THE RADIATIVE EQUILIBRIUM OF AN EXPANDING PLANETARY NEBULA

I. RADIATION PRESSURE IN LYMAN-α

S. Chandrasekhar
Yerkes Observatory
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ABSTRACT

In this paper the problem of the radiative equilibrium in Lyman-α of a differentially expanding planetary nebula is formulated de novo. Particular attention has been given to the formulation of the boundary conditions. It is shown that the transfer problem reduces to a novel boundary-value problem in hyperbolic partial differential equations. Explicit solutions have been found for the case in which the line absorption coefficient \( \sigma(\nu) \) has a rectangular form. On the basis of the solutions obtained, the question of the radiation pressure in Lyman-α has been re-examined. It is shown that when the Doppler shift due to the difference in velocities between the inner and the outer boundaries of the nebula exceeds the line width \( 2\Delta \nu \) of \( \sigma(\nu) \) by a factor \( 2\sqrt{3} \), the nebula can be divided into three parts: an inner, a central, and an outer part. In the central part the radiation pressure in Lyman-α is of the same order as that in the continuum, while in the inner and the outer parts it is not inappreciable. The bearing of this manner in which the radiation pressure in Lyman-α operates in an expanding atmosphere on the dynamics of a planetary nebula and also on other astrophysical problems is briefly indicated.

1. Introduction.—As is well known on Zanstra's theory\(^1\) of nebular luminosity, the hydrogen emission in a planetary nebula is traced to the conversion in the nebula of the incident ultraviolet radiation of the central star beyond the head of the Lyman series into radiation principally in the Lyman and the Balmer series. More specifically, if the optical thickness of the nebula for the ultraviolet radiation\(^2\) is sufficiently large, this conversion will be so nearly complete that for every ultraviolet quantum in the incident starlight, the nebula will emit a quantum in the first member of the Lyman series, namely, Lyman-α. This conversion of the radiation arises from the fact that, whenever an ultraviolet quantum is scattered, there is only a probability \( p \) of order \( \frac{1}{2} \) that it will be re-emitted as such.\(^3\) And on the occasions when the ultraviolet quantum is not re-emitted, a chain of absorptions and emissions takes place which so rapidly leads to the emission of a Lyman-α quantum that we may almost say that every time an ultraviolet quantum is scattered, there is a definite probability \( (1-p) \sim \frac{1}{2} \) that a Lyman-α quantum is emitted.\(^4\) The theory of radiative transfer of the ultraviolet radiation through the nebula has been developed along these lines by Ambražumian and others\(^5\) and shows that the conversion of the ultraviolet radiation into line radiation is so far advanced through most of the nebula that, if nothing else intervened, the nebula would be subject to enormous radiation pressure because the absorption coefficient in Lyman-α is several thousand times larger than in the continuum. The magnitude of the radiation pressure which may thus act on the nebula is so large that it will seriously endanger its permanence even over relatively limited periods of time. The consequences of this radiation pressure in Lyman-α

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\(^2\) In this paper we shall use the term “ultraviolet radiation” to denote all radiation beyond the head of the Lyman series.

\(^3\) G. Cilllé, M.N., 92, 820, 1932; 96, 777, 1936.

\(^4\) Cf. V. A. Ambražumian, M.N., 93, 50, 1932; also Bull. de l’Obs. cent. à Poulkovo, 13, No. 14, 3, 1933.

acting on the nebula are, indeed, such as to lead one to re-examine whether some factor has not been overlooked, which, when properly allowed for, will cut down the radiation pressure. In this connection, Zanstra\(^6\) has suggested that the very likely presence of differential motions in the nebula may very effectively reduce the radiation pressure which may act on the nebula. Unfortunately, Zanstra's particular considerations relating to this problem are vitiated by the use of an equation of transfer which, to put it bluntly, is incorrect under the circumstances envisaged. However, it appears that the methods which have been recently developed to treat an analogous problem in theory of the formation of absorption lines in a moving atmosphere\(^7\) are sufficiently general to enable one to solve the problem of radiative transfer in an expanding nebula. In this paper we therefore propose to study the problem of an expanding planetary nebula, particularly with a view to estimating the selective radiation pressure in Lyman-\(a\). In later papers it is our intention to extend the methods of this investigation to the consideration of a variety of related questions.

2. The equation of transfer for Lyman-\(a\) radiation and its approximate forms.—As was first pointed out by E. A. Milne,\(^8\) in studying the transfer of radiation in a planetary nebula we can ignore the curvature of the layers except when formulating the boundary conditions. We shall accordingly consider the atmosphere as stratified in parallel planes in which all properties are assumed to be constant over the planes \(z = \) constant. Let \(\rho(z)\) denote the density of the scattering material (neutral hydrogen atoms in our present context) at height \(z\), and \(w(z)\) the velocity of the material at the same height assumed parallel to the \(z\)-direction. Further, let \(\sigma(\nu)\) denote the mass-scattering coefficient in Lyman-\(a\) for the frequency \(\nu\) as judged by an observer at rest with respect to the material. We shall suppose that \(\sigma(\nu)\) differs appreciably from zero only in a small range of \(\nu\). However, it is in the essence of the problem that the half-width of \(\sigma(\nu)\) is of the same order as the Doppler shifts in frequency caused by the differential motions in the nebula. This last circumstance makes the change of frequency on scattering the only optical effect of the motion \(w(z)\) which has any importance and allows us to ignore all such effects as aberration, etc.

In writing down the equation of transfer for the problem of an expanding nebula, it is especially important that we be careful in referring all the frequencies to some chosen fixed observer. As will become apparent when we come to formulating our boundary conditions (§3), it is most convenient to choose our fixed observer as at rest with respect to the central star. Let \(I(\nu, z, \theta)\) then denote the specific intensity of the radiation at height \(z\) inclined at angle \(\theta\) to the positive normal and in the frequency \(\nu\) as judged by our fixed observer. This radiation will appear to an observer at rest with respect to the material at \(z\) as having a frequency

\[
\nu \left(1 - \frac{w}{c} \cos \theta\right).
\]

(1)

It will accordingly be scattered as such in all directions with a scattering coefficient of

\[
\sigma \left(\nu \left[1 - \frac{w}{c} \cos \theta\right]\right).
\]

(2)

The equation of transfer will accordingly take the form

\[
\cos \theta \frac{\partial I(\nu, z, \theta)}{\rho \partial z} = -\sigma \left(\nu \left[1 - \frac{w}{c} \cos \theta\right]\right) I(\nu, z, \theta) + j(\nu, z, \theta),
\]

(3)

\(^6\) M.N. 95, 16, 1934.

\(^7\) S. Chandrasekhar, Rev. Mod. Phys., 17, 138, 1945. This paper will be referred to as "Moving Atmospheres."

\(^8\) Zs. f. A. p., 1, 98, 1930.
where \( j (\nu, s, \theta) \) denotes the emission per unit time and per unit solid angle in the frequency \( \nu \) and in the direction \( \theta \). This emission will consist of two parts: that derived from the scattering of the radiation of appropriate frequencies from other directions into the direction considered and that derived from the conversion of the ultraviolet radiation at \( s \) into Lyman-\( \alpha \). The former is given by (cf. "Moving Atmospheres," eq. [5])

\[
\frac{1}{2} \sigma \left( \nu \left[ 1 - \frac{w}{c} \cos \theta \right] \right) \int_0^\pi I \left( \nu \left[ 1 - \frac{w}{c} \cos \theta + \frac{w}{c} \cos \chi \right], z, \chi \right) \sin \chi d\chi.
\]  

(4)

As for the latter, it will depend, among other things, on the probability \( (1 - p) \) with which an ultraviolet quantum is converted into a Lyman-\( \alpha \) quantum on scattering and on the source function characteristic of the ultraviolet radiation. Thus, if \( S_c \) denotes the flux of ultraviolet radiation incident per unit area of the inner boundary of the planetary nebula, the source function for the ultraviolet radiation is (cf. the references in n. 5),

\[
p \left\{ \frac{1}{2} \int_0^\pi I_c \sin \theta d\theta + \frac{1}{4} S_c e^{-(\tau_1 - \tau)} \right\},
\]  

(5)

where \( I_c \) denotes the intensity of the diffuse ultraviolet light, \( \tau \) the optical depth for the ultraviolet radiation measured from the outer boundary inward, and \( \tau_1 \) the total optical thickness of the nebula (also in ultraviolet light). To find the emission in Lyman-\( \alpha \) at frequency \( \nu \), consider the number of ultraviolet quanta absorbed in a slab of material of unit cross-section and height \( ds \) at \( s \). This is clearly given by

\[
\frac{1}{\hbar \nu} \left\{ J_c (\tau) + \frac{1}{4} S_c e^{-(\tau_1 - \tau)} \right\} \kappa_c \rho d s,
\]  

(6)

where \( \kappa_c \) denotes the absorption coefficient in the continuum, \( \nu_c \) a suitably averaged frequency to represent the ultraviolet radiation, and

\[
J_c = \frac{1}{2} \int_0^\pi I_c \sin \theta d\theta.
\]  

(7)

Since a Lyman-\( \alpha \) quantum is emitted with a probability \( (1 - p) \) every time an ultraviolet quantum is scattered, the total energy emitted per unit time in Lyman-\( \alpha \) (i.e., integrated over all the frequencies in the line) by the slab of material considered is

\[
(1 - p) \frac{\nu_0}{\nu_c} \left\{ J_c (\tau) + \frac{1}{4} S_c e^{-(\tau_1 - \tau)} \right\} \kappa_c \rho d s,
\]  

(8)

where \( \nu_0 \) denotes the frequency of Lyman-\( \alpha \). The fraction of the energy (8) which will appear in the frequency \( \nu \), as judged by the fixed observer, is

\[
\frac{\sigma \left( \nu \left[ 1 - \frac{w}{c} \cos \theta \right] \right)}{\int_0^\infty \sigma (\nu) d\nu}.
\]  

(9)

Hence the contribution to the emission \( j (\nu, s, \theta) \) by the conversion of the ultraviolet radiation is

\[
(1 - p) \frac{\nu_0}{\nu_c} \kappa_c \frac{2 \sigma_0 \Delta \nu}{2 \sigma_0 \Delta \nu} \left\{ J_c (\tau) + \frac{1}{4} S_c e^{-(\tau_1 - \tau)} \right\} \sigma \left( \nu \left[ 1 - \frac{w}{c} \cos \theta \right] \right),
\]  

(10)

where we have written

\[
2 \sigma_0 \Delta \nu = \int_0^\infty \sigma (\nu) d\nu.
\]  

(11)
The equation of transfer for the radiation of frequency \( \nu \) (as judged by the fixed observer) can accordingly be written in the form

\[
\mu \frac{\partial I (\nu, \mu, z)}{\rho \partial z} = - \sigma \left( \nu \left[ 1 - \frac{\omega}{c} \mu \right] \right) \left\{ I (\nu, \mu, z) \right\} - \frac{1}{2} \int_{-1}^{+1} I \left( \nu \left[ 1 - \frac{\omega}{c} \mu + \frac{\omega}{c} \mu' \right], z, \mu' \right) d\mu' - \frac{1}{2} E (z) \right\} \tag{12}
\]

where, for the sake of brevity, we have written

\[
E (z) = (1 - \rho) \frac{\rho_0}{\nu_c} \frac{K_c}{\sigma_0 \Delta \nu} \left( J_c (\tau) + \frac{1}{2} S_c e^{-(\tau - \nu)} \right). \tag{13}
\]

Further, in equation (12) we have written \( \mu \) and \( \mu' \) for \( \cos \theta \) and \( \cos \chi \), respectively.

In solving the equation of transfer (12), we shall adopt the method of approximation which has recently been developed in connection with the various problems of radiative transfer in the theory of stellar atmospheres.\(^8\) In this method we replace the integrals which appear in the equation of transfer by sums according to Gauss’s formula for numerical quadratures and replace the integrodifferential equation by a system of linear equations. Thus, considering equation (12), we replace it in the \( n \)th approximation by the system of \( 2n \) equations,

\[
\mu_i \frac{\partial I_i (\nu, z)}{\rho \partial z} = - \sigma \left( \nu \left[ 1 - \frac{\omega}{c} \mu_i \right] \right) \left\{ I_i (\nu, z) \right\} - \frac{1}{2} \Sigma a_i J_i \left( \nu \left[ 1 - \frac{\omega}{c} \mu_i + \frac{\omega}{c} \mu_j \right], z \right) - \frac{1}{2} E (z) \right\} \tag{14}
\]

where the \( \mu_i \)'s (\( i = \pm 1, \ldots, \pm n \)) are the zeros of the Legendre polynomial \( P_{2n} (\mu) \) and the \( a_i \)'s are the appropriate Gaussian weights. Further, in equation (14) we have written \( I_i (\nu, z) \) for \( I (\nu, z, \mu_i) \).

At this stage one further simplification of equation (14) is possible. In evaluating the Doppler shifts we need not distinguish between

\[
\nu \left( 1 - \frac{\omega}{c} \mu \right) \quad \text{and} \quad \nu - \nu_0 \frac{\omega}{c} \mu,
\]

where \( \nu_0 \) denotes the frequency of the center of the line. We may therefore replace equation (14) by the simpler one,

\[
\mu_i \frac{\partial I_i (\nu, z)}{\rho \partial z} = - \sigma \left( \nu - \nu_0 \frac{\omega}{c} \mu_i \right) \left\{ I_i (\nu, z) \right\} - \frac{1}{2} \Sigma a_i J_i \left( \nu - \nu_0 \frac{\omega}{c} \mu_i + \nu_0 \frac{\omega}{c} \mu_j, z \right) - \frac{1}{2} E (z) \right\} \tag{16}
\]

As in “Moving Atmospheres,” instead of considering the intensities \( I_i (i = \pm 1, \ldots, \pm n) \) for some fixed frequency \( \nu \), we shall consider them for the frequencies

\[
\nu_i = \nu + \nu_0 \frac{\omega}{c} \mu_i \quad (i = \pm 1, \ldots, \pm n), \tag{17}
\]

which are functions of \( z \). If we now let

\[
I_i (\nu_i, z) = \psi_i (\nu, z) \quad (i = \pm 1, \ldots, \pm n), \tag{18}
\]

\textit{S. Chandrasekhar, Ap. J., 100, 76, 117, 1944; 101, 95, 328, 348, 1945 (see particularly the first of these papers).}
the differential equations for the $\psi_i$'s become (cf. "Moving Atmospheres," eq. [16])

$$
\mu_i \frac{\partial \psi_i}{\partial z} - \mu_i \frac{v_0}{c} \frac{\partial}{\partial z} \frac{\partial \psi_i}{\partial \nu} = - \rho \sigma (\nu) \left( \psi_i - \frac{1}{2} \Sigma a_j \psi_j - \frac{1}{2} E \right).
$$

(19)

In our subsequent work we shall restrict ourselves to the first approximation. In this approximation

$$
\mu_1 = - \mu_{-1} = \frac{1}{\sqrt{3}} \quad \text{and} \quad a_1 = a_{-1} = 1,
$$

(20)

and equations (16) and (19) become

$$
\mu_1 \frac{\partial I_{+1}}{\rho \partial z} = - \frac{1}{2} \sigma \left( \nu - \frac{w}{c} \mu_1 \right) \left[ I_{+1} (\nu, z) - I_{-1} \left( \nu - 2 \frac{v_0}{c} \mu_1, z \right) - E (z) \right],
$$

(21)

$$
\mu_1 \frac{\partial I_{-1}}{\rho \partial z} = + \frac{1}{2} \sigma \left( \nu + \frac{v_0}{c} \mu_1 \right) \left[ I_{-1} (\nu, z) - I_{+1} \left( \nu + 2 \frac{v_0}{c} \mu_1, z \right) - E (z) \right],
$$

(22)

and

$$
\mu_1 \frac{\partial \psi_{+1}}{\partial z} - \mu_1 \frac{v_0}{c} \frac{\partial}{\partial z} \frac{\partial \psi_{+1}}{\partial \nu} = - \frac{1}{2} \rho \sigma (\nu) (\psi_{+1} - \psi_{-1} - E),
$$

(23)

$$
\mu_1 \frac{\partial \psi_{-1}}{\partial z} + \mu_1 \frac{v_0}{c} \frac{\partial}{\partial z} \frac{\partial \psi_{-1}}{\partial \nu} = + \frac{1}{2} \rho \sigma (\nu) (\psi_{-1} - \psi_{+1} - E),
$$

(24)

where it may be recalled that

$$
\psi_{+1} (\nu, z) = I_{+1} \left( \nu + \frac{v_0}{c}, z \right) \quad \text{and} \quad \psi_{-1} (\nu, z) = I_{-1} \left( \nu - \frac{v_0}{c}, z \right).
$$

(25)

3. The reduction to a boundary-value problem for the case $\sigma (\nu) =$ constant for $\nu_0 - \Delta \nu \leq \nu \leq \nu_0 + \Delta \nu$ and zero outside this interval and for a linear increase of $w$ with the optical depth.—In this paper we shall consider the problem of the expanding nebula for the case

$$
\sigma (\nu) = \sigma_0 = \text{constant for } \nu_0 - \Delta \nu \leq \nu \leq \nu_0 + \Delta \nu,
$$

(26)

and

$$
\frac{1}{\rho} \frac{d \nu}{d z} = \text{constant}.
$$

(27)

Further, we shall restrict ourselves to the first approximation.

When $\sigma (\nu)$ has the form (26), some care is required in the formulation of the boundary conditions; for, according to equations (21) and (22),

$$
\frac{\partial I_{+1}}{\partial z} \neq 0 \quad \text{only if} \quad \nu_0 - \Delta \nu + \mu_1 \frac{v_0}{c} \leq \nu \leq \nu_0 + \Delta \nu + \mu_1 \frac{v_0}{c},
$$

(28)

and

$$
\frac{\partial I_{-1}}{\partial z} \neq 0 \quad \text{only if} \quad \nu_0 - \Delta \nu - \mu_1 \frac{v_0}{c} \leq \nu \leq \nu_0 + \Delta \nu - \mu_1 \frac{v_0}{c}.
$$

(29)

Accordingly, in the $(\nu, w)$-plane the lines

$$
\nu = \nu_0 - \Delta \nu + \mu_1 \frac{v_0}{c} \quad \text{and} \quad \nu = \nu_0 + \Delta \nu + \mu_1 \frac{v_0}{c}
$$

(30)

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delimit the regions in which \( I_{\nu_1} \) is different from a constant from the regions in which it is a constant for varying \( s \). The situation is further clarified in Figure 1, where \( AD \) and \( BC \) represent the lines (30). Similarly, the lines (\( AF \) and \( BE \) in Fig. 1)

\[
\nu = \nu_0 - \Delta \nu - \mu \nu_0 \frac{w}{c} \quad \text{and} \quad \nu = \nu_0 + \Delta \nu - \mu \nu_0 \frac{w}{c}
\]

(31)
delimit the regions in which \( I_{\nu_1} \) is different from a constant from the regions in which it is a constant for varying \( s \).

![Diagram](image)

**Fig. 1**

In formulating the boundary conditions we must allow for the possibility that the inner boundary of the nebula may itself have a motion relative to our fixed observer, who, it will be remembered, is assumed to be at rest only with respect to the central star. Let \( w_i \) then denote the velocity of the nebula at its inner boundary. From Figure 1 it is apparent that we will have to distinguish three cases:

\[
\begin{align*}
\text{case I:} & \quad w_i = 0 \\
\text{case II:} & \quad \nu_0 - \Delta \nu + \mu \nu_0 \frac{w_i}{c} < \nu_0 \\
\text{case III:} & \quad \nu_0 - \Delta \nu + \mu \nu_0 \frac{w_i}{c} \geq \nu_0
\end{align*}
\]

(32)

These cases are, however, more conveniently distinguished in terms of the Doppler shift, \( 2D_{\nu}^{(i)} \), due to the velocity \( w_i \):

\[
2D_{\nu}^{(i)} = \nu_0 \frac{w_i}{c}.
\]

(33)

We have

\[
\begin{align*}
\text{case I:} & \quad D_{\nu}^{(i)} = 0 \\
\text{case II:} & \quad 0 < D_{\nu}^{(i)} < \frac{\Delta \nu}{2\mu_1} = \frac{\sqrt{3}}{2} \Delta \nu \\
\text{case III:} & \quad D_{\nu}^{(i)} \geq \frac{\Delta \nu}{2\mu_1} = \frac{\sqrt{3}}{2} \Delta \nu
\end{align*}
\]

(34)

These three cases correspond to situations in which the inner boundary of the nebula is at \( XY, X'Y' \), or \( X''Y'' \), respectively, as shown in Figure 1.

Now the physical boundary conditions which have to be translated into mathematical terms are, first, that there is no radiation incident on the nebula from the outside and, second, that

\[
I^* (\nu, \vartheta) = I^* (\nu, \pi - \vartheta)
\]

(35)
whenever \( I^* (\nu, \theta) \) refers to the intensity of a ray which, as judged by the fixed observer at the central star, has not suffered any change by absorptions or emissions by the intervening medium (see Fig. 2). This latter boundary condition arises from the geometry of the problem and the fact that the specific intensity along a ray does not change if no absorption or emission takes place.\(^{10}\) Accordingly, the boundary conditions with respect to which equations (21) and (22) have to be solved are (see Fig. 1):

\[
\begin{align*}
\text{case I} & \quad \begin{cases} 
I_{-1} (\nu) = 0 \text{ along } BE \text{ and } EF, \\
I_{+1} (\nu) = 0 \text{ along } BC, \\
I_{+1} (\nu) = I_{-1} (\nu) \text{ along } AB;
\end{cases} \\
\text{case II} & \quad \begin{cases} 
I_{-1} (\nu) = 0 \text{ along } B'E' \text{ and } EF, \\
I_{+1} (\nu) = 0 \text{ along } A'B'_1' \text{ and } B'_1'C', \\
I_{+1} (\nu) = I_{-1} (\nu) \text{ along } A'B'_1' ;
\end{cases} \\
\text{case III} & \quad \begin{cases} 
I_{-1} (\nu) = 0 \text{ along } B''E' \text{ and } EF, \\
I_{+1} (\nu) = 0 \text{ along } A''B''_1' \text{ and } B''_1'C'.
\end{cases}
\end{align*}
\]

When expressed for the intensities \( \psi_{+1} \) and \( \psi_{-1} \), the foregoing boundary conditions become

\[
\begin{align*}
\psi_{-1} (\nu, z_1) &= 0 \quad \text{for} \quad \nu_0 - \Delta \nu \leq \nu \leq \nu_0 + \Delta \nu, \\
\psi_{-1} (\nu_0 + \Delta \nu, z) &= 0 \quad \text{for} \quad 0 \leq z \leq z_1, \\
\psi_{+1} (\nu_0 + \Delta \nu, z) &= 0 \quad \text{for} \quad 0 \leq z \leq z_1.
\end{align*}
\]

for all three cases, while

\[
\begin{align*}
\psi_{-1} (\nu, 0) &= \psi_{+1} (\nu, 0) \quad \text{in case I}, \\
\psi_{+1} (\nu, 0) &= 0 \quad \text{for} \quad \nu_0 + \Delta \nu - 4 D_{\nu} (\nu) \mu_1 \leq \nu \leq \nu_0 + \Delta \nu, \\
\psi_{+1} (\nu, 0) &= \psi_{-1} (\nu + 4 D_{\nu} (\nu) \mu_1, 0) \quad \text{for} \quad \nu_0 - \Delta \nu \leq \nu < \nu_0 + \Delta \nu - 4 D_{\nu} (\nu) \mu_1, \\
\psi_{+1} (\nu, 0) &= 0 \quad \text{for} \quad \nu_0 - \Delta \nu \leq \nu \leq \nu_0 + \Delta \nu \quad \text{in case III}.
\end{align*}
\]

In the foregoing equations, \( z = 0 \) refers to the inner boundary of the nebula, while \( z = z_1 \) refers to the outer boundary.

\(^{10}\) Cf. Milne, op. cit. (n. 8.)
We shall now transform equations (23) and (24) to forms more convenient for their solution. Let $t$ denote the optical depth of the atmosphere measured from the outer boundary inward in terms of $\sigma_0$. Then

$$dt = -\rho \sigma_0 dz.$$  \hspace{1cm} (43)

It is, however, more convenient to use the variable

$$x = \frac{1}{2\mu_1} t = \frac{\sqrt{3}}{2} t$$ \hspace{1cm} (44)

instead of $t$. In terms of $x$, equations (23) and (24) become

$$\frac{\partial \psi_{+1}}{\partial x} - \mu_1 \frac{v_0}{c} \frac{dw}{dx} \frac{\partial \psi_{+1}}{\partial \nu} = \psi_{+1} - \psi_{-1} - E(x),$$ \hspace{1cm} (45)

and

$$\frac{\partial \psi_{-1}}{\partial x} + \mu_1 \frac{v_0}{c} \frac{dw}{dx} \frac{\partial \psi_{-1}}{\partial \nu} = \psi_{+1} - \psi_{-1} + E(x).$$ \hspace{1cm} (46)

Now the assumption (27) concerning the variation of $w$ clearly implies that the velocity is a linear function of $x$. Accordingly, we may write

$$w = \bar{w} + (w_1 - w_i) \frac{x}{x_1},$$ \hspace{1cm} (47)

where $w_1$ and $w_i$ denote the velocities at the outer ($x = 0$) and the inner ($x = x_1$) boundaries of the planetary nebula, respectively. As in "Moving Atmospheres," we shall express the difference in velocities between the inner and the outer boundaries of the nebula in terms of a Doppler width, $D\nu$, according to

$$2D\nu = \frac{v_0}{c} (w_1 - w_i).$$ \hspace{1cm} (48)

With these definitions,

$$\mu_1 \frac{v_0}{c} \frac{dw}{dx} = -2\mu_1 \frac{D\nu}{x_1} = -\frac{4}{3} \frac{D\nu}{t_i}.$$ \hspace{1cm} (49)

Equations (45) and (46) become

$$\frac{\partial \psi_{+1}}{\partial x} + 2\mu_1 \frac{D\nu}{x_1} \frac{\partial \psi_{+1}}{\partial \nu} = \psi_{+1} - \psi_{-1} - E(x)$$ \hspace{1cm} (50)

and

$$\frac{\partial \psi_{-1}}{\partial x} - 2\mu_1 \frac{D\nu}{x_1} \frac{\partial \psi_{-1}}{\partial \nu} = \psi_{+1} - \psi_{-1} + E(x).$$ \hspace{1cm} (51)

We now introduce the variable $y$, defined by

$$(v_0 + \Delta \nu) - \nu = 2\mu_1 \frac{D\nu}{x_1} y;$$ \hspace{1cm} (52)

$y$ therefore measures the frequency shifts from the violet edge of $\sigma(\nu)$ (as they enter the intensities $\psi_{+1}$ and $\psi_{-1}$) in units of

$$2\mu_1 \frac{D\nu}{x_1}.$$ \hspace{1cm} (53)
Equations (50) and (51) simplify to the forms

\[
\frac{\partial \psi_{+1}}{\partial x} - \frac{\partial \psi_{+1}}{\partial y} = \psi_{+1} - \psi_{-1} - E(x) \tag{54}
\]

and

\[
\frac{\partial \psi_{-1}}{\partial x} + \frac{\partial \psi_{-1}}{\partial y} = \psi_{+1} - \psi_{-1} + E(x). \tag{55}
\]

The range of the variables \( x \) and \( y \) in which the solution has to be sought is (cf. eq. [52])

\[
0 \leq x \leq x_1 \quad \text{and} \quad 0 \leq y \leq y_1 = \frac{1}{\mu_1} \frac{\Delta v}{Dv} x_1; \tag{56}
\]

and the boundary conditions with respect to which equations (54) and (55) have to be solved are (see Fig. 3):

\[
\begin{align*}
\psi_{-1} &= 0 \quad \text{on} \quad CD: \quad x = 0 \quad \text{and} \quad 0 \leq y \leq y_1, \\
\psi_{+1} &= 0 \quad \text{on} \quad CB: \quad y = 0 \quad \text{and} \quad 0 \leq x \leq x_1,
\end{align*}
\tag{57}
\]

in all three cases;

and on \( BA \)

\[
\psi_{+1} = \psi_{-1} \quad \text{for} \quad x = x_1 \quad \text{and} \quad 0 \leq y \leq y_1 \quad \text{(case I)}. \tag{58}
\]

\[
\psi_{+1}(y, x_1) = 0 \quad \text{for} \quad 0 \leq y \leq y^* = 2 \frac{Dv(i)}{Dv} x_1, \tag{59}
\]

and

\[
\psi_{+1}(y, x_1) = \psi_{-1}(y - y^*, x_1) \quad \text{for} \quad y^* < y \leq y_1 \quad \text{(case II)}. \tag{60}
\]

It is convenient to introduce one further transformation of the variable. Let

\[
\psi_{+1} = e^{-\nu F} \quad \text{and} \quad \psi_{-1} = e^{-\nu G}. \tag{61}
\]

Equations (54) and (55) become

\[
\frac{\partial F}{\partial x} - \frac{\partial F}{\partial y} = -G - e^{\nu E}(x) \tag{62}
\]

and

\[
\frac{\partial G}{\partial x} + \frac{\partial G}{\partial y} = +F + e^{\nu E}(x). \tag{63}
\]

The corresponding boundary conditions follow from equations (57)-(60). Thus,

\[
\begin{align*}
G &= 0 \quad \text{on} \quad CD: \quad x = 0 \quad \text{and} \quad 0 \leq y \leq y_1, \\
F &= 0 \quad \text{on} \quad CB: \quad y = 0 \quad \text{and} \quad 0 \leq x \leq x_1,
\end{align*}
\tag{64}
\]

in all three cases;

and on \( BA \)

\[
F = G \quad \text{for} \quad x = x_1 \quad \text{and} \quad 0 \leq y \leq y_1 \quad \text{(case I)}. \tag{65}
\]

\[
F(y, x_1) = 0 \quad \text{for} \quad 0 \leq y \leq y^*. \tag{66}
\]

and

\[
F(y, x_1) = e^{\nu y}G(y - y^*, x_1) \quad \text{for} \quad y^* < y \leq y_1 \quad \text{(case II)}. \tag{67}
\]

\[
F = 0 \quad \text{for} \quad x = x_1 \quad \text{and} \quad 0 \leq y \leq y_1 \quad \text{(case III)}. \tag{68}
\]
Finally, we may note that \( E(x) \) can be written in the form (cf. eq. [13])

\[
E(x) = \frac{1}{2\mu_1} \left( 1 - p \right) \frac{\nu_0}{\nu_c \Delta \nu} \alpha \left\{ T \left( a \beta \right) + \frac{1}{4} S_e e^{-\tau_1} e^{az} \right\} .
\]

(68)

where

\[
a x = \tau \quad \text{and} \quad a = 2\mu_1 \frac{K_e}{\sigma_0} .
\]

(69)

The quantity \( \alpha \) introduced in equation (68) is essentially the ratio of the absorption coefficients in the continuum and in the line; this ratio is generally of the order of \( 10^{-4} \), so that, if the need should arise, we may treat this as a small quantity. However, it should be remembered that the optical depth \( \tau_1 \) in Lyman-\( \alpha \), under the circumstances of our problem, is so large that \( a \tau_1 (= \tau_1) \) is of the order of unity.

4. The solution of the boundary-value problem for the case \( p = 0 \).—Before we can proceed with the solution of the boundary-value problem formulated in the preceding section, it is necessary for us to have an expression for the density of the diffuse ultraviolet radiation. This is given by the theory of radiative transfer of the ultraviolet radiation, and in a first approximation the solution for \( J_e \) has the form

\[
J_e(\tau) = A e^{-\lambda \tau} + B e^{+\lambda \tau} + \frac{3p}{4(2-3p)} S_e e^{-(\tau_1-\tau)} ,
\]

(70)

where \( A \) and \( B \) are certain constants and \( \lambda^2 = 3(1-p) \). While the solution of the boundary-value problem with this general form for \( J_e(\tau) \) is entirely feasible, we shall be avoiding a great deal of formal complexity without losing, at the same time, any of the essential features of the problem by considering the case \( p = 0 \). Indeed, it is known that the flux in the diffuse ultraviolet radiation is generally so small that the case \( p = 0 \) provides ample accuracy for most problems. Moreover, since in this investigation our prime interest is to evaluate the selective radiation pressure in Lyman-\( \alpha \), we shall not be restricting our treatment in any essential way if we put \( p = 0 \). It is evident that, when this is the case,

\[
J_e(\tau) = 0 \quad (p = 0).
\]

(71)

The expression for \( E(x) \) now reduces to (cf. eq. [68])

\[
E(x) = \left( \frac{\nu_0}{8\mu_1\nu_c \Delta \nu} S_e e^{-(\tau_1 - \tau)} \right) \alpha e^{ax} ;
\]

(72)

and equations (62) and (63) become

\[
\frac{\partial F}{\partial x} - \frac{\partial F}{\partial y} = -G - Q a e^{ax+ay}
\]

(73)

and

\[
\frac{\partial G}{\partial x} + \frac{\partial G}{\partial y} = +F + Q a e^{ax+ay},
\]

(74)

where we have written

\[
Q = \frac{\nu_0}{8\mu_1\nu_c \Delta \nu} S_e e^{-(\tau_1 - \tau)} .
\]

(75)

Equations (73) and (74) are nonhomogeneous. But they can be reduced to homogeneous forms, since a particular integral is readily found. Thus

\[
F = -Q \frac{2+a}{a} e^{ax+ay} \quad \text{and} \quad G = -Q \frac{2-a}{a} e^{ax+ay}
\]

(76)

\[\text{Cf. S. Chandrasekhar, Zs. f. A.}, 9, 266, 1935 \text{ (see eqs. [24] and [28]).}\]
are seen to satisfy equations (73) and (74). We accordingly write

$$F = Q \left( f - \frac{2 + a}{a} e^{ax+by} \right) \hspace{1cm} (77)$$

and

$$G = Q \left( g - \frac{2 - a}{a} e^{ax+by} \right) \hspace{1cm} (78)$$

and obtain for $f$ and $g$ the homogeneous equations

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = - g. \hspace{1cm} (79)$$

and

$$\frac{\partial g}{\partial x} + \frac{\partial g}{\partial y} = + f. \hspace{1cm} (80)$$

The boundary conditions with respect to which the foregoing equations have to be solved are (cf. eqs. [64]-[67] and [77] and [78]):

$$g = \frac{2 - a}{a} e^x \quad \text{on} \quad CD: x = 0 \quad \text{and} \quad 0 \leq y \leq y_1, \hspace{1cm} (81)$$

$$f = \frac{2 + a}{a} e^{ax} \quad \text{and} \quad g = \frac{2 - a}{a} e^{ax} \quad \text{on} \quad CB: y = 0 \quad \text{and} \quad 0 \leq x \leq x_1,$$

in all three cases, and on $AB \quad (x = x_1 \text{ and } 0 \leq y \leq y_1)$

$$f = g + 2 e^{ax_1+y} \quad \text{(case I)} \hspace{1cm} (82)$$

and

$$f = \frac{2 + a}{a} e^{ax_1+y} \quad \text{(case III)} \hspace{1cm} (83)$$

For case II there is a similar boundary condition on $AB$. But as we shall not be obtaining explicitly the solution for this case, we shall not continue to write down the boundary conditions for this case.

Eliminating $g$ between equations (79) and (80), we obtain

$$\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} + f = 0. \hspace{1cm} (84)$$

We require to solve this hyperbolic equation with the boundary conditions (cf. eqs. [79]-[83])

$$f = \frac{2 + a}{a} e^{ax}, \quad \frac{\partial f}{\partial x} = (2 + a) e^{ax}, \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{a^2 + a + 2}{a} e^{ax}$$

on $\quad BC: y = 0 \quad \text{and} \quad 0 \leq x \leq x_1.$ \hspace{1cm} (85)

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} - \frac{2 - a}{a} e^y \quad \text{on} \quad CD: x = 0 \quad \text{and} \quad 0 \leq y \leq y_1,$$

in all three cases, and on $AB \quad (x = x_1 \text{ and } 0 \leq y \leq y_1)$

$$f + \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} = 2 e^{ax_1+y} \quad \text{(case I)} \hspace{1cm} (86)$$

and

$$f = \frac{2 + a}{a} e^{ax_1+y} \quad \text{(case III)} \hspace{1cm} (87)$$
The solution to the boundary-value problem we have just formulated can be carried out in a manner quite analogous to that which was followed in the solution of a similar boundary-value problem in "Moving Atmospheres." Briefly, the method consists in applying Green's theorem to contours which are, in parts, the characteristics \( x - \xi = \pm (y - \eta) \), passing through suitably selected points \((\xi, \eta)\) and with the further choice of the Riemann function \( v(x, y; \xi, \eta) \) for the solution of the adjoint equation.12

Fig. 3

a) The solution in the region \( OCB \).—Let the characteristic \( x = y \) through \( C \) intersect \( AB \) at \( C' \) and the characteristic \( x_1 - x = y \) through \( B \) intersect \( CD \) at \( B' \). Further, let \( CC' \) and \( BB' \) intersect at \( O \) (see Fig. 3).

Now, since the function and its derivatives are specified along \( CB \), the solution in the region \( OCB \) (including the sides \( OC \) and \( OB \)) can be found directly by Riemann's method. Thus, applying Green's theorem to a contour such as \( EFGE \) where \( EF \) and \( EG \) are the characteristics through \( E = (\xi, \eta) \), we find that

\[
 f (\xi, \eta) = \frac{2 + a}{a} e^{a\xi} \cosh a\eta - \frac{1}{2} \int_{\eta-\gamma}^{\xi+\gamma} \left( f \frac{\partial v}{\partial y} - v \frac{\partial f}{\partial y} \right)_{y=0} dx. \tag{88}
\]

The Riemann function appropriate for the contour \( EFGE \) is

\[
 v(x, y; \xi, \eta) = I_0 \left( (y - \eta)^2 - (x - \xi)^2 \right)^{1/4} \tag{89}
\]

12 A more detailed description of the method will be found in "Moving Atmospheres," § 5.
Using this v in equation (88) and substituting also for f and \( \partial f/\partial y \) according to equation (85), we find

\[
f (\xi, \eta) = \frac{2 + a}{a} e^{\alpha z} \cosh \alpha \eta + \frac{a^2 + a + 2}{2a} \int_{\xi-\eta}^{\xi+\eta} e^{\alpha z} I_0 \left( \frac{\eta^2 - (x - \xi)^2}{4} \right) dx + \frac{2 + a}{2a} \eta \int_{\xi-\eta}^{\xi+\eta} e^{\alpha z} I_1 \left( \frac{\eta^2 - (x - \xi)^2}{4} \right) dx.
\]

Putting

\[
x - \xi = \eta \cos \vartheta
\]

into the two integrals on the right-hand side of equation (90), we obtain

\[
f (\xi, \eta) = \frac{2 + a}{a} e^{\alpha z} \cosh \alpha \eta + \frac{a^2 + a + 2}{2a} \eta e^{\alpha z} \int_0^\pi e^{\alpha \sin \vartheta} \varphi I_0 (\eta \sin \vartheta) \sin \vartheta d\vartheta + \frac{2 + a}{2a} \eta e^{\alpha z} \int_0^\pi e^{\alpha \sin \vartheta} \varphi I_1 (\eta \sin \vartheta) d\vartheta.
\]

The integrals occurring in the foregoing equation can be evaluated in the following manner:

Considering, first, the integral

\[
a \eta \int_0^\pi e^{\alpha \sin \vartheta} \varphi I_0 (\eta \sin \vartheta) \sin \vartheta d\vartheta,
\]

we replace \( e^{\alpha \cos \vartheta} \) and \( I_0 (\eta \sin \vartheta) \) by their respective series expansions and integrate term by term. We find

\[
a \eta \int_0^\pi e^{\alpha \sin \vartheta} \varphi I_0 (\eta \sin \vartheta) \sin \vartheta d\vartheta = 2a \eta \sum_{n=0}^\infty \frac{(a \eta)^{2n}}{(2n)!} \sum_{m=0}^\infty \frac{(\frac{1}{2} a \eta)^{2m}}{m! \Gamma (m + 1)} \int_0^{\pi/2} \cos^{2n} \vartheta \sin^{2m+1} \vartheta d\vartheta
\]

\[
= \sum_{n=0}^\infty \Gamma (n + \frac{1}{2}) \frac{a \eta^{2n+1}}{(2n)!} \sum_{m=0}^\infty \frac{(\frac{1}{2} a \eta)^{2m}}{m! \Gamma (m + n + \frac{1}{2})}
\]

\[
= \sum_{n=0}^\infty \Gamma (n + \frac{1}{2}) \frac{(2a^2 \eta)^n}{(2n)!} \sum_{m=0}^\infty \frac{(\frac{1}{2} a \eta)^{2m+n+1}}{m! \Gamma (m + n + \frac{1}{2})}
\]

\[
= \sum_{n=0}^\infty \Gamma (n + \frac{1}{2}) \frac{(2a^2 \eta)^n}{(2n)!} I_{n+1} (\eta)
\]

\[
= (2\pi \eta)^{\frac{1}{2}} \sum_{n=0}^\infty \frac{(\frac{1}{2} a^2 \eta)^n}{n!} I_{n+1} (\eta).
\]

But

\[
I_r \{ \eta \sqrt{(1 + a^2)} \} = (1 + a^2)^{\frac{1}{2}} \sum_{n=0}^\infty \frac{(\frac{1}{2} a^2 \eta)^n}{n!} I_{r+n} (\eta).
\]

This formula can be established by following the method used by G. N. Watson in his Theory of Bessel Functions, p. 141, Cambridge University Press, 1944, in proving a similar relation involving the Bessel functions with real arguments.
The last summation which occurs in equation (94) is therefore a special case of equation (95). We have

$$\sum_{n=0}^{\infty} \left( \frac{1}{2} a^2 \eta \right)^n \frac{1}{n!} I_{n+1} (\eta) \left[ \eta \sqrt{(1 + a^2)} \right].$$  \hfill (96)

Hence,

$$\eta \int_0^\infty e^{a \eta \cos \theta} I_0 (\eta \sin \theta) \sin \theta d\theta = \frac{(2\pi \eta)^{\frac{1}{4}}}{(1 + a^2)^{\frac{1}{4}}} I_1 \left\{ \eta \sqrt{(1 + a^2)} \right\};$$  \hfill (97)

or, substituting the explicit expression for $I_1$, we have

$$\eta \int_0^\infty e^{a \eta \cos \theta} I_0 (\eta \sin \theta) \sin \theta d\theta = \frac{2}{\sqrt{(1 + a^2)}} \sinh \left\{ \eta \sqrt{(1 + a^2)} \right\}.$$  \hfill (98)

Similarly,

$$a \eta \int_0^\infty e^{a \eta \cos \theta} I_1 (\eta \sin \theta) \sin \theta d\theta = 2 \sum_{n=0}^{\infty} \frac{(a \eta)^{2n+1}}{(2n)!} \sum_{m=0}^{\infty} \frac{\left( \frac{1}{2} \eta \right)^{2m+1}}{m! (m + 2)} \int_0^{\pi/2} \cos^{2n} \theta \sin^{2m+1} \theta d\theta$$

$$= \sum_{n=0}^{\infty} \Gamma \left( n + \frac{3}{2} \right) \frac{(a \eta)^{2n+1}}{(2n)!} \sum_{m=0}^{\infty} \frac{\left( \frac{1}{2} \eta \right)^{2m+1}}{(m + 1)! (m + n + \frac{3}{2})}$$

$$= \sum_{n=0}^{\infty} \Gamma \left( n + \frac{3}{2} \right) \frac{(2a^2 \eta)^{n+1}}{(2n)!} \sum_{m=0}^{\infty} \frac{\left( \frac{1}{2} \eta \right)^{2m+n+1}}{m! (m + n + \frac{3}{2})}$$

$$= \sum_{n=0}^{\infty} \Gamma \left( n + \frac{3}{2} \right) \frac{(2a^2 \eta)^{n+1}}{(2n)!} \left[ I_{n-\frac{1}{2}} (\eta) - \frac{\left( \frac{1}{2} \eta \right)^{n-\frac{1}{2}}}{\Gamma \left( n + \frac{3}{2} \right)} \right]$$

$$= a (2\pi \eta)^{\frac{1}{4}} \sum_{n=0}^{\infty} \frac{\left( \frac{1}{2} a^2 \eta \right)^{n+1}}{(2n)!} I_{n+1} (\eta) - 2a \sum_{n=0}^{\infty} \frac{(a \eta)^{2n}}{(2n)!}$$

$$= a (2\pi \eta)^{\frac{1}{4}} (1 + a^2)^{\frac{1}{4}} I_{n+1} \left\{ \eta \sqrt{(1 + a^2)} \right\} - 2a \cosh \eta.$$

Hence,

$$\eta \int_0^\infty e^{a \eta \cos \theta} I_1 (\eta \sin \theta) \sin \theta d\theta = 2 \cosh \left\{ \eta \sqrt{(1 + a^2)} \right\} - 2 \cosh \eta.$$  \hfill (100)

Substituting from equations (98) and (100) in equation (92), we obtain

$$f (\xi, \eta) = \frac{1}{a} e^{a^2} \left[ \frac{a^2 + a + 2}{\sqrt{(1 + a^2)}} \sinh \left\{ \eta \sqrt{(1 + a^2)} \right\} \right.$$  \hfill (101)

$$+ (2 + a) \cosh \left\{ \eta \sqrt{(1 + a^2)} \right\} \right].$$

This solves explicitly for $f$ in the region OCB. With this solution for $f$, $g$ can also be found in this region; for (cf. eq. [79])

$$g (\xi, \eta) = -\frac{\partial f}{\partial \xi} + \frac{\partial f}{\partial \eta}.$$  \hfill (102)
and equation (101) yields
\[
g (\xi, \eta) = \frac{1}{a} \cdot e^{x y} \left[ \frac{a^2 - a + 2}{(1 + a^2)} \sinh \left\{ \eta \sqrt{1 + a^2} \right\} + (2 - a) \cosh \left\{ \eta \sqrt{1 + a^2} \right\} \right]. \tag{103}
\]

The corresponding solutions for \(\psi_+\) and \(\psi_-\) are readily found. We have
\[
\psi_+ = \frac{Q}{a} e^{x - y} \left[ \frac{a^2 + a + 2}{(1 + a^2)} \sinh \left\{ y \sqrt{1 + a^2} \right\} + (2 + a) \cosh \left\{ y \sqrt{1 + a^2} \right\} \right] \tag{104}
\]
and
\[
\psi_- = \frac{Q}{a} e^{x - y} \left[ \frac{a^2 - a + 2}{(1 + a^2)} \sinh \left\{ y \sqrt{1 + a^2} \right\} + (2 - a) \cosh \left\{ y \sqrt{1 + a^2} \right\} \right]. \tag{105}
\]

b) The integral equation which insures the continuity of the solution along CO.—Along CD neither the function nor its derivatives are known: only a relation between them is given (cf. eq. [85]). But our knowledge of the solution along CO and the requirement that the solution be continuous along this line suffice to determine \(f(0, \eta)\) as the solution of an integral equation of Volterra’s type. To obtain this integral equation we apply Green’s theorem to a contour such as \(CHHI\), where \(H = (\eta, \eta)\) is a point on OC and \(HI\) is the characteristic \(\eta - x = y - \eta\) through \(H\). We find
\[
2f(\eta, \eta) = f(0, 2\eta) + \frac{2 + a}{a} + \int_0^{2\eta} \left( v \frac{\partial f}{\partial x} - f \frac{\partial v}{\partial x} \right) x = 0 \; dy. \tag{106}
\]

On the other hand, since (cf. eq. [85])
\[
\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} - \frac{2 - a}{a} e^{y} \tag{107}
\]
along CD, we have
\[
2f(\eta, \eta) = f(0, 2\eta) + \frac{2 + a}{a} - \frac{2 - a}{a} \int_0^{2\eta} e^{y} [v] x = 0 \; dy \tag{108}
\]
\[
+ \int_0^{2\eta} \left( v \frac{\partial f}{\partial y} \right) x = 0 \; dy - \int_0^{2\eta} f(0, y) \left( \frac{\partial v}{\partial x} \right) x = 0 \; dy.
\]

Integrating by parts the second of the three integrals occurring on the right-hand side of equation (108) and remembering that the Riemann function for this problem is always unity along the characteristics, we find that
\[
2f(\eta, \eta) = 2f(0, 2\eta) - \frac{2 - a}{a} \int_0^{2\eta} e^{y} [v] x = 0 \; dy \tag{109}
\]
\[
- \int_0^{2\eta} f(0, y) \left( \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} \right) x = 0 \; dy.
\]

The Riemann function appropriate for our present contour is
\[
v(x, y; \xi, \eta) = J_0 \left( \sqrt{1 - (y - x)^2} \right). \tag{110}
\]
With this choice of \( v \), equation (109) reduces to the form
\[
f(\eta, \eta) = f(0, 2\eta) - \frac{2 - a}{2a} \eta e^\eta \int_0^\eta e^\eta \cos \vartheta J_0(\eta \sin \vartheta) \sin \vartheta d\vartheta
\]
\[
- \frac{1}{2} \int_0^{2\eta} f(0, y) J_1\left(\left[\eta^2 - (y - \eta)^2\right]^{\frac{1}{2}}\right) \frac{yd\gamma}{\left[\eta^2 - (y - \eta)^2\right]^{\frac{1}{2}}}.
\]

(111)

The definite integral on the right-hand side of equation (111) can be evaluated. We have
\[
\eta \int_0^\eta e^\eta \cos \vartheta J_0(\eta \sin \vartheta) \sin \vartheta d\vartheta
\]
\[
= \sum_{n=0}^{\infty} \frac{\eta^{2n+1}}{(2n)!} \sum_{m=0}^{\infty} \left(-1\right)^m \frac{\left(\frac{1}{2} \eta\right)^{2m}}{m! \Gamma(m + 1)} \int_0^{\pi/2} \cos^{2n} \vartheta \sin^{2m+1} \vartheta d\vartheta
\]
\[
= \sum_{n=0}^{\infty} \frac{\Gamma\left(n + \frac{1}{2}\right)}{(2n)!} \frac{\eta^{2n+1}}{(2n)!} \sum_{m=0}^{\infty} \left(-1\right)^m \frac{\left(\frac{1}{2} \eta\right)^{2m}}{m! \Gamma(m + n + \frac{1}{2})}
\]
\[
= \sum_{n=0}^{\infty} \frac{\Gamma\left(n + \frac{1}{2}\right)}{(2n)!} \frac{(2\eta)^{n+\frac{1}{2}}}{(2n)!} \sum_{m=0}^{\infty} \left(-1\right)^m \frac{\left(\frac{1}{2} \eta\right)^{2m+n+\frac{1}{2}}}{m! \Gamma(m + n + \frac{3}{2})}
\]
\[
= (2\pi\eta)^{\frac{1}{2}} \sum_{n=0}^{\infty} \left\{ \frac{1}{\eta!} J_{n+\frac{1}{2}}(\eta) \right\}
\]
\[
= (2\pi\eta)^{\frac{1}{2}} \frac{(\frac{1}{2} \eta)^{\frac{1}{2}}}{\Gamma(1+\frac{1}{2})} = 2\eta.
\]

(112)

Equation (111) thus becomes
\[
\frac{1}{a} e^\eta \left[ \frac{a^2 + a + 2}{\sqrt{1 + a^2}} \sinh \{\eta \sqrt{1 + a^2}\} + (2 + a) \cosh \{\eta \sqrt{1 + a^2}\} \right] \sinh \{\eta \sqrt{1 + a^2}\} = f(0, 2\eta)
\]
\[
- \frac{1}{2a} \eta e^\eta = f(0, 2\eta) - \frac{1}{2} \int_0^{2\eta} f(0, y) J_1\left(\left[\eta^2 - (y - \eta)^2\right]^{\frac{1}{2}}\right) \frac{yd\gamma}{\left[\eta^2 - (y - \eta)^2\right]^{\frac{1}{2}}}.
\]

(113)

where we have also substituted for \( f(\eta, \eta) \) according to equation (101).

It is now seen that the right-hand side of equation (113) is simply the derivative of
\[
\frac{1}{a} e^\eta \left[ \frac{a^2 + a + 2}{\sqrt{1 + a^2}} \sinh \{\eta \sqrt{1 + a^2}\} + (2 + a) \cosh \{\eta \sqrt{1 + a^2}\} \right]
\]
\[
= \frac{2 - a}{a} \eta e^\eta \int_0^{2\eta} f(0, y) J_0\left(\left[\eta^2 - (y - \eta)^2\right]^{\frac{1}{2}}\right) d\gamma,
\]

(114)

with respect to \( \eta \). We can accordingly reduce equation (113) to an integral equation of Volterra's type. To perform this reduction we need only know the integral of the left-hand side of equation (113). We thus find that
\[
\frac{1}{a} e^\eta \left[ \frac{a^2 + a + 2}{\sqrt{1 + a^2}} \sinh \{\eta \sqrt{1 + a^2}\} + (2 + a) \cosh \{\eta \sqrt{1 + a^2}\} \right]
\]
\[
= \frac{2 - a}{a} (\eta - 1) e^\eta \int_0^{2\eta} f(0, y) J_0\left(\left[\eta^2 - (y - \eta)^2\right]^{\frac{1}{2}}\right) d\gamma,
\]

(115)

which is the required equation for \( f(0, y) \).
c) The solution of the integral equation (115).—To solve equation (115) we apply a Laplace transformation to this equation, i.e., we multiply both sides of the equation by $e^{-\eta}$ and integrate over $\eta$ from 0 to $\infty$. The right-hand side then becomes (cf. "Moving Atmospheres," eqs. [90]-[95])

$$\frac{1}{2s} \int_0^\infty f(0, y) \exp \left[ -y (s + s^{-1}) / 2 \right] dy,$$

while the evaluation of the Laplace transform of the left-hand side requires only elementary integrals. We find

$$\frac{1}{a} \left\{ \frac{(2 - a) (s - a)}{(s - a)^2 - (1 + a^2)} + \frac{a^2 - a + 2}{(s - a)^2 - (1 + a^2)} + \frac{(2 - a) (2 - s)}{a (s - 1)^2} \right\},$$

$$= \frac{1}{2s} \int_0^\infty f(0, y) \exp \left[ -y (s + s^{-1}) / 2 \right] dy.$$  \hspace{1cm} (117)

We can re-write the foregoing equation in the form

$$\frac{2 (s + 1) - a (s + 3) + 2a^2}{a (s^2 - 2sa - 1)} + \frac{(2 - a) (2 - s)}{a (s - 1)^2},$$

$$= \frac{1}{2s} \int_0^\infty f(0, y) \exp \left[ -y (s + s^{-1}) / 2 \right] dy.$$  \hspace{1cm} (118)

If we now let

$$s + s^{-1} = 2u,$$  \hspace{1cm} (119)

or, equivalently,

$$s = u + \sqrt{(u^2 - 1)},$$  \hspace{1cm} (120)

equation (118) provides the simple Laplace transform of $f(0, y)$, and the solution can, in principle, be found. However, since $a$ is a small quantity of the order of $10^{-4}$, the solution in the form of a series expansion in $a$ will suffice for most purposes. Accordingly, we expand the left-hand side of equation (118) in powers of $a$. We thus obtain

$$\frac{2 + a}{a} \frac{2s}{(s - 1)^2} + a \frac{4s}{(s - 1)^2} + O(a^2),$$

$$= \int_0^\infty f(0, y) \exp \left[ -y (s + s^{-1}) / 2 \right] dy.$$  \hspace{1cm} (121)

With the substitutions (119) and (120), equation (121) becomes

$$\frac{2 + a}{a} \frac{1}{u - 1} + \frac{2a}{(u - 1)^{3/2} \left[ \sqrt{(u + 1)} + \sqrt{(u - 1)} \right]} + O(a^2),$$

$$= \int_0^\infty f(0, y) e^{-uy} dy.$$  \hspace{1cm} (122)

Now the inverse Laplace transform of the first term on the left-hand side of equation (122) is

$$\frac{2 + a}{a} e^{uy};$$  \hspace{1cm} (123)
while writing the second term in the form

$$a\left[ \frac{1}{(u^2 - 1)\hat{i}} + \frac{2}{(u - 1)^{3/2}(u + 1)\hat{i}} - \frac{1}{u - 1} \right],$$

(124)

we see that its inverse Laplace transform is

$$a\{I_0(y) + 2y[I_0(y) + I_1(y)] - e^y\}.$$  

Hence,

$$f(0, y) = \frac{2 + a}{a}e^{-y}\{I_0(y) + 2y[I_0(y) + I_1(y)] - e^y\} + O(a^2).$$

(126)

The corresponding solution for $\psi_{+1}(0, y)$ is

$$\psi_{+1}(0, y) = Qa e^{-y}\{I_0(y) + 2y[I_0(y) + I_1(y)] - e^y\} + O(a^2)$$

$$\{0 \leq y \leq x_1\},$$

(127)

while $\psi_{-1}(0, y)$, of course, vanishes along this line, as required by the boundary conditions.

d) The integral equation insuring the continuity of the solution along OB and its solution (case I). Applying, next, Green’s theorem to a contour such as $JKBJ$, where $J = (x_1 - \eta, \eta)$ is a point on $OB$, and $JK$ is the characteristic $x - x_1 + \eta = y - \eta$ through $J$, we find in the usual manner that

$$2f(x_1 - \eta, \eta) = \frac{2 + a}{a}e^{ax_1} + f(x_1, 2\eta) - \int_0^{2\eta} \left( v \frac{\partial f}{\partial x} - f \frac{\partial v}{\partial x} \right)_{x=x_1} dy.$$  

(128)

In case I,

$$\frac{\partial f}{\partial x} = -f + \frac{\partial f}{\partial y} + 2e^{ax_1 + \eta}$$

(129)

along $AB$ (cf. eq. [86]). Using this in equation (128), we find, after some minor reductions, that

$$f(x_1 - \eta, \eta) = \frac{2 + a}{a}e^{ax_1} - e^{ax_1} \int_0^{2\eta} e^v[v]_{x=x_1} dy$$

$$+ \frac{1}{2} \int_0^{2\eta} f(x_1, y) \left( v + \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} \right)_{x=x_1} dy.$$  

(130)

The Riemann function appropriate to our present contour is

$$v(x, y; x_1 - \eta, \eta) = J_0([[(x - x_1 + \eta)^2 - (y - \eta)^2]^\frac{1}{2}].$$

(131)

With $v$ given by equation (131), equation (130) becomes

$$f(x_1 - \eta, \eta) = \frac{2 + a}{a}e^{ax_1} - \eta e^{ax_1 + \eta} \int_0^\pi e^\eta \cos \phi J_0(\eta \sin \phi) \sin \phi d\phi$$

$$+ \frac{1}{2} \int_0^{2\eta} f(x_1, y) J_0([\eta^2 - (y - \eta)^2]^\frac{1}{2}) dy$$

$$- \frac{1}{2} \int_0^{2\eta} f(x_1, y) J_1([\eta^2 - (y - \eta)^2]^\frac{1}{2}) \frac{2\eta - y}{[\eta^2 - (y - \eta)^2]^\frac{1}{2}} dy.$$  

(132)

But (cf. eq. [112])

$$\int_0^\pi e^\eta \cos \phi J_0(\eta \sin \phi) \sin \phi d\phi = 2.$$  

(133)
Using this result in equation (132) and substituting also for $f(x_1 - \eta, \eta)$ according to equation (101), we obtain

$$
\frac{1}{a} e^{a(x_1-\eta)} \left[ \frac{2 + a}{(1 + a^2)} \sinh \{ \eta \sqrt{(1 + a^2)} \} + (2 + a) \cosh \{ \eta \sqrt{(1 + a^2)} \} \right] \
- \frac{2 + a}{a} e^{ax_1} + 2 \eta e^{ax_1+\eta} = \frac{1}{2} \int_{0}^{2\eta} f(x_1, y) J_0 \left( \sqrt{\eta^2 - (y - \eta)^2} \right) \, dy \
- \frac{1}{2} \int_{0}^{2\eta} \frac{f(x_1, y)}{y} J_1 \left( \sqrt{2\eta y - y^2} \right) \sqrt{(2\eta y - y^2)} \, dy,
$$

which is again an integral equation of Volterra's type for $f(x_1, y)$.

To solve equation (134) we apply, as before, a Laplace transformation to this equation. We find (cf. eq. [116])

$$
\frac{1}{a} e^{ax_1} \left[ \frac{a^2 + a + 2}{(s + a)^2 - (1 + a^2)} + \frac{(2 + a)}{s + a} \right] - \frac{1}{a} e^{ax_1} \frac{2 + a}{s} \
+ \frac{2 e^{ax_1}}{(s - 1)^2} = \frac{1}{2s} \int_{0}^{\infty} f(x_1, y) \exp \left[ -y \left( s + s^{-1} \right) / 2 \right] \, dy \
- \frac{1}{2} \int_{0}^{\infty} d\eta \, e^{-\eta} \int_{0}^{2\eta} \frac{f(x_1, y)}{y} J_1 \left( \sqrt{2\eta y - y^2} \right) \sqrt{(2\eta y - y^2)} \, dy.
$$

Inverting the order of the integration in the double integral in equation (135) and introducing the variable

$$
2\eta y - y^2 = t^2
$$

we find that it reduces to

$$
\int_{0}^{\infty} \frac{dy}{y^2} f(x_1, y) e^{-\eta y/2} \int_{0}^{\infty} dt e^{t} J_1(t) e^{-s y/2}.
$$

But the integral over $t$ in (137) has the value\(^{14}\)

$$
\left( \frac{2}{s} \right)^2 e^{-\eta/2s}.
$$

Accordingly, the Laplace transform of $\int_{0}^{2\eta} \frac{f(x_1, y)}{y} J_1 \left( \sqrt{2\eta y - y^2} \right) \sqrt{(2\eta y - y^2)} \, dy$

$$
\int_{0}^{\infty} \frac{dy}{y^2} f(x_1, y) \exp \left[ -y \left( s + s^{-1} \right) / 2 \right] \, dy,
$$

and equation (135) becomes

$$
\frac{1}{a} e^{ax_1} \left[ \frac{2(s + 1) + a(s + 3) + 2a^2}{s^2 + 2s a - 1} - \frac{2 + a}{s} \right] \
\frac{s - 1}{2s^2} \int_{0}^{\infty} f(x_1, y) \exp \left[ -y \left( s + s^{-1} \right) / 2 \right] \, dy.
$$

With the substitution of equation (119) and (120) in equation (140), we shall obtain the Laplace transform of $f(x_1, y)$, and the solution becomes determinate. But again, as in

\(^{14}\) Cf. Watson, op. cit., p. 394, eq. (4).
he preceding subsection, we shall content ourselves with finding a solution in the form of a series expansion in \( a \). Thus, expanding the left-hand side of equation (140) in powers of \( a \) and after some rearranging of the terms, we find

\[
\frac{2}{x} e^{ax_1} \left[ \frac{(2+a)}{(s-1)^2} + 2a^2 \frac{s^2}{(s-1)^4} + O(a^3) \right]
= \int_0^\infty f(x_1, y) \exp \left[ -y (s + s^{-1}) / 2 \right] dy .
\]  

(141)

In terms of the variable \( u \) defined as in equation (119) or equation (120), the foregoing equation reduces to

\[
e^{ax_1} \left[ \frac{2+a}{a} \frac{1}{(u-1)} + \frac{a}{(u-1)^2} + O(a^2) \right] = \int_0^\infty f(x_1, y) e^{-yu} dy .
\]  

(142)

Hence,

\[
f(x_1, y) = \frac{2+a}{a} e^{ax_1+yu} + ay e^{ax_1+yu} + O(a^2) .
\]  

(143)

The corresponding solutions for \( \psi_+ \) and \( \psi_- \) are (cf. eq. [58])

\[
\psi_+ (x_1, y) = Qa y e^{ax_1} + O(a^2)
\]  

(144)

and

\[
\psi_- (x_1, y) = Qa y e^{ax_1} + O(a^2) .
\]  

(145)

e) The integral equation insuring the continuity of the solution along OB and its solution (case III).—In case III the boundary condition specified along AB is that (cf. eq. [87])

\[
f = \frac{2+a}{a} e^{ax_1+yu} ,
\]  

(146)

and we require to find \( (\partial f/\partial x)_{x=x_1} \). The solution proceeds along lines now familiar. Equation (128), which is valid also under our present conditions, gives

\[
2f(x_1 - \eta, \eta) = \frac{2+a}{a} e^{ax_1} (1 + e^{2\eta}) + \frac{2+a}{a} e^{ax_1} \int_0^{2\eta} e^u \left( \frac{\partial v}{\partial x} \right)_{x=x_1} dy - \int_0^{2\eta} \left( v \frac{\partial f}{\partial x} \right)_{x=x_1} dy .
\]  

(147)

The Riemann function is the same as in case I and is given in equation (131). Using this in equation (147), we find after some minor transformations that

\[
2f(x_1 - \eta, \eta) = \frac{2+a}{a} e^{ax_1} (1 + e^{2\eta}) - \frac{2+a}{a} \eta e^{ax_1+\eta} \int_0^\pi e^{\eta \cos \vartheta} J_1 (\sin \vartheta) d\vartheta - \int_0^{2\eta} \left( \frac{\partial f}{\partial x} \right)_{x=x_1} J_0 \left[ \eta^2 - (\eta - \eta^2) \right] dy .
\]  

(148)

But (cf. "Moving Atmospheres," eqs. [105] and [106])

\[
\eta \int_0^\pi e^{\eta \cos \vartheta} J_1 (\eta \sin \vartheta) d\vartheta = 2 (\cosh \eta - 1) .
\]  

(149)
Using this result in equation (148) and substituting also for \( f(x_1 - \eta, \eta) \) according to equation (101), we obtain
\[
\frac{2 + a}{a} e^{x_1 + \eta} - \frac{1}{a} e^{x_1 - \eta} \left[ \frac{a^2 + a + 2}{\sqrt{(1 + a^2)}} \sinh \left\{ \eta \sqrt{(1 + a^2)} \right\} + (2 + a) \right] \times \cosh \left\{ \eta \sqrt{(1 + a^2)} \right\} = \frac{1}{2} \int_0^{2\eta} \left( \frac{\partial f}{\partial x} \right)_{x=x_1} J_0 \left( \left[ \eta^2 - (y - \eta)^2 \right] \frac{1}{\eta} \right) dy.
\] (150)

This is the required Volterra integral equation for \( (\partial f/\partial x)_{x=x_1} \).

The solution of the integral equation (150) proceeds as in the other cases. We apply a Laplace transformation to the equation and obtain
\[
\frac{1}{a} e^{x_1} \left[ \frac{2 + a}{s - 1} - \frac{2(s + 1) + a(s + 3) + 2a^2}{s^2 + 2sa - 1} \right] = \frac{1}{2s} \int_0^\infty \left( \frac{\partial f}{\partial x} \right)_{x=x_1} \exp \left[ -y \left( s + s^{-1} / 2 \right) \right] dy.
\] (151)

And again, as in the earlier cases, we shall obtain a solution of this equation in the form of a series expansion in \( a \) by expanding the left-hand side of the equation in powers of \( a \). We thus find
\[
e^{x_1} \left[ \frac{4s}{(s - 1)^2} - a \frac{4s}{(s - 1)^3} + O(a^2) \right] = \int_0^\infty \left( \frac{\partial f}{\partial x} \right)_{x=x_1} \exp \left[ -y \left( s + s^{-1} / 2 \right) \right] dy.
\] (152)

With the substitutions (119) and (120), equation (152) becomes
\[
e^{x_1} \left[ \frac{2}{u - 1} - \frac{2a}{(u - 1)^{3/2}} \left[ \sqrt{(u + 1)} + \sqrt{(u - 1)} \right] + O(a^2) \right] = \int_0^\infty \left( \frac{\partial f}{\partial x} \right)_{x=x_1} e^{-yu} dy.
\] (153)

Equation (153) is seen to be similar in form to equation (122). We can accordingly write (cf. eqs. [124] and [125])
\[
\left( \frac{\partial f}{\partial x} \right)_{x=x_1} = 2e^{x_1 + \eta} + a e^{x_1} \left[ I_0(y) + 2y [I_0(y) + I_1(y)] - e^y \right] + O(a^2).
\] (154)

Since (cf. eq. [146])
\[
\left( \frac{\partial f}{\partial y} \right)_{x=x_1} = \frac{2 + a}{a} e^{x_1 + \eta},
\] (155)

we have
\[
g(x_1, y) = \frac{2}{a} e^{x_1 + y} + a e^{x_1} \left[ I_0(y) + 2y [I_0(y) + I_1(y)] - e^y \right] + O(a^2).
\] (156)

The corresponding solution for \( \psi_{-1}(x_1, y) \) is
\[
\psi_{-1}(x_1, y) = Q a e^{x_1 - y} \left[ I_0(y) + 2y [I_0(y) + I_1(y)] - e^y \right] + O(a^2)
\] (0 \leq y \leq x_1),
\] (157)

while \( \psi_{+1}(x_1, y) \) vanishes in accordance with the boundary conditions.

The solution in the region \( O'B'C'O' \) and its further continuation.—With the determination of \( f \) along \( CB' \) and \( f \) and its derivatives along \( BC' \), our knowledge of the
function and its derivatives along $B'C'B'C'$ is complete, and the solution in the region $O'B'C'O'B'C'$ becomes determinate; for, as in Riemann's method, by applying Green's theorem to contours such as $LMNL, PQRP, STCBUS$, we can find the solution in the regions $O'B'C, O'B'C$, and $O'B'O'C$. It is seen how a knowledge of the function along $COB$, together with the boundary conditions on $CB'$ and $BC'$, enables us to determine $f$ in the region $O'B'C'O'B'C'$, including the sides $B'O'$ and $O'C'$. It is now apparent that, in the same way, we can utilize our present knowledge of the function along $B'O'C'$ to extend the solution still further. We shall not, however, consider these further extensions of the solution in this paper but content ourselves with the solution which has been completed in the first square, $B'C'B'C'$. According to equation (56), this will suffice to determine the radiation field in all cases in which the ratio $Dv_\Delta \nu$ exceeds $\sqrt{3}$.

5. Formulae for the radiation pressure in an expanding atmosphere.—Consider a slab of material of unit cross-section and height $ds$. The normal force acting on this slab due to the absorption of radiation is

$$\frac{1}{c}2\pi \rho d z \int_{0}^{c} dv \sigma (v) \int_{0}^{T} d \sigma I \left( v + v_0 \frac{\mu}{c} \cos \vartheta, z, \vartheta \right) \cos \vartheta \sin \vartheta,$$

(158)

since the radiation in the direction $\vartheta$, which will appear to our fixed observer as having a frequency $v + v_0 (\mu/c) \cos \vartheta$, will be judged by an observer at rest with respect to the material as having a frequency $v$: it will accordingly be absorbed only as such.

The pressure $\Pi$ exerted by the radiation is therefore given by

$$\Pi (z) = \frac{1}{c}2\pi \rho \int_{0}^{c} dv \sigma (v) \int_{-1}^{+1} d \mu I \left( v + v_0 \frac{\mu}{c}, z, \vartheta \right) \mu.$$

(159)

Equation (159) is perfectly general. We shall now consider certain alternative forms of this equation suitable under various conditions and approximations.

First, replacing the integral over $\mu$ in equation (159) by a sum according to Gauss's formula for numerical quadratures, we obtain

$$\Pi (z) = \frac{1}{c}2\pi \rho \int_{0}^{c} \sigma (v) \sum_{i} a_{i} \mu_{i} I (v + v_0 \frac{\mu}{c}, z) dv.$$

(160)

The intensities which appear in equation (160) are exactly the intensities $\psi_i (v, z)$ as we have defined them in equations (17) and (18). We may therefore write

$$\Pi (z) = \frac{1}{c}2\pi \rho \int_{0}^{c} \sigma (v) \sum_{i} a_{i} \mu_{i} \psi_i (v, z) dv.$$

(161)

In the first approximation, equation (160) reduces to

$$\Pi (z) = \frac{1}{c}2\pi \mu_1 \rho \int_{0}^{c} \sigma (v) \left[ \psi_{+1} (v, z) - \psi_{-1} (v, z) \right] dv \left( \mu_1 = \frac{1}{\sqrt{3}} \right),$$

(162)

an equation which has an obvious physical interpretation.

For the particular form (26) for $\sigma (v)$, equation (162) further simplifies to

$$\Pi (z) = \frac{1}{c}2\pi \mu_1 \rho \sigma_0 \int_{v_{-\Delta v}}^{v_{+\Delta v}} \left[ \psi_{+1} (v, z) - \psi_{-1} (v, z) \right] dv,$$

(163)

or, expressing $v$ in the unit (53), we have

$$\Pi (x) = \frac{1}{c}4\pi \mu_1^2 \frac{Dv}{x_1} \rho \sigma_0 \int_{0}^{v_{1}} \left[ \psi_{+1} (x, y) - \psi_{-1} (x, y) \right] dy.$$

(164)
On the other hand, since (cf. eq. [44])

\[ x_1 = \frac{1}{2 \mu_1} l_1, \]  

(165)

where \( t_1 \) is the optical depth of the atmosphere in \( \sigma_0 \), we can re-write equation (164) in the form

\[ \Pi (x) = \frac{1}{c} 8 \pi \mu_1^2 D_\nu \frac{\rho_0}{t_1} \int_0^{\mu_1} [\psi_{+1} (x, y) - \psi_{-1} (x, y) ] \, dy. \]  

(166)

For the particular model considered in this paper,

\[ \frac{\rho \sigma_0}{t_1} = \frac{\rho_0 \kappa_c}{\tau_1}, \]  

(167)

where \( \kappa_c \) and \( \tau_1 \) refer to the ultraviolet continuum. Hence, in this case we may also write

\[ \Pi (x) = \frac{1}{c} 8 \pi \mu_1^2 D_\nu \frac{\rho_0 \kappa_c}{\tau_1} \int_0^{\mu_1} [\psi_{+1} (x, y) - \psi_{-1} (x, y) ] \, dy. \]  

(168)

6. The radiation pressure in Lyman-\( \alpha \) in an expanding planetary nebula.—With the solution of the transfer problem completed in the preceding sections, we are now in a position to answer the question raised in the introductory section, namely, as to how effective differential expansion can be in reducing the magnitude of the radiation pressure in Lyman-\( \alpha \) which will otherwise act. As we shall presently see, the case of greatest interest in this connection arises when

\[ D_\nu > \frac{2}{\mu_1} \Delta \nu \]  

(169)

or, alternatively, when (cf. eq. [56])

\[ y_1 < \frac{3}{2} x_1. \]  

(170)

When (170) is the case, for

\[ y_1 \leq x \leq x_1 - y_1, \]  

(171)

the solution for the radiation field is known explicitly and is given by equations (104) and (105) (see Figs. 3 and 4, where the regions so defined are indicated). The range (171) for \( x \) corresponds to the range

\[ \frac{y_1}{x_1} \leq \frac{l}{t_1} \leq 1 - \frac{y_1}{x_1} \]  

(172)
for the optical depth \( t \). Since (cf. eq. [56])

\[
\frac{y_1}{x_1} = \frac{1}{\mu_1} \frac{\Delta \nu}{D \nu},
\]

we can re-write (172) in the form

\[
t_1 \frac{1}{\mu_1} \frac{\Delta \nu}{D \nu} \leq t \leq t_1 \left( 1 - \frac{1}{\mu_1} \frac{\Delta \nu}{D \nu} \right).
\]

(174)

We shall refer to the part of the nebula included in the range of optical depths specified by (174) as the central part of the nebula. Similarly, we shall refer to the parts included in

\[
0 < t < t_1 \frac{1}{\mu_1} \frac{\Delta \nu}{D \nu}
\]

and

\[
t_1 \left( 1 - \frac{1}{\mu_1} \frac{\Delta \nu}{D \nu} \right) < t < t_1
\]

(175)

(176)
as the outer and the inner parts of the nebula, respectively. It should be emphasized at this point that this distinction between the outer, central, and inner parts of a nebula applies only to those cases in which the Doppler width, \( 2D \nu \), exceeds the line width, \( 2\Delta \nu \), by a factor \( 2\sqrt{3} \).

We shall now show how the selective radiation pressure in the central part of a nebula can be estimated. From equations (104) and (105) we find that

\[
\psi_{+1} - \psi_{-1} = 2Q e^{ax} \left\{ \frac{e^{-\nu_1}}{\sqrt{(1 + a^2)}} \sinh \left\{ y \sqrt{(1 + a^2)} \right\} + e^{+\nu} \cosh \left\{ y \sqrt{(1 + a^2)} \right\} - 1 \right\}
\]

(177)

and, since we have assumed that \( y_1 < \frac{1}{2}x_1 \), this solution is valid for the entire range of \( y \) for \( x \) in the range (171).

Now, according to equation (168), the radiation pressure in Lyman-\( \alpha \) is related directly to the integral of \( \psi_{+1} - \psi_{-1} \) over the line; and this can be readily found from the solution (177). We have

\[
\int_0^{y_i} (\psi_{+1} - \psi_{-1}) \, dy = 2Q e^{ax} \left\{ \frac{e^{-\nu_1}}{\alpha^2} \left[ 2 \cosh \left\{ y_1 \sqrt{(1 + a^2)} \right\} \right] \right. \\
+ \frac{2 + a^2}{\sqrt{(1 + a^2)}} \sinh \left\{ y_1 \sqrt{(1 + a^2)} \right\} - \frac{2}{\alpha^2} - y_1 \left\{ (y_1 \leq x \leq x_1 - y_1) \right\}
\]

(178)

Inserting this value in equation (168) and substituting also for \( Q \) according to equation (75), we obtain

\[
\Pi (x) = \left( \frac{\pi \rho \kappa e \nu_0}{c \nu_0 S_c} \right) \frac{D \rho}{\tau_1} \frac{\epsilon^{(\tau_1 - \tau)}}{\epsilon} \left\{ \frac{e^{-\nu_1}}{\alpha^2} \left[ 2 \cosh \left\{ y_1 \sqrt{(1 + a^2)} \right\} \right] \right. \\
+ \frac{2 + a^2}{\sqrt{(1 + a^2)}} \sinh \left\{ y_1 \sqrt{(1 + a^2)} \right\} - \frac{2}{\alpha^2} - y_1 \left\{ (y_1 \leq x \leq x_1 - y_1) \right\}
\]

(179)

where we have further used the relation \( ax = \tau \) (eq. [69]).
The full implications of equation (179) are best understood when we expand the quantity in braces in powers of \( a \) and retain the first nonvanishing term. We find in this manner that

\[
\Pi = \left( \frac{\pi \rho \kappa_c \rho_0}{c \nu_c} S_c \right) \frac{2}{c \nu_c} \frac{Dv}{\tau_1} \left\{ \alpha^2 \left[ y_1^2 - y_1 + \frac{1}{3} (1 - e^{-2y_1}) \right] + O(a^2) \right\}.
\]  

(180)

A further simplification of this equation is possible. Since \( y_1 \) is a large quantity of the order of \( 10^3 \) or \( 10^4 \), we may write, to a sufficient accuracy,

\[
\Pi = \left( \frac{\pi \rho \kappa_c \rho_0}{c \nu_c} S_c \right) \frac{1}{2} \mu_1 \frac{Dv}{\tau_1} a^2 y_1^2.
\]  

(181)

But (cf. eqs. [69] and [173])

\[
a y_1 = \frac{\Delta v}{\mu_1} a x_1 = \frac{\Delta v}{\mu_1} \tau_1.
\]  

(182)

Equation (181) therefore reduces to

\[
\Pi(x) = \left( \frac{\pi \rho \kappa_c \rho_0}{c \nu_c} S_c \right) \frac{\Delta v}{2} \frac{Dv}{\tau_1} e^{-(\tau_1 - \tau)} (y_1 \leq x \leq x_1 - y_1).
\]  

(183)

In other words, in the central part of a nebula the radiation pressures due to Lyman-\( a \) and the Lyman continuum are of the same order of magnitude.

Now for the case \( \rho = 0 \) (no re-emission in the Lyman continuum), the selective radiation pressure which will act in a static nebula can be readily written down. We must clearly have

\[
\Pi_{\text{static}} = \left( \frac{\pi \rho \sigma_0 \rho_0}{c \nu_c} S_c \right) [1 - e^{-(\tau_1 - \tau)}].
\]  

(184)

From a comparison of equations (183) and (184) the remarkable result emerges that if a differential expansion to the extent required by equation (169) exists, then in the central part of the nebula the radiation pressure in Lyman-\( a \) is effectively cut down by a factor of the order of \( 10^4 \).

We cannot, of course, expect that the large reduction in the radiation pressure achieved in the central part will be maintained throughout the nebula. We should rather expect certain "edge effects." To investigate the nature of these edge effects in detail, we need the radiation field in regions where the solution can be found only by quadratures (cf. §4, subsec. f). However, we can estimate the radiation pressure acting on the outer and the inner boundaries of the nebula, and these might provide some indications.

At \( \tau = 0 \) the inward intensity \( \psi_{-1}(0, y) \) vanishes, and the radiation pressure depends only on the integral of the outward intensity \( \psi_{+1}(0, y) \) over the line. Using the solution (127) for \( \psi_{+1}(0, y) \), we have, for the radiation pressure acting at \( \tau = 0 \),

\[
\Pi(0) \left( \frac{\pi \rho \kappa_c \rho_0}{c \nu_c} S_c \right) \mu_1^2 \frac{Dv}{\tau_1} e^{-\tau_1} \times \left[ a \int_0^{y_1} e^{-y} \left\{ I_0(y) + 2y \left[ I_0(y) + I_1(y) \right] - e^y \right\} dy + O(a^2) \right].
\]  

(185)

It does not appear that the integrals over the Bessel functions occurring in equation (185) can be evaluated explicitly. However, its asymptotic value for large \( y_1 \) can be readily found. We have

\[
\int_0^{y_1} e^{-y} \left\{ I_0(y) + 2y \left[ I_0(y) + I_1(y) \right] - e^y \right\} dy \rightarrow \frac{4}{(2\pi)^{1/2}} \int_0^{y_1} y^4 dy = \frac{8}{3 (2\pi)^{1/2}} y_1^{5/2}.
\]  

(186)
Using this result in equation (185), we obtain

\[ \Pi (0) \approx \left( \frac{\pi \rho \kappa_e v_0}{c \nu_e} S_e \right) \frac{8}{3} \left( \frac{\mu_{\Pi} \tau_1}{Dv} \right)^{1/3} \frac{Dv}{\tau_1} \frac{e^{-\tau_1}}{(\alpha y_1)^{3/2}}. \]  
(187)

Substituting for \( \alpha y_1 \) from equation (182), we finally have

\[ \Pi (0) \approx \left( \frac{\pi \rho \kappa_e v_0}{c \nu_e} S_e \right) \frac{8}{3} \left( \frac{\mu_{\Pi} \tau_1}{2 \pi a Dv} \right)^{1/4} e^{-\tau_1}. \]  
(188)

Accordingly, at \( \tau = 0 \), the radiation pressure due to Lyman-\( \alpha \) is cut down only by a factor of the order of \( \sqrt{(Dv/\alpha D\nu)} \). We may, therefore, expect that in the outer parts of a nebula the radiation pressure due to Lyman-\( \alpha \) will be appreciable, though it is not likely to exceed the radiation pressure due to the Lyman continuum by any very large factor.

The radiation pressure acting at \( \tau = \tau_1 \) will depend on the boundary conditions here. If we suppose that the inner boundary is at rest with respect to the central star (case I), the radiation pressure at \( \tau = \tau_1 \) vanishes identically, simply in virtue of the boundary conditions. But this will not clearly be true of the rest of the inner part, and it is likely that a true estimate of the radiation pressure which may act in these regions is to be found from that acting at \( \tau = \tau_1 \) in our case III. It will be recalled that this case arises when the Doppler shift, owing to the velocity at the inner boundary, exceeds the line width by a factor \( \sqrt{3}/2 \) (eq. [34]). When this happens, \( \psi_{+1} \) vanishes at \( \tau = \tau_1 \), but there is a net flux directed inward owing to the nonvanishing of \( \psi_{-1} \). Since the solution (157) for \( \psi_{-1}(x_1, y) \) in case III is similar in form to the solution (127) for \( \psi_{+1}(0, y) \), we can at once write down (cf. eq. [188])

\[ \Pi (\tau_1) \approx \left( \frac{\pi \rho \kappa_e v_0}{c \nu_e} S_e \right) \frac{8}{3} \left( \frac{\mu_{\Pi} \tau_1}{2 \pi a Dv} \right)^{1/4} \]  
(189)

which has the same validity as equation (188). A comparison of equations (188) and (189) suggests that, in general, we may expect that the order of magnitude of the radiation pressure acting in the outer and the inner parts will be about the same.

We may summarize the results of our discussion so far in the following general terms.

In a static nebula, selective radiation pressure in Lyman-\( \alpha \) is so large that we may expect a differential expansion to set in. This will have a tendency to reduce the magnitude of the radiation pressure acting. But the reduction, while it would set in gradually, becomes suddenly very effective when the differential expansion present exceeds a certain critical value. On the basis of our calculations we expect reduction factors of the order of \( 10^4 \) over at least parts of the nebula when the Doppler shift due to the difference in velocities at the inner and outer boundaries exceeds the undisplaced line width by a factor of the order of 3.5. When this happens, the nebula may be divided into a central part and the edges. In the central part, selective radiation pressure is cut down effectively to zero, while at the edges it may still be appreciable.

An exact discussion of the dynamics of a planetary nebula, which will properly take into account the rather complex manner in which radiation pressure in Lyman-\( \alpha \) acts, is likely to be a problem of considerable difficulty. But we may expect that a nebula, initially static, will "feel its way" to a state in which the parts we have described as central are fairly extensive; for such a state will have the character of quasi-stationariness, and dissipation, to the extent that it is present, will be confined only to the inner and the outer edges. We can expect such a quasi-stationary state to be reached, since the effectiveness with which selective radiation pressure acts is controlled very sensitively when the differential expansion has reached a certain stage.

7. The role of radiation pressure in Lyman-\( \alpha \) in other astrophysical problems.—The
manner and effectiveness of operation of the radiation pressure in Lyman-\(\alpha\) which we have described in the preceding section is likely to have a bearing on other astrophysical problems besides that of emission nebulae. Here we shall make reference to only two such groups of problems.

The first of these relates to the problem of gaseous shells surrounding Be stars. As Struve\textsuperscript{15} has pointed out, the clue to the variety of questions which a study of these interesting stars raises is probably to be found in the manner in which the radiation pressure in Lyman-\(\alpha\) acts on these shells. While a detailed discussion of these questions is beyond the scope of this investigation, we may refer particularly to the fact to which Struve has called attention, namely, that the atmospheres of these stars appear to consist of three separate layers: a normal reversing layer, a relatively stationary but extensive shell, and an extreme outer part which is expanding. This division of the atmosphere into three parts is strongly reminiscent of our own distinction of the inner, the central, and the outer parts of a planetary nebula. It is, moreover, not inconceivable that the very capriciousness of the phenomena exhibited by the Be stars is to be understood in terms of the extreme sensitiveness with which the effectiveness of the radiation pressure in Lyman-\(\alpha\) is controlled by the extent of the differential expansion present.

A second group of problems to which we wish to make reference relates to the spheres of ionized hydrogen surrounding luminous early-type stars. As B. Strömgren\textsuperscript{16} has particularly called attention, the conversion of the radiation in the Lyman continuum into radiation in Lyman-\(\alpha\) (in the manner of Zanstra’s theory) is a principal feature of this problem. We cannot, therefore, escape the conclusion that something of the sort which we expect to happen in a planetary nebula also happens in these ionized hydrogen regions. And it is interesting to speculate on the bearing of this, in turn, on the still larger problems of the interstellar clouds.

From the foregoing brief discussion it is apparent that the clue to the understanding of a great many astrophysical problems may lie in the very remarkable manner in which the radiation pressure in Lyman-\(\alpha\) operates in an expanding atmosphere.

\textsuperscript{15} Ap. J., 95, 134, 1942. The writer is indebted to Dr. O. Struve for illuminating discussions of these and other stellar spectroscopic problems.