

# ON THE RADIATIVE EQUILIBRIUM OF A STELLAR ATMOSPHERE. XV

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## ABSTRACT

In this paper the general equations of transfer for an atmosphere scattering radiation in accordance with Rayleigh's law and allowing for a partial elliptic polarization of the radiation field are formulated. It is shown that, under these conditions, we must consider, in addition to the intensities  $I_l$  and  $I_r$ , in two directions at right angles to each other in the plane of the electric and the magnetic vectors, the two further quantities

$$U = (I_l - I_r) \tan 2\chi \quad \text{and} \quad V = (I_l - I_r) \tan 2\beta \sec 2\chi,$$

where  $\chi$  denotes the inclination of the plane of polarization to the direction to which  $l$  refers and  $-\pi/2 \leq \beta \leq +\pi/2$  is an angle, the tangent of which is equal to the ratio of axes of the ellipse characterizing the state of polarization. (The sign of  $\beta$  depends on whether the polarization is right handed or left handed.)

It is found that the equations of transfer for  $I_l$ ,  $I_r$ , and  $U$  are of exactly the same forms as in cases in which only partial plane-polarization is contemplated and that the equation for  $V$  is independent of others. On Rayleigh's law,  $V$  is scattered in accordance with a phase function,  $\frac{1}{2} \cos \Theta$ .

The solution of the equation for  $V$  appropriate for the problem of diffuse reflection by a semi-infinite atmosphere is also given.

**1. Introduction.**—In earlier papers<sup>1</sup> of this series the general equations of transfer valid for an atmosphere scattering radiation in accordance with Rayleigh's law<sup>2</sup> and allowing for a partial plane-polarization of the radiation field have been formulated and solved for the case of an electron-scattering atmosphere and for the problem of diffuse reflection of a partially plane-polarized beam by a plane-parallel atmosphere. In this paper we shall extend this discussion to include the case of partial elliptic polarization of the radiation field.

**2. The parametric representation of partially elliptically polarized light and its resolution into two oppositely polarized streams. The composition of elliptically polarized streams with no mutual phase relationships.**—The most convenient characterization of an arbitrarily polarized light is due to Sir George Stokes,<sup>3</sup> and his representation with slight modifications will be used in this paper. However, in view of the general inaccessibility of Stokes's considerations, it may be useful to have them presented in a form suitable for our purposes.

Consider, first, an elliptically polarized beam with the plane of polarization<sup>4</sup> inclined at an angle  $\chi$  to a certain fixed direction,<sup>5</sup>  $l$  (say). Let  $\beta$  denote the angle whose tangent is equal to the ratio of the axes of the ellipse traced by the end-point of the electric vector, the numerical value of  $\beta$  being supposed to lie between the limits 0 and  $\pi/2$  and

<sup>1</sup> *Ap. J.*, 103, 351, 1946; 104, 110, 1946; and 105, 151, 164, 1947. These papers will be referred to as "Papers X, XI, XIII, and XIV," respectively. We shall also have occasion to refer to Paper IX (*Ap. J.* 103, 165, 1946).

<sup>2</sup> As in Papers XI, XIII, and XIV, we shall include under "Rayleigh's law" only that part of it which pertains to the state of polarization and the angular distribution of the scattered radiation.

<sup>3</sup> *Trans. Cambridge Phil. Soc.*, 9, 399, 1852; or *Mathematical and Physical Papers of Sir George Stokes* (Cambridge, England, 1901), 3, 233-51.

<sup>4</sup> We shall take this to coincide with the plane of vibration of the electric vector.

<sup>5</sup> These directions are referred in the plane containing the electric and the magnetic vectors.

the sign of  $\beta$  being positive or negative according to whether the polarization is right handed or left handed. Finally, let  $\xi^{(0)}$  denote a quantity proportional to the mean amplitude of the electric vector, whose square is equal to the intensity of the beam:

$$I = [\xi^{(0)}]^2. \quad (1)$$

Then, for the elliptically polarized beam with the specified characteristics, the amplitudes  $\xi_x$  and  $\xi_{x+\pi/2}$  in the directions  $x$  and  $x + \pi/2$  can be represented in the forms

$$\xi_x = \xi^{(0)} \cos \beta \sin \omega t \quad \text{and} \quad \xi_{x+\pi/2} = \xi^{(0)} \sin \beta \cos \omega t, \quad (2)$$

where  $\omega$  denotes the circular frequency of the light considered.

From equations (2) it follows that the amplitudes  $\xi_l$  and  $\xi_r$  in the direction  $l$  and in the direction  $r$  at right angles to  $l$  are

$$\xi_l = \xi^{(0)} (\cos \beta \cos \chi \sin \omega t - \sin \beta \sin \chi \cos \omega t) \quad (3)$$

and

$$\xi_r = \xi^{(0)} (\cos \beta \sin \chi \sin \omega t + \sin \beta \cos \chi \cos \omega t). \quad (4)$$

We can re-write the foregoing expressions for  $\xi_l$  and  $\xi_r$  in the forms

$$\xi_l = \xi_l^{(0)} \sin (\omega t - \epsilon_1) \quad \text{and} \quad \xi_r = \xi_r^{(0)} \sin (\omega t - \epsilon_2), \quad (5)$$

where

$$\left. \begin{aligned} \xi_l^{(0)} &= \xi^{(0)} (\cos^2 \beta \cos^2 \chi + \sin^2 \beta \sin^2 \chi)^{1/2}, \\ \xi_r^{(0)} &= \xi^{(0)} (\cos^2 \beta \sin^2 \chi + \sin^2 \beta \cos^2 \chi)^{1/2}, \end{aligned} \right\} \quad (6)$$

and

$$\tan \epsilon_1 = \tan \chi \tan \beta; \quad \tan \epsilon_2 = -\cot \chi \tan \beta. \quad (7)$$

The intensities  $I_l$  and  $I_r$  in the directions  $l$  and  $r$  are therefore given by

$$I_l = [\xi_l^{(0)}]^2 = I (\cos^2 \beta \cos^2 \chi + \sin^2 \beta \sin^2 \chi) \quad (8)$$

and

$$I_r = [\xi_r^{(0)}]^2 = I (\cos^2 \beta \sin^2 \chi + \sin^2 \beta \cos^2 \chi). \quad (9)$$

Furthermore, from equations (6) and (7) it follows that

$$2 \xi_l^{(0)} \xi_r^{(0)} \cos (\epsilon_1 - \epsilon_2) = I \cos 2\beta \sin 2\chi \quad (10)$$

and

$$2 \xi_l^{(0)} \xi_r^{(0)} \sin (\epsilon_1 - \epsilon_2) = I \sin 2\beta. \quad (11)$$

Thus, whenever the amplitudes of an elliptically polarized beam in two directions at right angles to each other can be expressed in the form (5), we can at once write the relations

$$I = I_l + I_r = [\xi_l^{(0)}]^2 + [\xi_r^{(0)}]^2, \quad (12)$$

$$Q = I_l - I_r = I \cos 2\beta \cos 2\chi = [\xi_l^{(0)}]^2 - [\xi_r^{(0)}]^2, \quad (13)$$

$$U = (I_l - I_r) \tan 2\chi = I \cos 2\beta \sin 2\chi = 2 \xi_l^{(0)} \xi_r^{(0)} \cos \delta, \quad (14)$$

and

$$V = (I_l - I_r) \tan 2\beta \sec 2\chi = I \sin 2\beta = 2 \xi_l^{(0)} \xi_r^{(0)} \sin \delta, \quad (15)$$

where

$$\delta = \epsilon_1 - \epsilon_2 \quad (16)$$

denotes the difference in phase with which the components of the electric vector vibrate in the directions  $l$  and  $r$ .

It will be observed that, according to the definitions of the quantities  $Q$ ,  $U$ , and  $V$  for an elliptically polarized beam,

$$I = (Q^2 + U^2 + V^2)^{1/2}. \quad (17)$$

Now Stokes has shown that any elliptically polarized beam (and therefore, as it will appear, *any* partially polarized beam) can be resolved into two other elliptically polarized beams of specified states of polarization. Thus, a beam of intensity  $I$  polarized in the direction  $\chi$  and of an ellipticity corresponding to  $\beta$  can be expressed as the result of superposition of two beams polarized in the directions  $\chi_1$  and  $\chi_2$  and with ellipticities corresponding to  $\beta_1$  and  $\beta_2$ , and of intensities

$$I_1 = \frac{\sin^2(\beta_2 - \beta) \cos^2(\chi_2 - \chi) + \cos^2(\beta_2 + \beta) \sin^2(\chi_2 - \chi)}{\sin^2(\beta_2 - \beta_1) \cos^2(\chi_2 - \chi_1) + \cos^2(\beta_2 + \beta_1) \sin^2(\chi_2 - \chi_1)} I \quad (18)$$

and

$$I_2 = \frac{\sin^2(\beta_1 - \beta) \cos^2(\chi_1 - \chi) + \cos^2(\beta_1 + \beta) \sin^2(\chi_1 - \chi)}{\sin^2(\beta_2 - \beta_1) \cos^2(\chi_2 - \chi_1) + \cos^2(\beta_2 + \beta_1) \sin^2(\chi_2 - \chi_1)} I, \quad (19)$$

respectively. Of these various modes of resolution, the one of greatest interest is the resolution into *oppositely polarized beams* when

$$\beta_2 = -\beta_1 \quad \text{and} \quad \chi_2 = \chi_1 + \frac{\pi}{2}, \quad (20)$$

i.e., when the ellipses described are similar, their major axes perpendicular to each other, and the direction of revolution of one contrary to that in the other. The importance of this concept of opposite polarization arises from the fact that *oppositely polarized beams cannot interfere with one another*.

For the case of resolution into oppositely polarized beams, equations (18) and (19) become

$$I_1 = I [\sin^2(\beta_1 + \beta) \sin^2(\chi_1 - \chi) + \cos^2(\beta_1 - \beta) \cos^2(\chi_1 - \chi)] \quad (21)$$

and

$$I_2 = I [\sin^2(\beta_1 - \beta) \cos^2(\chi_1 - \chi) + \cos^2(\beta_1 + \beta) \sin^2(\chi_1 - \chi)]. \quad (22)$$

When  $\beta_1 = 0$ , we have the resolution of the beam into plane-polarized components at right angles to each other, and equations (21) and (22) become equivalent to equations (8) and (9).

The expression (21) for  $I_1$  can be expanded into the form

$$I_1 = \frac{1}{2} [I + I \cos 2\beta \cos 2\chi \cos 2\beta_1 \cos 2\chi_1 + I \cos 2\beta \sin 2\chi \cos 2\beta_1 \sin 2\chi_1 + I \sin 2\beta \cos 2\chi \cos 2\beta_1 \sin 2\chi_1 + I \sin 2\beta \sin 2\chi \cos 2\beta_1 \cos 2\chi_1 + I \cos 2\beta \cos 2\chi \sin 2\beta_1 \sin 2\chi_1 + I \sin 2\beta \sin 2\chi \sin 2\beta_1 \cos 2\chi_1 + I \cos 2\beta \sin 2\chi \sin 2\beta_1 \cos 2\chi_1 + I \sin 2\beta \cos 2\chi \sin 2\beta_1 \sin 2\chi_1 + I \sin 2\beta \sin 2\chi \sin 2\beta_1 \sin 2\chi_1] \quad (23)$$

or, in terms of the quantities  $Q$ ,  $U$ , and  $V$  defined as in equations (13)–(15), we have

$$I_1 = \frac{1}{2} [I + Q \cos 2\beta_1 \cos 2\chi_1 + U \cos 2\beta_1 \sin 2\chi_1 + V \sin 2\beta_1]. \quad (24)$$

The corresponding expression for  $I_2$  can be obtained by simply changing the sign of  $\beta_1$  and replacing  $\chi_1$  by  $\chi_1 + \pi/2$ . We have

$$I_2 = \frac{1}{2} [I - Q \cos 2\beta_1 \cos 2\chi_1 - U \cos 2\beta_1 \sin 2\chi_1 - V \sin 2\beta_1]. \quad (25)$$

Turning, next, to the consideration of the result of the superposition of a number of *independent* streams of elliptically polarized light, i.e., of polarized streams which have no phase relationships, we start with the following principle enunciated by Stokes: "When any number of polarized streams from different sources mix together, after having been variously modified by reflection, refraction, transmission through doubly refracting media, tourmalines, etc., the intensity of the mixture is equal to the sum of the intensities due to the separate streams."

From this principle it follows, in particular, that, if each of the separate streams is resolved into two states of opposite polarization, the resultant mixture will have intensities in the two states which are the sum of respective intensities of the separate streams. Thus, if  $I^{(n)}$ ,  $\chi^{(n)}$ , and  $\beta^{(n)}$  define the intensity, the plane of polarization, and the ellipticity of a typical stream and if  $I_1^{(n)}$  and  $I_2^{(n)}$  are its intensities in the states of polarization  $(\beta_1, \chi_1)$  and  $(-\beta_1, \chi_1 + \pi/2)$ , respectively, then

$$I_1 = \Sigma I_1^{(n)} \quad \text{and} \quad I_2 = \Sigma I_2^{(n)} \quad (26)$$

where  $I_1$  and  $I_2$  refer to the mixture.

Using equation (24) to express the intensities of the various streams in the component  $(\beta_1, \chi_1)$ , we have

$$I_1 = \frac{1}{2} [\Sigma I^{(n)} + \Sigma Q^{(n)} \cos 2\beta_1 \cos 2\chi_1 + \Sigma U^{(n)} \cos 2\beta_1 \sin 2\chi_1 + \Sigma V^{(n)} \sin 2\beta_1]. \quad (27)$$

The intensity of the resultant mixture in the state of polarization  $(\chi_1, \beta_1)$  can therefore be expressed in the form

$$I_1 = \frac{1}{2} [I + Q \cos 2\beta_1 \cos 2\chi_1 + U \cos 2\beta_1 \sin 2\chi_1 + V \sin 2\beta_1], \quad (28)$$

where

$$I = \Sigma I^{(n)}; \quad Q = \Sigma Q^{(n)}; \quad U = \Sigma U^{(n)}; \quad V = \Sigma V^{(n)}. \quad (29)$$

We have, of course, a similar expression for  $I_2$ .

From equations (28) and (29) it follows that a beam resulting from the superposition of a number of independent streams of elliptically polarized light can again be characterized by a set of parameters,  $I$ ,  $Q$ ,  $U$ , and  $V$ , which are the sums of the respective parameters characterizing the individual streams. Moreover, any two polarized beams characterized by the same set of parameters,  $I$ ,  $Q$ ,  $U$ , and  $V$ , will be *optically equivalent* in the sense that "they will present exactly the same appearance on being viewed through a crystal followed by a Nicol's prism or other analyzer" (Stokes). On the other hand, it should be noted that, for a polarized beam obtained in this manner, the relation (17) (derived for an elliptically polarized beam) will not, in general, be valid, indicating the fact that the mixture of a number of independent streams of elliptically polarized light will, in general, lead to a beam which is only partially polarized. However, it is clear that, under the circumstances of partial polarization, we can regard the light as a mixture of an unpolarized beam of natural light, of intensity

$$I^{(u)} = I - (Q^2 + U^2 + V^2)^{1/2}, \quad (30)$$

and a polarized beam, of intensity

$$I^{(p)} = (Q^2 + U^2 + V^2)^{1/2}, \quad (31)$$

the plane of polarization and the ratio of the axes of the ellipse of this polarized part being given by

$$\tan 2\chi = \frac{U}{Q} \quad \text{and} \quad \sin 2\beta = \frac{V}{(Q^2 + U^2 + V^2)^{1/2}}. \quad (32)$$

An alternative way of regarding a partially polarized beam, defined in terms of the parameters  $I$ ,  $Q$ ,  $U$ , and  $V$ , is to express it as the resultant of two streams in the states of

opposite polarization  $(\beta, \chi)$  and  $(-\beta, \chi + \pi/2)$ , where  $\chi$  and  $\beta$  are given by equations (32). The intensities of the component streams in the two states can be readily written down when it is remembered that unpolarized, or natural, light is equivalent to any two independent oppositely polarized streams of half the intensity. The unpolarized part (30) of the beam is, therefore, equivalent to a mixture of two independent polarized beams, each of intensity  $\frac{1}{2}I^{(u)}$ , in the states of polarization  $(\beta, \chi)$  and  $(-\beta, \chi + \pi/2)$ . Combining the former with the polarized part  $I^{(p)}$  (eq. [31]) in the same state of polarization, we conclude that a beam characterized by the parameters  $I$ ,  $Q$ ,  $U$ , and  $V$  is equivalent to two independent beams of elliptically polarized light, of intensities

$$I^{(+)} = \frac{1}{2} [I + (Q^2 + U^2 + V^2)^{1/2}] \quad (33)$$

and

$$I^{(-)} = \frac{1}{2} [I - (Q^2 + U^2 + V^2)^{1/2}] , \quad (34)$$

in the states of polarization

$$(\beta, \chi) \quad \text{and} \quad \left(-\beta, \chi + \frac{\pi}{2}\right) , \quad (35)$$

respectively, where

$$\chi = \frac{1}{2} \tan^{-1} \frac{U}{Q} \quad \text{and} \quad \beta = \frac{1}{2} \sin^{-1} \frac{V}{(Q^2 + U^2 + V^2)^{1/2}} . \quad (36)$$

The particular importance of this resolution arises from the fact that we may regard the two streams  $I^{(+)}$  and  $I^{(-)}$  as *independent*.

3. *The source functions  $\mathfrak{J}_l$ ,  $\mathfrak{J}_r$ ,  $\mathfrak{J}_U$ , and  $\mathfrak{J}_V$ .*—From the discussion of Stokes's representation of partially polarized light in the preceding section, it follows that, for the purposes of formulating the equation of transfer, we may uniquely characterize the radiation field at any given point by the four intensities  $I_l(\vartheta, \varphi)$ ,  $I_r(\vartheta, \varphi)$ ,  $U(\vartheta, \varphi)$ , and  $V(\vartheta, \varphi)$ , where  $\vartheta$  and  $\varphi$  are the polar angles, referred to an appropriately chosen co-ordinate system through the point under consideration (see Fig. 1 in Paper X), and  $l$  and  $r$  refer to the directions in the meridian plane and at right angles to it, respectively.

To evaluate the source functions,  $\mathfrak{J}_l(\vartheta, \varphi)$ ,  $\mathfrak{J}_r(\vartheta, \varphi)$ ,  $\mathfrak{J}_U(\vartheta, \varphi)$ , and  $\mathfrak{J}_V(\vartheta, \varphi)$ , appropriate to any point in the atmosphere, we shall first consider the contributions to these source functions arising from the scattering of the radiation in the direction  $(\vartheta', \varphi')$ .

The radiation in the direction  $(\vartheta', \varphi')$  will be characterized by the intensities  $I_l(\vartheta', \varphi')$ ,  $I_r(\vartheta', \varphi')$ ,  $U(\vartheta', \varphi')$ , and  $V(\vartheta', \varphi')$ . Equivalently, we may also characterize it by the intensities

$$I^{(+)}(\vartheta', \varphi') = \frac{1}{2} \{I_l + I_r + [(I_l - I_r)^2 + U^2 + V^2]^{1/2}\}_{\vartheta', \varphi'} \quad (37)$$

and

$$I^{(-)}(\vartheta', \varphi') = \frac{1}{2} \{I_l + I_r - [(I_l - I_r)^2 + U^2 + V^2]^{1/2}\}_{\vartheta', \varphi'} , \quad (38)$$

in the states of opposite polarization,

$$(\beta, \chi) \quad \text{and} \quad \left(-\beta, \chi + \frac{\pi}{2}\right) , \quad (39)$$

respectively, where

$$\left. \begin{aligned} \tan 2\chi &= \frac{U(\vartheta', \varphi')}{I_l(\vartheta', \varphi') - I_r(\vartheta', \varphi')} \\ \text{and} \quad \sin 2\beta &= \frac{V(\vartheta', \varphi')}{\sqrt{[I_l(\vartheta', \varphi') - I_r(\vartheta', \varphi')]^2 + U^2(\vartheta', \varphi') + V^2(\vartheta', \varphi')}} . \end{aligned} \right\} (40)$$

And, as we have already explained in § 2, in this type of resolution the polarized components  $I^{(+)}(\vartheta', \varphi')$  and  $I^{(-)}(\vartheta', \varphi')$  may be regarded as uncorrelated in their phases.

We may accordingly consider the scattering of the radiation in the direction  $(\vartheta', \varphi')$  as the result of scattering of the two independent, oppositely polarized components,  $I^{(+)}(\vartheta', \varphi')$  and  $I^{(-)}(\vartheta', \varphi')$ . Each of these polarized components will give rise to polarized scattered beams, which will again have no correlation in phases. In other words, the radiation scattered in the direction  $(\vartheta, \varphi)$  from other directions can be considered as a mixture of a very large number of independent, elliptically polarized streams and can, therefore, be combined according to the additive law of composition of the parameters  $I_l$ ,  $I_r$ ,  $U$ , and  $V$ .

Consider, then, the scattering of the elliptically polarized component  $I^{(+)}(\vartheta', \varphi')$  in the direction  $(\vartheta', \varphi')$  and confined to an element of solid angle,  $d\omega'$ , into the direction  $(\vartheta, \varphi)$ . Introducing the qualities,  $\xi$ 's, as in § 2, we may express the amplitudes  $\xi_x^{(+)}$  and  $\xi_{x+\pi/2}^{(+)}$  in the forms

$$\xi_x^{(+)} = \xi^{(+, 0)} \cos \beta \sin \omega t \quad \text{and} \quad \xi_{x+\pi/2}^{(+)} = \xi^{(+, 0)} \sin \beta \cos \omega t, \quad (41)$$

where  $\xi^{(+, 0)}$  is so defined that (cf. eq. [1])

$$I^{(+)}(\vartheta', \varphi') = [\xi^{(+, 0)}]^2. \quad (42)$$

The amplitude,  $\xi_x^{(+)}$ , when scattered in the direction  $(\vartheta, \varphi)$  will lead to amplitudes in the meridian plane through  $(\vartheta, \varphi)$  and at right angles to it which are proportional, respectively, to (cf. Paper XI, eqs. [31] and [32])

$$\xi_x^{(+)} [(l, l) \cos \chi + (r, l) \sin \chi] \quad (43)$$

and

$$\xi_x^{(+)} [(l, r) \cos \chi + (r, r) \sin \chi], \quad (44)$$

where (cf. Paper XI, eq. [33]),

$$\left. \begin{aligned} (l, l) &= \sin \vartheta \sin \vartheta' + \cos \vartheta \cos \vartheta' \cos (\varphi' - \varphi), \\ (r, l) &= +\cos \vartheta \sin (\varphi' - \varphi), \\ (l, r) &= -\cos \vartheta' \sin (\varphi' - \varphi), \\ (r, r) &= \cos (\varphi' - \varphi). \end{aligned} \right\} \quad (45)$$

Similarly, the amplitude  $\xi_{x+\pi/2}^{(+)}$  will lead to scattered amplitudes in the directions, parallel, respectively, perpendicular to the meridian plane through  $(\vartheta, \varphi)$  of amounts proportional to

$$\xi_{x+\pi/2}^{(+)} [-(l, l) \sin \chi + (r, l) \cos \chi] \quad (46)$$

and

$$\xi_{x+\pi/2}^{(+)} [-(l, r) \sin \chi + (r, r) \cos \chi]. \quad (47)$$

The phase relationship between  $\xi_x^{(+)}$  and  $\xi_{x+\pi/2}^{(+)}$  will be maintained in these scattered amplitudes and must, therefore, be added as amplitudes with the correct phase differences. Therefore, the elliptically polarized component,  $I^{(+)}(\vartheta', \varphi')$ , of the radiation in the direction  $(\vartheta', \varphi')$ , when scattered in the direction  $(\vartheta, \varphi)$ , will give rise to an elliptically polarized beam, the amplitudes of which in the meridian plane and at right angles to it will be proportional, respectively, to

$$\xi_l^{(s)} = \xi^{(+, 0)} (A_x \cos \beta \sin \omega t + A_{x+\pi/2} \sin \beta \cos \omega t) \quad (48)$$

and

$$\xi_r^{(s)} = \xi^{(+, 0)} (B_x \cos \beta \sin \omega t + B_{x+\pi/2} \sin \beta \cos \omega t), \quad (49)$$

where, for the sake of brevity, we have written

$$\left. \begin{aligned} A_x &= + (l, l) \cos \chi + (r, l) \sin \chi, \\ B_x &= + (l, r) \cos \chi + (r, r) \sin \chi, \\ A_{x+\pi/2} &= - (l, l) \sin \chi + (r, l) \cos \chi, \\ B_{x+\pi/2} &= - (l, r) \sin \chi + (r, r) \cos \chi. \end{aligned} \right\} \quad (50)$$

We can now re-write equations (48) and (49) in the standard form (cf. eq. [5]),

$$\left. \begin{aligned} \xi_l^{(s)} &= \xi_l^{(s, 0)} \sin (\omega t - \epsilon_1), \\ \xi_r^{(s)} &= \xi_r^{(s, 0)} \sin (\omega t - \epsilon_2), \end{aligned} \right\} \quad (51)$$

where

$$\left. \begin{aligned} \xi_l^{(s, 0)} &= \xi^{(+, 0)} (A_x^2 \cos^2 \beta + A_{x+\pi/2}^2 \sin^2 \beta)^{1/2}, \\ \xi_r^{(s, 0)} &= \xi^{(+, 0)} (B_x^2 \cos^2 \beta + B_{x+\pi/2}^2 \sin^2 \beta)^{1/2}, \end{aligned} \right\} \quad (52)$$

and

$$\tan \epsilon_1 = - \frac{A_{x+\pi/2}}{A_x} \tan \beta; \quad \tan \epsilon_2 = - \frac{B_{x+\pi/2}}{B_x} \tan \beta. \quad (53)$$

With the amplitudes of the scattered radiation expressed in the form (51), we can write down the contributions,  $d\mathfrak{J}_i^{(+)}(\vartheta, \varphi; \vartheta', \varphi')$ ,  $d\mathfrak{J}_r^{(+)}(\vartheta, \varphi; \vartheta', \varphi')$ ,  $d\mathfrak{J}_U^{(+)}(\vartheta, \varphi; \vartheta', \varphi')$ , and  $d\mathfrak{J}_V^{(+)}(\vartheta, \varphi; \vartheta', \varphi')$ , to the source functions,  $\mathfrak{J}_i(\vartheta, \varphi)$ ,  $\mathfrak{J}_r(\vartheta, \varphi)$ ,  $\mathfrak{J}_U(\vartheta, \varphi)$ , and  $\mathfrak{J}_V(\vartheta, \varphi)$ , arising from the scattering of the elliptically polarized component,  $I^{(+)}(\vartheta', \varphi')$ , of the radiation in the direction  $(\vartheta', \varphi')$  and confined to an element of solid angle,  $d\omega'$ , in the following forms (cf. eqs. [8]–[15]):

$$\left. \begin{aligned} d\mathfrak{J}_i^{(+)}(\vartheta, \varphi; \vartheta', \varphi') &= \frac{3}{8\pi} [\xi_l^{(s, 0)}]^2 d\omega', \\ d\mathfrak{J}_r^{(+)}(\vartheta, \varphi; \vartheta', \varphi') &= \frac{3}{8\pi} [\xi_r^{(s, 0)}]^2 d\omega', \\ d\mathfrak{J}_U^{(+)}(\vartheta, \varphi; \vartheta', \varphi') &= \frac{3}{8\pi} [2 \xi_l^{(s, 0)} \xi_r^{(s, 0)} \cos(\epsilon_1 - \epsilon_2)] d\omega', \\ d\mathfrak{J}_V^{(+)}(\vartheta, \varphi; \vartheta', \varphi') &= \frac{3}{8\pi} [2 \xi_l^{(s, 0)} \xi_r^{(s, 0)} \sin(\epsilon_1 - \epsilon_2)] d\omega'. \end{aligned} \right\} \quad (54)$$

The various quantities which occur on the right-hand sides of the foregoing expressions can be evaluated according to equations (50)–(53). Thus

$$\left. \begin{aligned} [\xi_l^{(s, 0)}]^2 &= [\xi^{(+, 0)}]^2 (A_x^2 \cos^2 \beta + A_{x+\pi/2}^2 \sin^2 \beta) \\ &= I^{(+)}(\vartheta', \varphi') \{ (l, l)^2 [\cos^2 \chi \cos^2 \beta + \sin^2 \chi \sin^2 \beta] \\ &\quad + (r, l)^2 [\sin^2 \chi \cos^2 \beta + \cos^2 \chi \sin^2 \beta] + (l, l)(r, l) \sin 2\chi \cos 2\beta \}; \end{aligned} \right\} \quad (55)$$

or, using equations (8), (9), and (14), we have

$$[\xi_l^{(s, 0)}]^2 = (l, l)^2 I_l^{(+)}(\vartheta', \varphi') + (r, l)^2 I_r^{(+)}(\vartheta', \varphi') + (l, l)(r, l) U^{(+)}(\vartheta', \varphi'). \quad (56)$$

Similarly,

$$[\xi_r^{(s, 0)}]^2 = (l, r)^2 I_l^{(+)}(\vartheta', \varphi') + (r, r)^2 I_r^{(+)}(\vartheta', \varphi') + (l, r)(r, r) U^{(+)}(\vartheta', \varphi'). \quad (57)$$

Again, according to equations (50), (52), and (53),

$$\left. \begin{aligned} 2 \xi_l^{(s, 0)} \xi_r^{(s, 0)} \cos(\epsilon_1 - \epsilon_2) &= 2 I^{(+)}(\vartheta', \varphi') (A_x B_x \cos^2 \beta + A_{x+\pi/2} B_{x+\pi/2} \sin^2 \beta) \\ &= 2 I^{(+)}(\vartheta', \varphi') \{ (l, l)(l, r) [\cos^2 x \cos^2 \beta + \sin^2 x \sin^2 \beta] \\ &\quad + (r, l)(r, r) [\sin^2 x \cos^2 \beta + \cos^2 x \sin^2 \beta] \\ &\quad + [(l, l)(r, r) + (r, l)(l, r)] \cos x \sin x \cos 2\beta \} \end{aligned} \right\} \quad (58)$$

or

$$\left. \begin{aligned} 2 \xi_l^{(s, 0)} \xi_r^{(s, 0)} \cos(\epsilon_1 - \epsilon_2) &= 2 (l, l)(l, r) I_l^{(+)}(\vartheta', \varphi') + 2 (r, l)(r, r) I_r^{(+)}(\vartheta', \varphi') \\ &\quad + [(l, l)(r, r) + (r, l)(l, r)] U^{(+)}(\vartheta', \varphi') \end{aligned} \right\} \quad (59)$$

And, finally,

$$\left. \begin{aligned} 2 \xi_l^{(s, 0)} \xi_r^{(s, 0)} \sin(\epsilon_1 - \epsilon_2) &= I^{(+)}(\vartheta', \varphi') (A_x B_{x+\pi/2} - B_x A_{x+\pi/2}) \sin 2\beta \\ &= I^{(+)}(\vartheta', \varphi') [(l, l)(r, r) - (r, l)(l, r)] \sin 2\beta \end{aligned} \right\} \quad (60)$$

or

$$2 \xi_l^{(s, 0)} \xi_r^{(s, 0)} \sin(\epsilon_1 - \epsilon_2) = V^{(+)}(\vartheta', \varphi') [(l, l)(r, r) - (r, l)(l, r)]. \quad (61)$$

Equations (54) now become

$$d \mathfrak{J}_l^{(+)}(\vartheta, \varphi; \vartheta', \varphi') = \frac{3}{8\pi} \{ (l, l)^2 I_l^{(+)}(\vartheta', \varphi') + (r, l)^2 I_r^{(+)}(\vartheta', \varphi') \\ + (l, l)(r, l) U^{(+)}(\vartheta', \varphi') \} d\omega', \quad (62)$$

$$d \mathfrak{J}_r^{(+)}(\vartheta, \varphi; \vartheta', \varphi') = \frac{3}{8\pi} \{ (l, r)^2 I_l^{(+)}(\vartheta', \varphi') + (r, r)^2 I_r^{(+)}(\vartheta', \varphi') \\ + (l, r)(r, r) U^{(+)}(\vartheta', \varphi') \} d\omega', \quad (63)$$

$$d \mathfrak{J}_U^{(+)}(\vartheta, \varphi; \vartheta', \varphi') = \frac{3}{8\pi} \{ 2(l, l)(l, r) I_l^{(+)}(\vartheta', \varphi') + 2(r, l)(r, r) I_r^{(+)}(\vartheta', \varphi') \\ + [(l, l)(r, r) + (r, l)(l, r)] U^{(+)}(\vartheta', \varphi') \} d\omega', \quad (64)$$

and

$$d \mathfrak{J}_V^{(+)}(\vartheta, \varphi; \vartheta', \varphi') = \frac{3}{8\pi} [(l, l)(r, r) - (r, l)(l, r)] V^{(+)}(\vartheta', \varphi') d\omega'. \quad (65)$$

The corresponding contributions to the various source functions arising from the scattering of the other polarized component,  $I^{(-)}(\vartheta', \varphi')$ , in the opposite state of polarization,  $(-\beta, x + \pi/2)$ , can be obtained by simply writing  $(-)$  in place of  $(+)$  in the foregoing equations. And, since the intensities  $I_l$ ,  $I_r$ ,  $U$ , and  $V$  are simply additive when streams of polarized light with no correlation in their phases are mixed, it is clear that the contributions to the various source functions arising from the scattering of the radiation [ $I_l(\vartheta', \varphi')$ ,  $I_r(\vartheta', \varphi')$ ,  $U(\vartheta', \varphi')$ ,  $V(\vartheta', \varphi')$ ] in the direction  $(\vartheta', \varphi')$  and confined to an element of solid angle,  $d\omega'$ , are

$$d \mathfrak{J}_l(\vartheta, \varphi; \vartheta', \varphi') = \frac{3}{8\pi} \{ (l, l)^2 I_l(\vartheta', \varphi') + (r, l)^2 I_r(\vartheta', \varphi') \\ + (l, l)(r, l) U(\vartheta', \varphi') \} d\omega', \quad (66)$$

$$d \mathfrak{J}_r(\vartheta, \varphi; \vartheta', \varphi') = \frac{3}{8\pi} \{ (l, r)^2 I_l(\vartheta', \varphi') + (r, r)^2 I_r(\vartheta', \varphi') \\ + (l, r)(r, r) U(\vartheta', \varphi') \} d\omega', \quad (67)$$

$$d \mathfrak{J}_U(\vartheta, \varphi; \vartheta', \varphi') = \frac{3}{8\pi} \{ 2(l, l)(l, r) I_l(\vartheta', \varphi') + 2(r, l)(r, r) I_r(\vartheta', \varphi') \\ + [(l, l)(r, r) + (r, l)(l, r)] U(\vartheta', \varphi') \} d\omega', \quad (68)$$

and

$$d\mathfrak{J}_V(\vartheta, \varphi; \vartheta', \varphi') = \frac{3}{8\pi} [ (l, l)(r, r) - (r, l)(l, r) ] V(\vartheta', \varphi') d\omega'. \quad (69)$$

Integrating these equations over the unit sphere, we obtain the required source functions. We have

$$\mathfrak{J}_l(\mu, \varphi) = \frac{3}{8\pi} \int_{-1}^{+1} \int_0^{2\pi} \left\{ (l, l)^2 I_l(\mu', \varphi') + (r, l)^2 I_r(\mu', \varphi') + (l, l)(r, l) U(\mu', \varphi') \right\} d\mu' d\varphi', \quad (70)$$

$$\mathfrak{J}_r(\mu, \varphi) = \frac{3}{8\pi} \int_{-1}^{+1} \int_0^{2\pi} \left\{ (l, r)^2 I_l(\mu', \varphi') + (r, r)^2 I_r(\mu', \varphi') + (l, r)(r, r) U(\mu', \varphi') \right\} d\mu' d\varphi', \quad (71)$$

$$\mathfrak{J}_U(\mu, \varphi) = \frac{3}{8\pi} \int_{-1}^{+1} \int_0^{2\pi} \left\{ 2(l, l)(l, r) I_l(\mu', \varphi') + 2(r, l)(r, r) I_r(\mu', \varphi') + [(l, l)(r, r) + (r, l)(l, r)] U(\mu', \varphi') \right\} d\mu' d\varphi', \quad (72)$$

and

$$\mathfrak{J}_V(\mu, \varphi) = \frac{3}{8\pi} \int_{-1}^{+1} \int_0^{2\pi} [ (l, l)(r, r) - (r, l)(l, r) ] V(\mu', \varphi') d\mu' d\varphi', \quad (73)$$

where the direction cosines,  $\mu$  and  $\mu'$ , have been used in place of  $\cos \vartheta$  and  $\cos \vartheta'$ .

Comparing equations (70)–(72) with the corresponding equations obtained in Paper XI (eqs. [49]–[51]) for a partially plane-polarized radiation field, we observe that they are of identical forms. The intensities  $I_l$ ,  $I_r$ , and  $U$ , therefore, satisfy the same equations of transfer as in cases in which only partial plane-polarization is contemplated. The equations of transfer for  $I_l$ ,  $I_r$ , and  $U$  can therefore be expressed quite generally in vector form as in Paper XIV, § 2. However, when elliptic polarization is contemplated,<sup>6</sup> an additional equation for  $V$  must be considered. This equation is, however, independent of the others and is given by (cf. eq. [73])

$$\frac{dV(s, \mu, \varphi)}{\rho \sigma ds} = -V(s, \mu, \varphi) + \frac{3}{8\pi} \int_{-1}^{+1} \int_0^{2\pi} [ (l, l)(r, r) - (r, l)(l, r) ] \times V(s, \mu', \varphi') d\mu' d\varphi', \quad (74)$$

where  $s$  measures the linear distance in the direction  $(\vartheta, \varphi)$  and  $\sigma$  denotes the mass-scattering coefficient.

Finally, we may notice that, according to equations (45),

$$(l, l)(r, r) - (r, l)(l, r) = \mu\mu' + (1 - \mu^2)(1 - \mu'^2)^{1/2} \cos(\varphi - \varphi'). \quad (75)$$

The intensity  $V$  is therefore scattered in accordance with a phase function  $\frac{3}{2} \cos \Theta$ .

4. The solution of the equation of transfer for  $V$  for the problem of diffuse reflection by a semi-infinite plane-parallel atmosphere.—In the problem of diffuse reflection we distinguish, as usual, between the part of the incident radiation which penetrates to various depths and the diffuse scattered radiation. Similarly, we also distinguish between the contributions to the source function arising from the scattering of the reduced incident radiation prevailing at any level and from the scattering of the diffuse radiation.

If  $\pi V_0$  denotes the flux in  $V$  incident as a parallel beam on a plane-parallel atmosphere

<sup>6</sup> It is, of course, evident—and it is, indeed, required by the equations of transfer—that, on Rayleigh scattering, partial elliptic polarization in the radiation field can be induced only by the direct incidence of such radiation.

at an angle  $\cos^{-1} \mu_0$  normal to the boundary, the equation of transfer (74) can be re-written in the following form:

$$\left. \begin{aligned} \mu \frac{dV(\tau, \mu, \varphi)}{d\tau} &= V(\tau, \mu, \varphi) \\ -\frac{3}{8\pi} \int_{-1}^{+1} \int_0^{2\pi} [\mu\mu' + (1 - \mu^2)^{1/2}(1 - \mu'^2)^{1/2} \cos(\varphi - \varphi')] V(\mu', \varphi') d\mu' d\varphi' \\ &- \frac{3}{8} V_0 [-\mu\mu_0 + (1 - \mu^2)^{1/2}(1 - \mu_0^2)^{1/2} \cos \varphi] e^{-\tau/\mu_0} \end{aligned} \right\} \quad (76)$$

From equation (76) it follows that the solution  $V(\tau, \mu, \varphi)$  must be expressible in the form

$$V(\tau, \mu, \varphi) = V^{(0)}(\tau, \mu) + V^{(1)}(\tau, \mu) \cos \varphi, \quad (77)$$

where, as the notation implies,  $V^{(0)}$  and  $V^{(1)}$  are functions of  $\tau$  and  $\mu$  only. Equation (76) now breaks up into the following two equations:

$$\mu \frac{dV^{(0)}}{d\tau} = V^{(0)} - \frac{3}{4} \mu \int_{-1}^{+1} V(\tau, \mu') \mu' d\mu' + \frac{3}{8} V_0 \mu \mu_0 e^{-\tau/\mu_0} \quad (78)$$

and

$$\left. \begin{aligned} \mu \frac{dV^{(1)}}{d\tau} &= V^{(1)} - \frac{3}{8} (1 - \mu^2)^{1/2} \int_{-1}^{+1} V^{(1)}(\tau, \mu') (1 - \mu'^2)^{1/2} d\mu' \\ &- \frac{3}{8} V_0 (1 - \mu^2)^{1/2} (1 - \mu'^2)^{1/2} e^{-\tau/\mu_0} \end{aligned} \right\} \quad (79)$$

Equations (78) and (79) are of the standard form considered in Paper IX, § 4. We can therefore write down at once the angular distribution of the radiation reflected from a semi-infinite atmosphere. We have

$$V^{(0)}(0, \mu) = -\frac{3}{8} V_0 \mu \mu_0 H_v(\mu) H_v(\mu_0) \frac{\mu_0}{\mu + \mu_0} \quad (80)$$

and

$$V^{(1)}(0, \mu) = \frac{3}{8} V_0 (1 - \mu^2)^{1/2} (1 - \mu'^2)^{1/2} H_v^{(1)}(\mu) H_v^{(1)}(\mu_0) \frac{\mu_0}{\mu + \mu_0}, \quad (81)$$

where, in the  $n$ th approximation of the method of solution of the earlier papers,  $H_v$  and  $H_v^{(1)}$  are  $H$ -functions (cf. Paper XIV) defined in terms of the roots of the characteristic equations,

$$1 = \frac{3}{2} \sum_{j=1}^n \frac{a_j \mu_j^2}{1 - k^2 \mu_j^2} \quad (82)$$

and

$$1 = \frac{3}{4} \sum_{j=1}^n \frac{a_j (1 - \mu_j^2)}{1 - k^2 \mu_j^2}, \quad (83)$$

respectively.

It will be noticed that the characteristic equation defining  $H_v^{(1)}$  is the same as the one defining  $H_r(\mu)$  in Papers X and XI. Hence  $H_v^{(1)}(\mu)$  is identical with  $H_r(\mu)$ .

Combining solutions (80) and (81) in accordance with equation (77), we have the law of reflection:

$$\left. \begin{aligned} V(\mu, \varphi; \mu_0, \varphi_0) &= \frac{3}{8} V_0 [-\mu\mu_0 H_v(\mu) H_v(\mu_0) \\ &+ (1 - \mu^2)^{1/2} (1 - \mu_0^2)^{1/2} H_r(\mu) H_r(\mu_0) \cos(\varphi - \varphi_0)] \frac{\mu_0}{\mu + \mu_0} \end{aligned} \right\} \quad (84)$$

A table of the function  $H_r(\mu)$  in the third approximation will be found in Paper XI (Table 3). A similar tabulation of the function  $H_v$  is now provided (Table 1).

TABLE 1  
THE FUNCTION  $H_v(\mu)$  IN THE THIRD APPROXIMATION

$\mu$	$H_v(\mu)$	$\mu$	$H_v(\mu)$	$\mu$	$H_v(\mu)$
0.....	1.000	0.35.....	1.107	0.70.....	1.168
0.05.....	1.021	.40.....	1.118	0.75.....	1.175
.10.....	1.039	.45.....	1.128	0.80.....	1.182
.15.....	1.055	.50.....	1.137	0.85.....	1.188
.20.....	1.070	.55.....	1.146	0.90.....	1.193
.25.....	1.084	.60.....	1.154	0.95.....	1.199
0.30.....	1.096	0.65.....	1.161	1.00.....	1.204

$$k_1 = 0.8540755; \quad k_2 = 1.3690329; \quad k_3 = 4.1105114.$$

Finally, we may remark that, according to the ideas developed in Paper XIV, the exact solution for  $V(\mu, \varphi; \mu_0, \varphi_0)$  can be obtained by simply redefining the functions  $H_v$  and  $H_r$ , which occur in equation (84) as solutions of the functional equations,

$$H_v(\mu) = 1 + \frac{3}{4} \mu H_v(\mu) \int_0^1 \frac{H_v(\mu')}{\mu + \mu'} \mu'^2 d\mu' \quad (85)$$

and

$$H_r(\mu) = 1 + \frac{3}{8} \mu H_r(\mu) \int_0^1 \frac{H_r(\mu')}{\mu + \mu'} (1 - \mu'^2) d\mu'. \quad (86)$$

Tables of solutions of these and other functional equations will be found in Paper XVI.