}(\mu)(3-\mu^2)\,\mu d\mu$	$\sqrt{0.3}=0.5477226$	0.547735
	$\frac{3}{8}\int_0^1 H^{(1)}(\mu)\,\mu^2(1-\mu^2)\,d\mu$	$1-\sqrt{0.9}=0.05131670$	0.051317
	$\frac{3}{32}\int_0^1 H^{(2)}(\mu)(1-\mu^2)^2 d\mu$	$1-\sqrt{0.9}=0.05131670$	0.051316
II...	$\int_0^1 H_l(\mu)(1-\mu^2)\,d\mu$	$\frac{4}{3}=1.333333$	1.3332
	$\int_0^1 H_l(\mu)(1-\mu^2)\,\mu d\mu$	$\frac{4}{3}\sqrt{0.2}=0.5962848$	0.59628
	$\frac{3}{8}\int_0^1 H_r(\mu)(1-\mu^2)\,d\mu$	$1-\frac{1}{\sqrt{2}}=0.2928932$	0.29288
	$\frac{3}{4}\int_0^1 H_v(\mu)\,\mu^2 d\mu$	$1-\frac{1}{\sqrt{2}}=0.2928932$	0.292893
	$\frac{3}{8}\int_0^1 H^{(1)}(\mu)(1-\mu^2)(1+2\mu^2)\,d\mu$	$1-\sqrt{0.3}=0.45227744$	0.45226
	$\frac{3}{16}\int_0^1 H^{(2)}(\mu)(1+\mu^2)^2 d\mu$	$1-\sqrt{0.3}=0.45227744$	0.45227