ON THE RADIATIVE EQUILIBRIUM OF A STELLAR ATMOSPHERE. IX

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ABSTRACT

In this paper the problem of diffuse reflection by a semi-infinite plane-parallel atmosphere is considered along the lines of the earlier papers of this series. Explicit solutions are obtained for the cases when the scattering of radiation by the atmosphere takes place in accordance with the phase functions $\lambda(1 + x \cos \Theta)$, $(0 < \lambda \leqslant 1,\ -1 \leqslant x \leqslant 1)$, and $(1 + \cos^2 \Theta)$. It is shown how simple, closed expressions can be found for the angular distribution of the reflected radiation in a general nth approximation. Only certain simple algebraic equations need be solved for their "characteristic roots" to bring the solutions to their numerical forms.

Tables of certain constants and functions required for the practical use of the solutions are provided.

1. *Introduction.*—The phenomenon of diffuse reflection by a semi-infinite plane-parallel atmosphere is of particular interest for astrophysics. It occurs in the study of planetary illumination and of the reflection effect in eclipsing binaries. And it is basic for the interpretation of reflection nebulae. While these various aspects of the phenomenon have been the subject of numerous investigations, it is fair to say that, except in the context of the reflection effect in binaries,[1] the fundamental problem in the theory of radiative transfer has not received an adequately satisfactory treatment. However, interest in the general problem has been revived by a series of recent papers by V. A. Ambarzumian,[2] who has tried to eliminate the explicit solution of the equation of transfer by concentrating on the angular distribution of the reflected radiation alone. In this manner he has been able to reduce the problem of characterizing the reflected radiation to the solution of a number of relatively simple integral equations, which he then seeks to solve numerically by an iteration method. In this paper we shall show how, for the particular cases considered by Ambarzumian, the method which has been developed in the earlier papers of this series[3] can be successfully applied to yield explicit solutions for the angular distribution of the reflected radiation. To reduce these solutions to their numerical forms, it is necessary only to solve certain algebraic equations[4] for "characteristic roots." The method presented in this paper has, accordingly, an advantage over Ambarzumian's in that, in addition to reducing the necessary numerical work very considerably, it also yields simple, closed expressions for the solution in a general nth approximation.

As we have already indicated, the basic problem is that of the radiative equilibrium of a semi-infinite plane-parallel atmosphere exposed to a parallel beam of radiation of flux πF per unit area, normal to itself, and incident at an angle β, normal to the boundary of the atmosphere (see Fig. 1 in paper VIII). Moreover, in considering the general problem of diffuse reflection, it is necessary that we do not restrict ourselves to the case of isotropic scattering but allow for the anisotropy of the scattered radiation in accordance with a "phase function" $p(\cos \Theta)$. The meaning of this phase function is that

$$p\,(\cos \Theta)\,\frac{d\omega}{4\pi} \tag{1}$$

[1] Cf. S. Chandrasekhar, *Ap. J.*, **101**, 348, 1945; also, C. U. Cesco and J. Sahade (in press).

[2] *J. Physics Acad. Sci. U.S.S.R.*, **8**, 64, 1944, and references given in this paper.

[3] See particularly *Ap. J.*, **100**, 76, 117, 1944, and **101**, 328, 348, 1945. These papers will be referred to as "II," "III," "VII," and "VIII," respectively.

[4] The degree of these equations depends on the order of the approximation in which the solutions are sought.

governs the probability that a pencil of radiation will be scattered in a direction inclined at an angle Θ to the incident direction and confined to an element of solid angle $d\omega$. On these assumptions, the equation of transfer, in a standard notation, is

$$\left.\begin{aligned}\cos\vartheta\,\frac{dI(\tau,\vartheta,\varphi)}{d\tau} = I(\tau,\vartheta,\varphi) - \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} I(\tau,\vartheta',\varphi')\,p\,(\cos\vartheta\cos\vartheta' \\ +\sin\vartheta\sin\vartheta'\cos[\varphi-\varphi'])\sin\vartheta' d\vartheta' d\varphi' - \tfrac{1}{4}Fe^{-\tau\sec\beta}p\,(-\cos\vartheta\cos\beta \\ +\sin\vartheta\sin\beta\cos\varphi)\,,\end{aligned}\right\} \quad (2)$$

where it will be noted that we have assumed (as it entails no loss of generality) that the radiation πF is incident along the direction $\vartheta = \pi - \beta$ and $\varphi = 0$.

In this paper we shall restrict ourselves to the consideration of the following two phase functions:

$$p\,(\cos\Theta) = \lambda\,(1 + x\cos\Theta) \qquad (0 < \lambda \leqslant 1, 0 \leqslant |x| \leqslant 1)\,, \quad (3)$$

where λ and x are two constants and

$$p\,(\cos\Theta) = \tfrac{3}{4}\,(1 + \cos^2\Theta)\,. \quad (4)$$

The phase function (4) corresponds, of course, to Rayleigh's law of scattering. But the phase function (3), in addition to introducing an asymmetry in the backward and the forward scattering, allows also for the conversion on scattering of the radiant into other forms of energy in terms of the "albedo," λ. The study of diffuse reflection with the phase function (3) is particularly suitable for the analysis of planetary illumination.[5] This is, moreover, also the case for which Ambarzumian has obtained some numerical results. We shall, accordingly, study this case in some detail. In a later paper we shall outline the method for solving the equation of transfer (2) with a general phase function and relate our method to Ambarzumian's.

I. DIFFUSE REFLECTION IN ACCORDANCE WITH THE PHASE FUNCTION $\lambda(1 + x\cos\Theta)$

2. *The reduction of the equation of transfer.*—For a phase function of the form (3), equation (2) becomes

$$\left.\begin{aligned}\cos\vartheta\,\frac{dI(\tau,\vartheta,\varphi)}{d\tau} = I(\tau,\vartheta,\varphi) - \frac{\lambda}{4\pi}\int_0^{\pi}\int_0^{2\pi} I(\tau,\vartheta',\varphi')\,[1 + x\,(\cos\vartheta\cos\vartheta' \\ +\sin\vartheta\sin\vartheta'\cos[\varphi-\varphi'])\,]\sin\vartheta' d\vartheta' d\varphi' - \frac{\lambda}{4}Fe^{-\tau\sec\beta}[1 + x\,(-\cos\vartheta\cos\beta \\ +\sin\vartheta\sin\beta\cos\varphi)\,]\,.\end{aligned}\right\} \quad (5)$$

The form of equation (5) immediately suggests that we seek a solution in the form

$$I(\tau,\vartheta,\varphi) = I^{(0)}(\tau,\vartheta) + I^{(1)}(\tau,\vartheta)\cos\varphi\,. \quad (6)$$

Substituting this form for $I(\tau, \vartheta, \varphi)$ in equation (5), we find that the equation breaks up into two equations for $I^{(0)}$ and $I^{(1)}$, respectively. We have

$$\left.\begin{aligned}\mu\frac{dI^{(0)}}{d\tau} = I^{(0)} - \tfrac{1}{2}\lambda\int_{-1}^{+1} I^{(0)}(\tau,\mu')\,d\mu' - \tfrac{1}{2}x\lambda\mu\int_{-1}^{+1} I^{(0)}(\tau,\mu')\,\mu' d\mu' \\ - \tfrac{1}{4}\lambda Fe^{-\tau\sec\beta}(1 - x\mu\cos\beta)\end{aligned}\right\} \quad (7)$$

[5] See a forthcoming paper in which the solutions obtained in this paper are used to interpret the known data on planetary illumination.

and

$$\left.\begin{aligned}\mu\frac{dI^{(1)}}{d\tau}=I^{(1)}-\tfrac{1}{4}x\lambda\sqrt{(1-\mu^2)}\int_{-1}^{+1}I^{(1)}(\tau,\mu')\sqrt{(1-\mu'^2)}\,d\mu'\\-\tfrac{1}{4}x\lambda F e^{-\tau\sec\beta}\sin\beta\sqrt{(1-\mu^2)}\,,\end{aligned}\right\}\quad(8)$$

where we have written μ for $\cos\vartheta$. We shall now show how the two foregoing equations for $I^{(0)}$ and $I^{(1)}$ can be solved.

3. *The solution of equation* (7) *in the* n*th approximation.*—As in paper II, we replace the integrals which occur on the right-hand side of equation (7) by sums according to Gauss's formula of numerical quadratures and obtain an equivalent system of linear equations. In the nth approximation this is

$$\left.\begin{aligned}\mu_i\frac{dI_i^{(0)}}{d\tau}=I_i^{(0)}-\tfrac{1}{2}\lambda\Sigma a_jI_j^{(0)}-\tfrac{1}{2}x\lambda\mu_i\Sigma a_j\mu_jI_j^{(0)}\\-\tfrac{1}{4}\lambda F e^{-\tau\sec\beta}\,(1-x\mu_i\cos\beta)\qquad(i=\pm1,\ldots,\pm n)\,,\end{aligned}\right\}\quad(9)$$

where the various symbols have the same meanings as in paper II.

In solving the system of equations represented by equation (9) we first seek the general solution of the associated homogeneous system

$$\mu_i\frac{dI_i}{d\tau}=I_i-\tfrac{1}{2}\lambda\Sigma a_jI_j-\tfrac{1}{2}x\lambda\mu_i\Sigma a_j\mu_jI_j\qquad(i=\pm1,\ldots,\pm n)\quad(10)$$

and then add to it a particular integral of the nonhomogeneous system.

To obtain the different linearly independent solutions of the system (10), we proceed as follows: Setting

$$I_i=g_ie^{-k\tau}\qquad(i=\pm1,\ldots,\pm n)\quad(11)$$

in equation (10), where the g_i's and k are constants, unspecified for the present, we obtain

$$(1+\mu_ik)\,g_i=\tfrac{1}{2}\lambda\Sigma a_jg_j+\tfrac{1}{2}x\lambda\mu_i\Sigma a_j\mu_jg_j\,.\quad(12)$$

Equation (12) implies that g_i must be expressible in the form

$$g_i=\frac{A+B\mu_i}{1+\mu_ik}\qquad(i=\pm1,\ldots,\pm n)\,,\quad(13)$$

where A and B are two constants independent of i. Substituting equation (13) back into equation (12), we find

$$A+B\mu_i=\tfrac{1}{2}\lambda\Sigma\frac{a_j(A+B\mu_j)}{1+\mu_jk}+\tfrac{1}{2}x\lambda\mu_i\Sigma\frac{a_j\mu_j(A+B\mu_j)}{1+\mu_jk}\,.\quad(14)$$

Since this equation must be valid for all i's ,we must require that

$$A=\tfrac{1}{2}\lambda\,(AD_0+BD_1)\quad(15)$$

and

$$B=\tfrac{1}{2}x\lambda\,(AD_1+BD_2)\quad(16)$$

where we have introduced the quantity

$$D_m=\Sigma\frac{a_j\mu_j{}^m}{1+\mu_jk}\,.\quad(17)$$

These D_m's satisfy the recursion formula (VIII, eq. [52])

$$D_m = \frac{1}{k}\left(\frac{2}{m}\epsilon_{m,\,\text{odd}} - D_{m-1}\right); \tag{18}$$

in particular,

$$D_1 = \frac{1}{k}(2 - D_0) \quad \text{and} \quad D_2 = -\frac{1}{k} D_1 = -\frac{1}{k^2}(2 - D_0). \tag{19}$$

Returning to equations (15) and (16), we can re-write them in the forms

$$(2 - \lambda D_0) A - \lambda D_1 B = 0 \tag{20}$$

and

$$x\lambda D_1 A + (x\lambda D_2 - 2) B = 0. \tag{21}$$

In order that A and B do not vanish identically, we must require that

$$(2 - \lambda D_0)(x\lambda D_2 - 2) + x\lambda^2 D_1^2 = 0. \tag{22}$$

Using the recurrence relation (18), the foregoing equation can be reduced to give

$$2 = \lambda\left[D_0 - \frac{x(1-\lambda)}{k} D_1\right], \tag{23}$$

or equivalently (cf. eq. [19])

$$2 = \lambda\left[D_0 + x(1-\lambda) D_2\right]. \tag{24}$$

In other words, k must be a root of the equation

$$2 = \lambda \Sigma \frac{a_j[1 + x(1-\lambda)\mu_j^2]}{1 + \mu_j k}; \tag{25}$$

or, since $a_j = a_{-j}$, and $\mu_j = -\mu_{-j}$,

$$1 = \lambda \sum_{j=1}^{n} \frac{a_j[1 + x(1-\lambda)\mu_j^2]}{1 - \mu_j^2 k^2}. \tag{26}$$

This is the characteristic equation for k. Equation (26) is of order n in k^2 and for $\lambda \neq 1$[6] admits of $2n$ distinct nonvanishing roots, which must occur in pairs as

$$\pm k_a \qquad (a = 1, \ldots, n). \tag{27}$$

From equation (20) (or [21]) we now conclude that

$$B = \frac{2 - \lambda D_0}{\lambda D_1} A, \tag{28}$$

or, according to equation (23), that

$$B = -\frac{x(1-\lambda)}{k} A. \tag{29}$$

Hence (cf. eq. [13])

$$g_i = \text{constant}\ \frac{1 - x(1-\lambda)\mu_i/k}{1 + \mu_i k} \qquad (i = \pm 1, \ldots, \pm n). \tag{30}$$

Thus the homogeneous system of equations (10) admits the $2n$ linearly independent integrals

$$I_i = \text{constant}\ \frac{1 \mp x(1-\lambda)\mu_i/k_a}{1 \pm \mu_i k_a} e^{\mp k_a \tau} \qquad \begin{pmatrix} i = \pm 1, \ldots, \pm n \\ a = 1, \ldots, n \end{pmatrix}. \tag{31}$$

[6] We treat the case $\lambda = 1$ separately (cf. n. 8, p. 177).

The general solution can therefore be written in the form

$$\left.\begin{aligned} I_i = \tfrac{1}{4}\lambda F \Bigg\{ & \sum_{a=1}^{n} \frac{M_a\left[1 - x\,(1-\lambda)\,\mu_i/k_a\right]}{1+\mu_i k_a}\, e^{-k_a\tau} \\ & + \sum_{a=1}^{n} \frac{M_{-a}\left[1 + x\,(1-\lambda)\,\mu_i/k_a\right]}{1-\mu_i k_a}\, e^{+k_a\tau}\Bigg\} \qquad (i = \pm 1, \ldots, \pm n)\,, \end{aligned}\right\} \tag{32}$$

where $M_{\pm a}$ ($a = 1, \ldots, n$) are $2n$ constants of integration.

To complete the solution of the nonhomogeneous system (9), we need a particular integral. This can be found in the following manner:

Setting

$$I_i^{(0)} = \tfrac{1}{4}\lambda F h_i e^{-\tau \sec\beta} \qquad (i = \pm 1, \ldots, \pm n) \tag{33}$$

in equation (9) (the h_i's are certain constants unspecified for the present), we verify that we must have

$$(1 + \mu_i \sec\beta)\, h_i = \tfrac{1}{2}\lambda \Sigma a_j h_j + \tfrac{1}{2} x \lambda \mu_i \Sigma a_j \mu_j h_j + 1 - x\mu_i \cos\beta\,. \tag{34}$$

Equation (34) implies that the constants h_i must be expressible in the form

$$h_i = \frac{\gamma + \mu_i \delta}{1 + \mu_i \sec\beta} \qquad (i = \pm 1, \ldots, \pm n)\,, \tag{35}$$

where the constants γ and δ have to be determined in accordance with the relation

$$\gamma + \mu_i \delta = \left[\tfrac{1}{2}\lambda\,(\gamma E_0 + \delta E_1) + 1\right] + \mu_i\left[\tfrac{1}{2} x \lambda\,(\gamma E_1 + \delta E_2) - x\cos\beta\right], \tag{36}$$

where we have used E_m to denote

$$E_m = \Sigma \frac{a_j \mu_j^m}{1 + \mu_j \sec\beta}\,. \tag{37}$$

From equation (36) we conclude that the equations which determine γ and δ are

$$(2 - \lambda E_0)\,\gamma - \lambda E_1 \delta - 2 = 0 \tag{38}$$

and

$$x\lambda E_1 \gamma + (x\lambda E_2 - 2)\,\delta - 2x\cos\beta = 0\,. \tag{39}$$

Solving these equations, we find

$$\gamma = \frac{1}{1 - \lambda \displaystyle\sum_{j=1}^{n} \frac{a_j\left[1 + x\,(1-\lambda)\,\mu_j^2\right]}{1 - \mu_j^2 \sec^2\beta}} \tag{40}$$

and

$$\delta = -\gamma x\,(1-\lambda)\cos\beta\,. \tag{41}$$

In reducing the solutions for γ and δ to the foregoing forms, use has been made of the recursion formula (cf. eq. [18])

$$E_m = \cos\beta\left(\frac{2}{m}\,\epsilon_{m,\,\mathrm{odd}} - E_{m-1}\right), \tag{42}$$

which the E_m's satisfy.

The expression (40) for γ has a simple representation in terms of the roots $k_1^2, \ldots, k_n^2$ of the characteristic equation (26);[7] for, considering the function

$$T(z) = 1 - \lambda \sum_{j=1}^{n} \frac{a_j [1 + x(1-\lambda)\mu_j^2]}{1 - \mu_j^2 z}, \tag{43}$$

we observe that it vanishes for

$$z = k_\alpha^2 \qquad (\alpha = 1, \ldots, n). \tag{44}$$

Accordingly,

$$\prod_{j=1}^{n} (1 - \mu_j^2 z) T(z), \tag{45}$$

which is a polynomial of degree n in z, cannot differ from

$$\prod_{\alpha=1}^{n} (z - k_\alpha^2) \tag{46}$$

except by a constant factor. The constant of proportionality can be determined by comparing the coefficients of the highest powers. In this manner we find that

$$T(z) = (-1)^n \mu_1^2 \ldots \mu_n^2 \frac{\prod_{\alpha=1}^{n} (z - k_\alpha^2)}{\prod_{j=1}^{n} (1 - \mu_j^2 z)}. \tag{47}$$

Hence,

$$\gamma = \frac{1}{T(\sec^2 \beta)} = \frac{(-1)^n}{\mu_1^2 \ldots \mu_n^2} \frac{\prod_{j=1}^{n} (1 - \mu_j^2 \sec^2 \beta)}{\prod_{\alpha=1}^{n} (\sec^2 \beta - k_\alpha^2)}, \tag{48}$$

or, somewhat differently,

$$\gamma = \frac{(-1)^n}{\mu_1^2 \ldots \mu_n^2} \frac{\prod_{j=1}^{n} (\cos^2 \beta - \mu_j^2)}{\prod_{\alpha=1}^{n} (1 - k_\alpha^2 \cos^2 \beta)}. \tag{49}$$

In terms of the functions (cf. II, eqs. [58] and [59])

$$P(\mu) = \prod_{j=1}^{n} (\mu - \mu_j) \tag{50}$$

and

$$R(\mu) = \prod_{\alpha=1}^{n} (1 - k_\alpha \mu), \tag{51}$$

[7] The analysis which follows is similar to that in paper VIII, following eq. (40).

we can express γ alternatively in the form

$$\gamma = \frac{1}{\mu_1^2 \ldots \mu_n^2} \frac{P(\cos\beta)\,P(-\cos\beta)}{R(\cos\beta)\,R(-\cos\beta)} . \tag{52}$$

When we return to equations (33) and (35), it is seen that the nonhomogeneous system (9) admits the particular integral (cf. eq. [41])

$$I_i^{(0)} = \tfrac{1}{4}\lambda F e^{-\tau \sec\beta} \gamma \frac{1 - x(1-\lambda)\mu_i \cos\beta}{1 + \mu_i \sec\beta} \qquad (i = \pm 1, \ldots, \pm n) . \tag{53}$$

Adding to this particular integral the general solution of the homogeneous system which is bounded for $\tau \to \infty$, we have

$$\left.\begin{aligned} I_i^{(0)} = \tfrac{1}{4}\lambda F \Bigg[\sum_{a=1}^{n} & \frac{M_a [1 - x(1-\lambda)\mu_i/k_a]\, e^{-k_a \tau}}{1 + \mu_i k_a} \\ & + \frac{\gamma [1 - x(1-\lambda)\mu_i \cos\beta]\, e^{-\tau \sec\beta}}{1 + \mu_i \sec\beta} \Bigg] \qquad (i = \pm 1, \ldots, \pm n) , \end{aligned}\right\} \tag{54}$$

where the constants M_a $(a = 1, \ldots, n)$ have to be determined from the boundary conditions at $\tau = 0$.

At $\tau = 0$ we have no incident radiation derived from the material. Accordingly, we should require that

$$I_{-i}^{(0)} = 0 \qquad \text{at} \qquad \tau = 0 \qquad \text{and for} \quad i = 1, \ldots, n . \tag{55}$$

Hence the equations which determine M_a are

$$\left.\sum_{a=1}^{n} \frac{M_a [1 + x(1-\lambda)\mu_i/k_a]}{1 - \mu_i k_a} + \frac{\gamma [1 + x(1-\lambda)\mu_i \cos\beta]}{1 - \mu_i \sec\beta} = 0 \qquad (i = 1, \ldots, n) . \right\} \tag{56}$$

If we now let $G(\mu)$ denote

$$G(\mu) = \sum_{a=1}^{n} \frac{M_a [1 + x(1-\lambda)\mu/k_a]}{1 - \mu k_a} + \frac{\gamma [1 + x(1-\lambda)\mu \cos\beta]}{1 - \mu \sec\beta} , \tag{57}$$

then

$$G(\mu_i) = 0 \qquad (i = 1, \ldots, n) . \tag{58}$$

The angular distribution of the part of the reflected radiation corresponding to $I^{(0)}$ (cf. eq. [6]) can be found from the source function

$$\mathfrak{J}^{(0)} = \tfrac{1}{2}\lambda \Sigma a_j I_j^{(0)} + \tfrac{1}{2} x \lambda \mu \Sigma a_j \mu_j I_j^{(0)} + \tfrac{1}{4} F e^{-\tau \sec\beta} (1 - x\mu \cos\beta) , \tag{59}$$

according to the formula

$$I^{(0)}(0, \mu) = \int_0^\infty \mathfrak{J}^{(0)}(\tau)\, e^{-\tau/\mu} \frac{d\tau}{\mu} . \tag{60}$$

The quantities on the right-hand side of equation (59) can readily be evaluated in terms of the solution (54). We find

$$I^{(0)}(0, \mu) = \tfrac{1}{4}\lambda F G(-\mu) . \tag{61}$$

This is in agreement with solution (54) for $\tau = 0$ and at the points of the Gaussian division, $\mu = u_i$.

We shall now show how an explicit formula for $G(\mu)$ can be found without having to solve for the constants M_a.

Consider the function

$$(1-\mu \sec \beta) R(\mu) G(\mu) = (1-\mu \sec \beta) \prod_{a=1}^{n} (1-k_a \mu) G(\mu) . \tag{62}$$

This is a polynomial of degree $n+1$ in μ which vanishes for $\mu = \mu_i$, $i = 1, \ldots, n$. Consequently, there must exist a proportionality of the form

$$(1-\mu \sec \beta) R(\mu) G(\mu) \propto P(\mu)(\mu + c) , \tag{63}$$

where c is some constant. The constant of proportionality can be found from a comparison of the coefficients of the highest powers of μ on either side. On the left hand the coefficient of μ^{n+1} is

$$(-1)^n k_1 \ldots k_n x (1-\lambda) \left[\sum_{a=1}^{n} \frac{M_a}{k_a^2} \sec \beta + \gamma \cos \beta \right], \tag{64}$$

while on the right-hand side it is unity. Hence,

$$\left.\begin{aligned} G(\mu) = (-1)^n k_1 \ldots k_n x (1-\lambda) & \left[\sum_{a=1}^{n} \frac{M_a}{k_a^2} \sec \beta + \gamma \cos \beta \right] \\ & \times \frac{P(\mu)}{R(\mu)} \frac{\mu + c}{1-\mu \sec \beta} . \end{aligned}\right\} \tag{65}$$

Now, according to equation (57),

$$\lim_{\mu \to \cos \beta} (1-\mu \sec \beta) G(\mu) = \gamma [1 + x(1-\lambda) \cos^2 \beta] . \tag{66}$$

Substituting for γ and $G(\mu)$ from equations (52) and (65) in equation (66), we obtain

$$\left.\begin{aligned} (-1)^n k_1 \ldots k_n x (1-\lambda) & \left[\sum_{a=1}^{n} \frac{M_a}{k_a^2} \sec \beta + \gamma \cos \beta \right] \frac{P(\cos \beta)}{R(\cos \beta)} (\cos \beta + c) \\ & = \frac{1 + x(1-\lambda)\cos^2 \beta}{\mu_1^2 \ldots \mu_n^2} \frac{P(\cos \beta) P(-\cos \beta)}{R(\cos \beta) R(-\cos \beta)} , \end{aligned}\right\} \tag{67}$$

or

$$\left.\begin{aligned} (-1)^n k_1 \ldots k_n x (1-\lambda) & \left[\sum_{a=1}^{n} \frac{M_a}{k_a^2} \sec \beta + \gamma \cos \beta \right] \\ & = \frac{1 + x(1-\lambda)\cos^2 \beta}{\mu_1^2 \ldots \mu_n^2} \frac{P(-\cos \beta)}{R(-\cos \beta)} \frac{1}{\cos \beta + c} . \end{aligned}\right\} \tag{68}$$

In virtue of this relation, equation (65) becomes

$$G(\mu) = \frac{1 + x(1-\lambda)\cos^2 \beta}{\mu_1^2 \ldots \mu_n^2} \frac{P(-\cos \beta) P(\mu)}{R(-\cos \beta) R(\mu)} \frac{\mu + c}{(\cos \beta + c)(1-\mu \sec \beta)} . \tag{69}$$

Equation (69) specifies $G(\mu)$ completely except for the constant c, which remains to be determined.

From equations (57) and (69) it follows that

$$G(0) = \sum_{a=1}^{n} M_a + \gamma = (-1)^n \frac{1 + x(1-\lambda)\cos^2\beta}{\mu_1 \ldots \mu_n} \frac{P(-\cos\beta)}{R(-\cos\beta)} \frac{c}{\cos\beta + c} . \tag{70}$$

On the other hand, since (cf. eq. [57])

$$[1 + x(1-\lambda) k_a^{-2}] M_a = \lim_{\mu \to k_a^{-1}} (1 - k_a \mu) G(\mu) , \tag{71}$$

we have

$$M_a = \frac{1 + x(1-\lambda)\cos^2\beta}{\mu_1^2 \ldots \mu_n^2} \frac{P(-\cos\beta)}{R(-\cos\beta)} \frac{c + k_a^{-1}}{\cos\beta + c} m_a , \tag{72}$$

where

$$m_a = \frac{P(k_a^{-1})}{[1 + x(1-\lambda) k_a^{-2}](1 - k_a^{-1}\sec\beta) R_a(k_a^{-1})} . \tag{73}$$

In equation (73) we have introduced the function $R_a(\mu)$, which is obtained from $R(\mu)$ by omitting the factor $(1 - k_a \mu)$ in its product representation. Thus (cf. eq. [51])

$$R_a(\mu) = \prod_{\beta \neq a} (1 - k_\beta \mu) . \tag{74}$$

Substituting now for M_a and γ according to equations (72) and (52) in equation (70), we obtain, after some minor reductions, the following equation, which, as we shall see, determines c:

$$\left. \begin{aligned} c\left[(-1)^n \mu_1 \ldots \mu_n - \sum_{a=1}^{n} m_a - \frac{1}{1 + x(1-\lambda)\cos^2\beta} \frac{P(\cos\beta)}{R(\cos\beta)} \right] & \\ = \sum_{a=1}^{n} \frac{m_a}{k_a} + \frac{\cos\beta}{1 + x(1-\lambda)\cos^2\beta} \frac{P(\cos\beta)}{R(\cos\beta)} . & \end{aligned} \right\} \tag{75}$$

In order that we may use equation (75) to obtain an explicit formula for c, we have to sum the two series

$$\sum_{a=1}^{n} m_a \quad \text{and} \quad \sum_{a=1}^{n} \frac{m_a}{k_a} . \tag{76}$$

Considering first Σm_a, we have to evaluate

$$\sum_{a=1}^{n} \frac{P(k_a^{-1})}{[1 + x(1-\lambda) k_a^{-2}](1 - k_a^{-1}\sec\beta) R_a(k_a^{-1})} . \tag{77}$$

We re-write this in the form

$$\left. \begin{aligned} \sum_{a=1}^{n} m_a &= -\cos\beta \sum_{a=1}^{n} \frac{k_a P(k_a^{-1})}{[1 + x(1-\lambda) k_a^{-2}](1 - k_a\cos\beta) R_a(k_a^{-1})} \\ &= -\frac{\cos\beta}{R(\cos\beta)} \sum_{a=1}^{n} \frac{k_a P(k_a^{-1}) R_a(\cos\beta)}{[1 + x(1-\lambda) k_a^{-2}] R_a(k_a^{-1})} \\ &= -\frac{\cos\beta}{R(\cos\beta)} f(\cos\beta) \end{aligned} \right\} \tag{78}$$

where

$$f(z)=\sum_{a=1}^{n}\frac{k_a P(k_a^{-1})R_a(z)}{[1+x(1-\lambda)k_a^{-2}]R_a(k_a^{-1})}. \tag{79}$$

As defined in the foregoing equation, $f(z)$ is a polynomial of degree $n-1$ in z which assumes for $z=k_a^{-1}$ the values

$$f(k_a^{-1})=\frac{k_a}{1+x(1-\lambda)k_a^{-2}}P(k_a^{-1}) \quad (a=1,\ldots,n). \tag{80}$$

Hence,

$$z[1+x(1-\lambda)z^2]f(z)-P(z) \tag{81}$$

is a polynomial of degree $n+2$ in z which vanishes for $z=k_a^{-1}$ $(a=1,\ldots,n)$. There must, accordingly, be a relation of the form

$$z[1+x(1-\lambda)z^2]f(z)=P(z)+R(z)(\xi z^2+\eta z+\zeta), \tag{82}$$

where ξ, η, and ζ are certain constants. These constants can be found in the following manner:

First putting $z=0$ in equation (82), we conclude that

$$\zeta=(-1)^{n+1}\mu_1\ldots\mu_n. \tag{83}$$

Next, comparing the coefficients of z^{n+2} on either side of equation (82), we have

$$\left.\begin{aligned}-(-1)^n k_1\ldots k_n x(1-\lambda)\sum_{a=1}^{n}\frac{P(k_a^{-1})}{[1+x(1-\lambda)k_a^{-2}]R_a(k_a^{-1})}\\ =(-1)^n k_1\ldots k_n\xi,\end{aligned}\right\} \tag{84}$$

or

$$\xi=-x(1-\lambda)\sum_{a=1}^{n}\frac{P(k_a^{-1})}{[1+x(1-\lambda)k_a^{-2}]R_a(k_a^{-1})}. \tag{85}$$

And, finally, equating the coefficients of z^{n+1} on both sides of equation (82), we have

$$\left.\begin{aligned}(-1)^{n-2}x(1-\lambda)k_1\ldots k_n\left[\left(\sum_{a=1}^{n}\frac{1}{k_a}\right)\sum_{a=1}^{n}\frac{P(k_a^{-1})}{[1+x(1-\lambda)k_a^{-2}]R_a(k_a^{-1})}\right.\\ \left.-\sum_{a=1}^{n}\frac{P(k_a^{-1})}{k_a[1+x(1-\lambda)k_a^{-2}]R_a(k_a^{-1})}\right]=(-1)^{n-1}k_1\ldots k_n\left(\sum_{a=1}^{n}\frac{1}{k_a}\right)\xi\\ +(-1)^n k_1\ldots k_n\eta.\end{aligned}\right\} \tag{86}$$

Substituting for ξ from equation (85) in the foregoing equation, we find

$$\eta=-x(1-\lambda)\sum_{a=1}^{n}\frac{P(k_a^{-1})}{k_a[1+x(1-\lambda)k_a^{-2}]R_a(k_a^{-1})}. \tag{87}$$

Returning to equation (78), we now have, according to equations (82) and (83),

$$\left.\begin{aligned}\sum_{a=1}^{n}m_a=-\frac{1}{1+x(1-\lambda)\cos^2\beta}\left[\frac{P(\cos\beta)}{R(\cos\beta)}\right.\\ \left.+\xi\cos^2\beta+\eta\cos\beta+(-1)^{n+1}\mu_1\ldots\mu_n\right].\end{aligned}\right\} \tag{88}$$

With this expression for Σm_a, the terms in the square brackets on the left-hand side of equation (75) can be reduced to

$$-x(1-\lambda)\frac{\cos\beta}{1+x(1-\lambda)\cos^2\beta}(\rho\cos\beta+\sigma), \tag{89}$$

where

$$\rho=\sum_{a=1}^{n}\frac{P(k_a^{-1})}{[1+x(1-\lambda)k_a^{-2}]R_a(k_a^{-1})}+(-1)^{n+1}\mu_1\ldots\mu_n \tag{90}$$

and

$$\sigma=\sum_{a=1}^{n}\frac{P(k_a^{-1})}{k_a[1+x(1-\lambda)k_a^{-2}]R_a(k_a^{-1})} \tag{91}$$

are two constants, depending only on the characteristic roots k_a.

Considering next the summation $\Sigma m_a/k_a$ which occurs on the right-hand side of equation (75), we have to evaluate (cf. eq. [73])

$$\sum_{a=1}^{n}\frac{m_a}{k_a}=\sum_{a=1}^{n}\frac{P(k_a^{-1})}{k_a[1+x(1-\lambda)k_a^{-2}](1-k_a^{-1}\sec\beta)R_a(k_a^{-1})}. \tag{92}$$

We re-write this in the form (cf. eq. [78])

$$\sum_{a=1}^{n}\frac{m_a}{k_a}=-\frac{\cos\beta}{R(\cos\beta)}g(\cos\beta), \tag{93}$$

where

$$g(z)=\sum_{a=1}^{n}\frac{P(k_a^{-1})R_a(z)}{[1+x(1-\lambda)k_a^{-2}]R_a(k_a^{-1})} \tag{94}$$

is a polynomial of degree $(n-1)$ in z. It is seen that

$$g(k_a^{-1})=\frac{P(k_a^{-1})}{1+x(1-\lambda)k_a^{-2}}. \tag{95}$$

Hence,

$$[1+x(1-\lambda)z^2]\,g(z)-P(z) \tag{96}$$

is a polynomial of degree $n+1$ in z which vanishes for $z=k_a^{-1}$ $(a=1,\ldots,n)$. We conclude that

$$[1+x(1-\lambda)z^2]\,g(z)=P(z)+R(z)(az+b), \tag{97}$$

where a and b are constants. To determine them, we first set $z=0$ in equation (97) and find (cf. eq. [90])

$$b=\sum_{a=1}^{n}\frac{P(k_a^{-1})}{[1+x(1-\lambda)k_a^{-2}]R_a(k_a^{-1})}+(-1)^{n+1}\mu_1\ldots\mu_n=\rho. \tag{98}$$

Next, comparing the coefficients of the highest powers of z on either side of equation (97), we find (cf. eq. [91])

$$a=-x(1-\lambda)\sum_{a=1}^{n}\frac{P(k_a^{-1})}{k_a[1+x(1-\lambda)k_a^{-2}]R_a(k_a^{-1})}=-x(1-\lambda)\,\sigma. \tag{99}$$

We have thus shown that

$$[1+x(1-\lambda)z^2]\,g(z)=P(z)-R(z)[x(1-\lambda)\sigma z-\rho]. \tag{100}$$

Hence (cf. eq. [93]),

$$\sum_{a=1}^{n}\frac{m_a}{k_a}=-\frac{\cos\beta}{1+x(1-\lambda)\cos^2\beta}\left[\frac{P(\cos\beta)}{R(\cos\beta)}-x(1-\lambda)\sigma\cos\beta+\rho\right]. \tag{101}$$

Combining equations (75), (89), and (101), we now obtain

$$\left.\begin{aligned}&-cx(1-\lambda)\frac{\cos\beta}{1+x(1-\lambda)\cos^2\beta}(\rho\cos\beta+\sigma)\\&\qquad=\frac{\cos\beta[x(1-\lambda)\sigma\cos\beta-\rho]}{1+x(1-\lambda)\cos^2\beta},\end{aligned}\right\} \tag{102}$$

or

$$c=-\frac{x(1-\lambda)\sigma\cos\beta-\rho}{x(1-\lambda)(\rho\cos\beta+\sigma)}, \tag{103}$$

which is our formula for c. With c given by equation (103) we readily verify that

$$\cos\beta+c=\rho\frac{1+x(1-\lambda)\cos^2\beta}{x(1-\lambda)(\rho\cos\beta+\sigma)} \tag{104}$$

and

$$\left.\begin{aligned}&\frac{1+x(1-\lambda)\cos^2\beta}{\cos\beta+c}(\mu+c)\\&\qquad=\frac{1}{\rho}[x(1-\lambda)(\rho\cos\beta+\sigma)\mu-x(1-\lambda)\sigma\cos\beta+\rho]\\&\qquad=\frac{1}{\rho}[\rho-x(1-\lambda)\{\sigma(\cos\beta-\mu)-\rho\mu\cos\beta\}].\end{aligned}\right\} \tag{105}$$

Accordingly, equation (69) becomes

$$G(\mu)=\frac{1}{\mu_1^2\ldots\mu_n^2\rho}\frac{P(-\cos\beta)P(\mu)}{R(-\cos\beta)R(\mu)}\frac{\rho-x(1-\lambda)[\sigma(\cos\beta-\mu)-\rho\mu\cos\beta]}{1-\mu\sec\beta}, \tag{106}$$

which is our formula for $G(\mu)$.

The angular distribution of the reflected radiation corresponding to the part $I^{(0)}$ is given by (cf. eq. [61])

$$I^{(0)}(0,\mu)=\tfrac{1}{4}\lambda FG(-\mu). \tag{107}$$

With $G(\mu)$ given by equation (106) we can express $I^{(0)}(0,\mu)$ in the form

$$\left.\begin{aligned}I^{(0)}(0,\mu)&=\tfrac{1}{4}\lambda FH^{(0)}(\mu)H^{(0)}(\cos\beta)\frac{\cos\beta}{\cos\beta+\mu}\\&\qquad\times\left[1-x(1-\lambda)\left\{\frac{\sigma}{\rho}(\cos\beta+\mu)+\mu\cos\beta\right\}\right],\end{aligned}\right\} \tag{108}$$

where we have introduced the function $H^{(0)}(\mu)$ defined by

$$H^{(0)}(\mu)=\frac{(-1)^n}{\mu_1\ldots\mu_n}\frac{P(-\mu)}{R^{(0)}(-\mu)}. \tag{109}$$

In equation (109) we have added a superscript "0" to R to emphasize the fact that this function is defined in terms of the characteristic roots appropriate to the system of equations governing $I_j^{(0)}$.

For $\lambda = 1$ the solution (108) for the angular distribution reduces to

$$I^{(0)}(0, \mu) = \tfrac{1}{4} F H^{(0)}(\mu) H^{(0)}(\cos\beta) \frac{\cos\beta}{\cos\beta + \mu}, \tag{110}$$

and it may be readily verified that this is *identical* with the solution found in paper VIII for the reflection effect in eclipsing binaries.[8]

4. *The solution of a general type of integrodifferential equation of which equation (8) is a special case.*—Equation (8) is typical of a large class of equations which occurs in this theory. It is of the general type

$$\mu \frac{dI}{d\tau} = I - \epsilon\psi(\mu) \int_{-1}^{+1} I(\tau, \mu')\psi(\mu')\, d\mu' - \epsilon F e^{-\tau \sec\beta}\psi(\cos\beta)\psi(\mu), \tag{111}$$

where ϵ is a constant and $\psi^2(\mu)$ is a polynomial of degree $2m$ in μ. It will therefore be convenient to have the solution of this general equation appropriate to the conditions of our problem.

Since $\psi^2(\mu)$ is rational, it suggests that we express $I(\tau, \mu)$ in the form

$$I(\tau, \mu) = \phi(\tau, \mu)\psi(\mu) \tag{111'}$$

and obtain for ϕ the equation

$$\mu \frac{d\phi}{d\tau} = \phi - \epsilon \int_{-1}^{+1} \phi(\tau, \mu')\psi^2(\mu')\, d\mu' - \epsilon F e^{-\tau \sec\beta}\psi(\cos\beta). \tag{112}$$

To solve equation (112) in the nth approximation, we replace it by the system of $2n$ linear equations

$$\mu_i \frac{d\phi_i}{d\tau} = \phi_i - \epsilon \Sigma a_j \psi^2(\mu_j)\phi_j - \epsilon F e^{-\tau \sec\beta}\psi(\cos\beta) \qquad (i = \pm 1, \ldots, \pm n), \tag{113}$$

where ϕ_i denotes $\phi(\tau, \mu_i)$ and the rest of the symbols have their usual meanings. In this connection, it should be noted that, in order that we may be consistent in our scheme of approximation, it is necessary that the order of the approximation

$$n \geqslant m, \tag{114}$$

where it may be recalled that $2m$ is the degree of the polynomial $\psi^2(\mu)$.

Considering first the homogeneous system

$$\mu_i \frac{d\phi_i}{d\tau} = \phi_i - \epsilon \Sigma a_j \psi^2(\mu_j)\phi_j \qquad (i = \pm 1, \ldots, \pm n), \tag{115}$$

[8] It should, however, be pointed out in this equation that for $\lambda = 1$ the characteristic equation allows only $n - 1$ distinct nonvanishing roots for k^2. (In fact, the characteristic equation [26] reduces to the one considered in paper II *independently* of x for $\lambda = 1$.) Accordingly, in this case there exist only $(2n - 2)$ independent integrals of the form

$$I_i = \frac{\text{constant}}{1 \pm \mu_i k_a} e^{\mp k_a \tau} \qquad \begin{pmatrix} i = \pm 1, \ldots, \pm n \\ a = 1, \ldots, n-1 \end{pmatrix}$$

for the homogeneous system (10). On the other hand, when $\lambda = 1$, equation (10) admits the further integral.

$$I_i = b\left(\tau + \frac{1}{1 - \frac{1}{3}x}\mu_i + Q\right) \qquad (i = \pm 1, \ldots, \pm n)$$

with two arbitrary constants b and Q. Nevertheless, it can be shown that the procedure of formally putting $\lambda = 1$ in eq. (108) actually leads to the correct solution.

associated with equation (113), we readily verify that it admits $2n$ linearly independent integrals of the form

$$\phi_i = \frac{\text{constant}}{1 \pm \mu_i k_a} e^{\mp k_a \tau} \qquad \begin{pmatrix} i = \pm 1, \ldots, \pm n \\ a = 1, \ldots, n \end{pmatrix}, \tag{116}$$

where $\pm k_a$, $(a = 1, \ldots, n)$ are $2n$ distinct[9] nonvanishing roots of the characteristic equation

$$1 = 2\epsilon \sum_{j=1}^{n} \frac{a_j \psi^2(\mu_j)}{1 - \mu_j^2 k^2}. \tag{117}$$

To find a particular integral of the nonhomogeneous equation (113), we set

$$\phi_i = \epsilon F \psi(\cos\beta)\, h_i e^{-\tau \sec\beta} \qquad (i = \pm 1, \ldots, \pm n), \tag{118}$$

where the h_i's are constants unspecified for the present. Inserting this form for ϕ_i in equation (113), we find that

$$h_i (1 + \mu_i \sec\beta) = \epsilon \Sigma a_j h_j \psi^2(\mu_j) + 1. \tag{119}$$

The constants h_i must therefore be expressible in the form

$$h_i = \frac{\gamma}{1 + \mu_i \sec\beta}, \tag{120}$$

the constant γ in turn being determined by the condition

$$\gamma = \epsilon\gamma \Sigma \frac{a_j \psi^2(\mu_j)}{1 + \mu_j \sec\beta} + 1, \tag{121}$$

or

$$\gamma = \frac{1}{1 - 2\epsilon \sum_{j=1}^{n} \dfrac{a_j \psi^2(\mu_j)}{1 - \mu_j^2 \sec^2\beta}}. \tag{122}$$

By arguments similar to those adopted in the reduction of analogous equations (VIII, eq. [40] and eq. [40] in the preceding section) it can be shown that the formula for γ can be reduced to the form

$$\gamma = \frac{1}{\mu_1^2 \ldots \mu_n^2} \frac{P(\cos\beta) P(-\cos\beta)}{R(\cos\beta) R(-\cos\beta)}, \tag{123}$$

where it should be noted that

$$R(\mu) = \prod_{a=1}^{n} (1 - k_a \mu) \tag{124}$$

has to be evaluated in terms of the characteristic roots of the system under consideration.

[9] An exceptional case may arise if

$$\sum_{j=1}^{n} a_j \psi^2(\mu_j) = \frac{1}{2\epsilon},$$

when $k^2 = 0$ will be a root of the characteristic equation. However, this is not likely to happen in practice. And, even if it does, it can be shown that our final solution (141) for the angular distribution of the emergent radiation will continue to be valid.

When we return to equations (118) and (120), it is seen that equation (113) admits the particular integral

$$\phi_i = \frac{\epsilon\gamma F\psi(\cos\beta)}{1+\mu_i \sec\beta} e^{-\tau \sec\beta} \qquad (i=\pm 1, \ldots, \pm n) . \quad (125)$$

Adding to this particular integral the general solution of the homogeneous system (115) which is compatible with the boundedness of the solution for $\tau \to \infty$, we have

$$\phi_i = \epsilon F\psi(\cos\beta)\left[\sum_{a=1}^{n} \frac{M_a e^{-k_a\tau}}{1+\mu_i k_a} + \frac{\gamma e^{-\tau\sec\beta}}{1+\mu_i \sec\beta}\right] \qquad (i=\pm 1, \ldots, \pm n) , \quad (126)$$

where the M_a's ($a = 1, \ldots, n$) are n constants of integration, to be determined from the boundary conditions at $\tau = 0$, namely, that here

$$\phi_{-i} = 0 \qquad (i = 1, \ldots, n) . \quad (127)$$

In terms of the function

$$G(\mu) = \sum_{a=1}^{n} \frac{M_a}{1-\mu k_a} + \frac{\gamma}{1-\mu \sec\beta} , \quad (128)$$

the boundary conditions are

$$G(\mu_i) = 0 \qquad (i=1, \ldots, n) . \quad (129)$$

The angular distribution of the emergent radiation can also be expressed in terms of $G(\mu)$, for (cf. eqs. [111] and [126])

$$I(0, \mu) = \epsilon F\psi(\cos\beta)\psi(\mu)G(-\mu) . \quad (130)$$

We shall now show how an explicit formula for $G(\mu)$ can be found without having to solve for the constants M_a.

When we consider the function

$$(1-\mu\sec\beta)R(\mu)G(\mu) , \quad (131)$$

it is seen that it is a polynomial of degree n in μ which vanishes for $\mu = \mu_i$ ($i = 1, \ldots, n$). It cannot therefore differ from $P(\mu)$ except by a constant factor, and the constant factor can be found from a comparison of the coefficients of the highest power of μ. In this manner we find that

$$G(\mu) = (-1)^n k_1 \ldots k_n \left[\sum_{a=1}^{n} \frac{M_a}{k_a} \sec\beta + \gamma\right] \frac{P(\mu)}{R(\mu)} \frac{1}{1-\mu\sec\beta} . \quad (132)$$

On the other hand, since (cf. eq. [128])

$$\gamma = \lim_{\mu \to \cos\beta} (1-\mu\sec\beta)G(\mu) , \quad (133)$$

we have, according to equations (123) and (132),

$$\left.\begin{aligned} \frac{1}{\mu_1^2 \ldots \mu_n^2} \frac{P(-\cos\beta)P(\cos\beta)}{R(-\cos\beta)R(\cos\beta)} = (-1)^n k_1 \ldots k_n \left[\sum_{a=1}^{n} \frac{M_a}{k_a}\sec\beta + \gamma\right] \\ \times \frac{P(\cos\beta)}{R(\cos\beta)} . \end{aligned}\right\} \quad (134)$$

In other words,

$$(-1)^n k_1 \ldots . k_n \left[\sum_{a=1}^{n} \frac{M_a}{k_a} \sec \beta + \gamma \right] = \frac{1}{\mu_1^2 \ldots . \mu_n^2} \frac{P(-\cos \beta)}{R(-\cos \beta)} . \tag{135}$$

In virtue of this relation, equation (132) becomes

$$G(\mu) = \frac{1}{\mu_1^2 \ldots . \mu_n^2} \frac{P(\mu) P(-\cos \beta)}{R(\mu) R(-\cos \beta)} \frac{1}{1 - \mu \sec \beta} . \tag{136}$$

Consequently, the formula giving the angular distribution of the reflected radiation can be expressed in the form (cf. eq. [130])

$$I(0, \mu) = \epsilon F \psi(\cos \beta) \psi(\mu) H(\mu) H(\cos \beta) \frac{\cos \beta}{\cos \beta + \mu} , \tag{137}$$

where (cf. eq. [109])

$$H(\mu) = \frac{(-1)^n}{\mu_1 \ldots . \mu_n} \frac{P(-\mu)}{R(-\mu)} . \tag{138}$$

5. *The solution of equation (8) in the* n*th approximation.*—To apply the results of the preceding section to the solution of equation (8), we have only to set

$$\epsilon = \tfrac{1}{4} x \lambda \quad \text{and} \quad \psi^2(\mu) = 1 - \mu^2 . \tag{139}$$

Moreover, according to equation (114), solutions must be sought in approximations *higher* than the first.

The characteristic equation is (cf. eq. [117])

$$1 = \tfrac{1}{2} x \lambda \sum_{j=1}^{n} \frac{a_j (1 - \mu_j^2)}{1 - \mu_j^2 k^2} , \tag{140}$$

and the angular distribution of the reflected radiation corresponding to the part $I^{(1)}$ of I is given by

$$I^{(1)}(0, \mu) = \tfrac{1}{4} x \lambda F \sin \vartheta \sin \beta H^{(1)}(\mu) H^{(1)}(\cos \beta) \frac{\cos \beta}{\cos \beta + \mu} . \tag{141}$$

In equation (141) we have added a superscript "1" to H to emphasize the fact that the $R(\mu)$ occurring in the definition of $H(\mu)$ (cf. eq. [138]) has to be evaluated in terms of the roots of the characteristic equation (140).

6. *Angular distribution of the reflected radiation: numerical results.*—Combining the results of the preceding sections, we can express the angular distribution of the reflected radiation in the form (cf. eqs. [90], [91], [108], and [141])

$$\left. \begin{aligned} I(0, \mu) = \tfrac{1}{4} \lambda F \Big[& H^{(0)}(\mu) H^{(0)}(\mu') \Big\{ 1 - x(1 - \lambda) \Big(\frac{\sigma}{\rho} (\mu + \mu') + \mu \mu' \Big) \Big\} \\ & + x (1 - \mu^2)^{\frac{1}{2}} (1 - \mu'^2)^{\frac{1}{2}} H^{(1)}(\mu) H^{(1)}(\mu') \cos \varphi \Big] \frac{\mu'}{\mu + \mu'} , \end{aligned} \right\} \tag{142}$$

where we have written μ' for $\cos \beta$. For specified values of x and λ the solution becomes determinate in terms of the positive nonvanishing roots $k_a^{(0)}$ and $k_a^{(1)}$ $(a = 1, \ldots . , n)$ of the characteristic equations

$$1 = \lambda \sum_{j=1}^{n} \frac{a_j [1 + x(1 - \lambda) \mu_j^2]}{1 - \mu_j^2 k^2} \tag{143}$$

and

$$1=\tfrac{1}{2}x\lambda\sum_{j=1}^{n}\frac{a_j(1-\mu_j^2)}{1-\mu_j^2k^2}\,, \tag{144}$$

respectively. In particular, the functions $H^{(i)}(\mu)$ (i = 0 and 1) have the representation

$$H^{(i)}(\mu)=\frac{1}{\mu_1\cdot\ldots\cdot\mu_n}\,\frac{\prod_{j=1}^{n}(\mu+\mu_j)}{\prod_{a=1}^{n}(1-k_a^{(i)}\mu)}\qquad(i=0,1)\,. \tag{145}$$

For the purposes of the practical evaluation of the roots $k_a^{(0)}$ and $k_a^{(1)}$ it is convenient to transform equations (143) and (144) in the following manner (cf. VII, p. 336, n. 6): Letting

$$\Delta_{2m}^{(0)}=\sum_{j=1}^{n}\frac{a_j[1+x(1-\lambda)\mu_j^2]\mu_j^{2m}}{1-\mu_j^2k^2}\quad\text{and}\quad\Delta_{2m}^{(1)}=\sum_{j=1}^{n}\frac{a_j(1-\mu_j^2)\mu_j^{2m}}{1-\mu_j^2k^2}\,, \tag{146}$$

we readily establish the recursion formulae

$$\Delta_{2m}^{(0)}=\frac{1}{k^2}\left[\Delta_{2m-2}^{(0)}-\left\{\frac{1}{2m-1}+\frac{x(1-\lambda)}{2m+1}\right\}\right]\qquad(m\leqslant 2n) \tag{147}$$

and

$$\Delta_{2m}^{(1)}=\frac{1}{k^2}\left[\Delta_{2m-2}^{(1)}-\frac{2}{4m^2-1}\right]\qquad(m\leqslant 2n)\,; \tag{148}$$

these, together with the relations

$$\Delta_0^{(0)}=\frac{1}{\lambda}\quad\text{and}\quad\Delta_0^{(1)}=\frac{2}{x\lambda}\,, \tag{149}$$

determine the Δ's very simply. And in terms of these Δ's the characteristic equations are expressible in the form

$$\sum_{m=0}^{\infty}\Delta_{2m}^{(i)}p_{2m}=0\,, \tag{150}$$

where the p_{2m}'s are the coefficients of μ^{2m} in the Legendre polynomial $P_{2n}(\mu)$. It will be noticed that, in contrast to equations (143) and (144), equation (150) does not require an explicit knowledge of the Gaussian weights and divisions.

In Tables 1 through 8 the functions $H^{(0)}(\mu)$ and $H^{(1)}(\mu)$ are tabulated for various values of the parameters which enter into them. Certain other auxiliary quantities, such as the characteristic roots, are also tabulated. Except for the case $x = 0$, all the quantities tabulated are those in the second approximation. However, for the case $x = 0$, the solutions have also been found in the third approximation. It would appear, from an inspection particularly of Table 4, that the solutions in the second approximation provide an accuracy of 1–2 per cent over the entire range of the variables.

A comparison of our results with Ambarzumian's tabulation for the case $x = 1$ indicates that his method of solving his integral equations leads to errors which exceed 5 per cent over certain ranges of the variables.

TABLE 1

THE CHARACTERISTIC ROOTS $k_1^{(0)}$ AND $k_2^{(0)}$ AND $(1-\lambda)\sigma/\rho$ FOR VARIOUS VALUES OF x AND λ

λ	$x=1.0$			$x=0.5$			$x=-0.5$			$x=-1.0$		
	$k_1^{(0)}$	$k_2^{(0)}$	$(1-\lambda)\sigma/\rho$	$k_1^{(0)}$	$k_2^{(0)}$	$(1-\lambda)\sigma/\rho$	$k_1^{(0)}$	$k_2^{(0)}$	$(1-\lambda)\sigma/\rho$	$k_1^{(0)}$	$k_2^{(0)}$	$(1-\lambda)\sigma/\rho$
0.9.........	0.4401	2.0534	0.1348	0.4847	2.0543	0.1259	0.5633	2.0561	0.1129	0.5987	2.0571	0.1079
0.8.........	0.6114	2.1394	.1477	0.6637	2.1426	.1402	0.7567	2.1491	.1285	0.7987	2.1525	.1238
0.7.........	0.7346	2.2299	.1438	0.7864	2.2358	.1380	0.8794	2.2481	.1286	0.9215	2.2545	.1246
0.6.........	0.8313	2.3243	.1318	0.8785	2.3327	.1277	0.9640	2.3503	.1206	1.0029	2.3595	.1175
0.5.........	0.9103	2.4219	.1153	0.9507	2.4322	.1124	1.0246	2.4535	.1074	1.0585	2.4645	.1053
0.4.........	0.9765	2.5223	.0956	1.0090	2.5333	.0938	1.0691	2.5559	.0906	1.0970	2.5675	.0892
0.3.........	1.0328	2.6249	.0738	1.0570	2.6352	.0728	1.1023	2.6564	.0710	1.1237	2.6672	.0702
0.2.........	1.0815	2.7291	.0503	1.0972	2.7375	.0499	1.1275	2.7544	.0491	1.1420	2.7629	.0487
0.1.........	1.1239	2.8347	0.0256	1.1315	2.8396	0.0256	1.1466	2.8494	0.0253	1.1540	2.8543	0.0253

TABLE 2

THE FUNCTION $H^{(0)}(\mu)$ FOR VARIOUS VALUES OF λ AND FOR $x=1.0$ IN THE SECOND APPROXIMATION

μ	$\lambda=0.9$	$\lambda=0.8$	$\lambda=0.7$	$\lambda=0.6$	$\lambda=0.5$	$\lambda=0.4$	$\lambda=0.3$	$\lambda=0.2$	$\lambda=0.1$
0.............	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.1...........	1.148	1.121	1.100	1.082	1.066	1.051	1.037	1.024	1.012
0.2...........	1.275	1.221	1.180	1.145	1.115	1.088	1.064	1.041	1.020
0.3...........	1.387	1.306	1.246	1.197	1.155	1.117	1.084	1.054	1.026
0.4...........	1.488	1.380	1.302	1.240	1.187	1.141	1.100	1.064	1.031
0.5...........	1.579	1.445	1.350	1.276	1.214	1.160	1.114	1.072	1.034
0.6...........	1.663	1.503	1.393	1.307	1.237	1.177	1.125	1.079	1.037
0.7...........	1.739	1.555	1.430	1.334	1.257	1.191	1.134	1.084	1.040
0.8...........	1.810	1.602	1.463	1.358	1.274	1.203	1.143	1.089	1.042
0.9...........	1.876	1.645	1.493	1.380	1.289	1.214	1.150	1.094	1.044
1.0...........	1.937	1.684	1.520	1.399	1.303	1.223	1.156	1.097	1.046

TABLE 3

THE FUNCTION $H^{(0)}(\mu)$ FOR VARIOUS VALUES OF λ AND FOR $x=0.5$ IN THE SECOND APPROXIMATION

μ	$\lambda=0.9$	$\lambda=0.8$	$\lambda=0.7$	$\lambda=0.6$	$\lambda=0.5$	$\lambda=0.4$	$\lambda=0.3$	$\lambda=0.2$	$\lambda=0.1$
0.............	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.1...........	1.143	1.115	1.094	1.077	1.061	1.047	1.034	1.022	1.011
0.2...........	1.265	1.209	1.168	1.135	1.106	1.081	1.058	1.037	1.018
0.3...........	1.371	1.288	1.229	1.182	1.142	1.107	1.076	1.048	1.023
0.4...........	1.465	1.356	1.280	1.220	1.170	1.128	1.091	1.057	1.027
0.5...........	1.551	1.416	1.323	1.252	1.194	1.145	1.102	1.064	1.031
0.6...........	1.628	1.468	1.361	1.280	1.215	1.159	1.112	1.070	1.033
0.7...........	1.698	1.515	1.394	1.304	1.232	1.172	1.120	1.075	1.036
0.8...........	1.763	1.557	1.424	1.325	1.247	1.182	1.127	1.080	1.038
0.9...........	1.823	1.595	1.450	1.344	1.260	1.192	1.133	1.083	1.039
1.0...........	1.878	1.629	1.474	1.361	1.272	1.200	1.139	1.087	1.041

TABLE 4

THE FUNCTION $H^{(0)}(\mu)$ IN THE SECOND AND THIRD APPROXIMATIONS FOR VARIOUS VALUES OF λ AND FOR $x=0.0$*

μ	$\lambda=0.95$		$\lambda=0.9$		$\lambda=0.8$		$\lambda=0.7$		$\lambda=0.6$	
	Second Approximation	Third Approximation	Second Approximation	Third Approximation	Second Approximation	Third Approximation	Second Approximation	Third Approximation	Second Approximation	Third Approximation
0.	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.1	1.158	1.171	1.138	1.150	1.110	1.120	1.089	1.097	1.072	1.078
0.2	1.297	1.314	1.255	1.270	1.199	1.211	1.158	1.168	1.125	1.133
0.3	1.421	1.439	1.356	1.373	1.272	1.285	1.214	1.224	1.168	1.176
0.4	1.533	1.552	1.445	1.462	1.335	1.348	1.260	1.270	1.202	1.210
0.5	1.636	1.655	1.525	1.541	1.389	1.401	1.299	1.309	1.231	1.242
0.6	1.731	1.749	1.597	1.612	1.437	1.448	1.333	1.342	1.256	1.269
0.7	1.819	1.836	1.662	1.677	1.479	1.490	1.362	1.371	1.277	1.292
0.8	1.901	1.918	1.722	1.736	1.517	1.527	1.388	1.396	1.295	1.312
0.9	1.978	1.994	1.777	1.790	1.551	1.560	1.412	1.419	1.311	1.329
1.0	2.050	2.065	1.828	1.840	1.582	1.591	1.432	1.439	1.326	1.345

μ	$\lambda=0.5$		$\lambda=0.4$		$\lambda=0.3$		$\lambda=0.2$		$\lambda=0.1$	
	Second Approximation	Third Approximation	Second Approximation	Third Approximation	Second Approximation	Third Approximation	Second Approximation	Third Approximation	Second Approximation	Third Approximation
0.	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.1	1.056	1.062	1.043	1.047	1.031	1.034	1.020	1.022	1.009	1.010
0.2	1.098	1.104	1.074	1.079	1.052	1.056	1.033	1.036	1.016	1.017
0.3	1.130	1.136	1.097	1.102	1.069	1.072	1.043	1.045	1.021	1.022
0.4	1.155	1.161	1.116	1.120	1.081	1.084	1.051	1.053	1.024	1.025
0.5	1.176	1.182	1.131	1.135	1.092	1.094	1.057	1.059	1.027	1.028
0.6	1.194	1.199	1.143	1.147	1.100	1.103	1.062	1.064	1.029	1.030
0.7	1.209	1.214	1.154	1.157	1.107	1.110	1.067	1.068	1.031	1.032
0.8	1.222	1.227	1.163	1.166	1.113	1.115	1.070	1.072	1.033	1.034
0.9	1.234	1.238	1.171	1.174	1.119	1.121	1.074	1.075	1.034	1.035
1.0	1.244	1.248	1.178	1.181	1.123	1.125	1.076	1.078	1.036	1.036

* The characteristic roots for this case are those which have been tabulated in a different connection by C. U. Cesco, S. Chandrasekhar, and J. Sahade (*Ap. J.*, **100**, 355, 1944; esp. p. 358, Table 1). However, it should be noted that λ, as used in the present paper, is $1-\lambda$, as used in the paper just quoted.

TABLE 5

THE FUNCTION $H^{(0)}(\mu)$ FOR VARIOUS VALUES OF λ AND FOR $x=-0.5$ IN THE SECOND APPROXIMATION

μ	$\lambda=0.9$	$\lambda=0.8$	$\lambda=0.7$	$\lambda=0.6$	$\lambda=0.5$	$\lambda=0.4$	$\lambda=0.3$	$\lambda=0.2$	$\lambda=0.1$
0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.1	1.134	1.105	1.084	1.067	1.052	1.039	1.028	1.018	1.008
0.2	1.246	1.189	1.148	1.116	1.090	1.067	1.047	1.030	1.014
0.3	1.343	1.258	1.199	1.155	1.118	1.088	1.061	1.038	1.018
0.4	1.427	1.316	1.241	1.186	1.141	1.104	1.072	1.045	1.021
0.5	1.502	1.366	1.277	1.211	1.160	1.117	1.081	1.050	1.024
0.6	1.570	1.409	1.307	1.233	1.175	1.128	1.088	1.055	1.026
0.7	1.630	1.447	1.334	1.252	1.188	1.137	1.095	1.058	1.027
0.8	1.686	1.482	1.357	1.268	1.200	1.145	1.100	1.061	1.029
0.9	1.736	1.512	1.377	1.282	1.210	1.152	1.104	1.064	1.030
1.0	1.783	1.540	1.395	1.294	1.218	1.158	1.108	1.066	1.031

TABLE 6

THE FUNCTION $H^{(0)}(\mu)$ FOR VARIOUS VALUES OF λ AND FOR $x=-1.0$ IN THE SECOND APPROXIMATION

μ	$\lambda=0.9$	$\lambda=0.8$	$\lambda=0.7$	$\lambda=0.6$	$\lambda=0.5$	$\lambda=0.4$	$\lambda=0.3$	$\lambda=0.2$	$\lambda=0.1$
0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.1	1.130	1.101	1.079	1.062	1.048	1.036	1.025	1.016	1.007
0.2	1.238	1.180	1.139	1.107	1.082	1.060	1.042	1.026	1.012
0.3	1.330	1.244	1.186	1.142	1.107	1.079	1.054	1.034	1.016
0.4	1.411	1.298	1.225	1.170	1.128	1.093	1.064	1.039	1.018
0.5	1.482	1.344	1.257	1.193	1.144	1.104	1.071	1.044	1.020
0.6	1.545	1.384	1.284	1.212	1.157	1.113	1.077	1.047	1.022
0.7	1.601	1.419	1.307	1.229	1.169	1.121	1.083	1.050	1.023
0.8	1.653	1.450	1.328	1.243	1.178	1.128	1.087	1.053	1.024
0.9	1.700	1.477	1.346	1.255	1.187	1.134	1.090	1.055	1.025
1.0	1.743	1.502	1.362	1.266	1.194	1.139	1.094	1.057	1.026

TABLE 7

THE CHARACTERISTIC ROOTS $k_1^{(1)}$ AND $k_2^{(1)}$ FOR VARIOUS VALUES OF $x\lambda$

$x\lambda$	$k_1^{(1)}$	$k_2^{(1)}$	$x\lambda$	$k_1^{(1)}$	$k_2^{(1)}$	$x\lambda$	$k_1^{(1)}$	$k_2^{(1)}$	$x\lambda$	$k_1^{(1)}$	$k_2^{(1)}$
1.0	1.1212	2.4875	0.5	1.1454	2.7222	−0.1	1.1638	2.9835	−0.6	1.1742	3.1866
0.9	1.1270	2.5357	0.4	1.1491	2.7672	−0.2	1.1661	3.0251	−0.7	1.1759	3.2259
0.8	1.1323	2.5833	0.3	1.1525	2.8116	−0.3	1.1683	3.0662	−0.8	1.1775	3.2646
0.7	1.1370	2.6302	0.2	1.1556	2.8554	−0.4	1.1704	3.1068	−0.9	1.1791	3.3030
0.6	1.1414	2.6765	0.1	1.1585	2.8987	−0.5	1.1723	3.1470	−1.0	1.1805	3.3409

TABLE 8

THE FUNCTION $H^{(1)}(\mu)$ FOR VARIOUS VALUES OF $x\lambda$ IN THE SECOND APPROXIMATION

μ	$x\lambda=1.0$	$x\lambda=0.9$	$x\lambda=0.8$	$x\lambda=0.7$	$x\lambda=0.6$	$x\lambda=0.5$	$x\lambda=0.4$	$x\lambda=0.3$	$x\lambda=0.2$	$x\lambda=0.1$
0.	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.1	1.040	1.035	1.031	1.027	1.023	1.019	1.015	1.011	1.007	1.003
0.2	1.067	1.060	1.052	1.045	1.038	1.031	1.024	1.018	1.012	1.006
0.3	1.088	1.077	1.067	1.058	1.049	1.040	1.031	1.023	1.015	1.007
0.4	1.103	1.091	1.079	1.068	1.057	1.046	1.036	1.027	1.018	1.009
0.5	1.115	1.101	1.088	1.075	1.063	1.052	1.040	1.030	1.019	1.009
0.6	1.125	1.110	1.095	1.082	1.068	1.056	1.044	1.032	1.021	1.010
0.7	1.133	1.117	1.102	1.087	1.073	1.059	1.046	1.034	1.022	1.011
0.8	1.140	1.123	1.107	1.091	1.076	1.062	1.049	1.036	1.023	1.011
0.9	1.146	1.128	1.111	1.095	1.079	1.064	1.050	1.037	1.024	1.012
1.0	1.151	1.133	1.115	1.098	1.082	1.067	1.052	1.038	1.025	1.012

μ	$x\lambda=-0.1$	$x\lambda=-0.2$	$x\lambda=-0.3$	$x\lambda=-0.4$	$x\lambda=-0.5$	$x\lambda=-0.6$	$x\lambda=-0.7$	$x\lambda=-0.8$	$x\lambda=-0.9$	$x\lambda=-1.0$
0.	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.1	0.997	0.993	0.990	0.987	0.983	0.980	0.977	0.974	0.971	0.968
0.2	0.994	0.989	0.983	0.978	0.973	0.968	0.963	0.958	0.954	0.949
0.3	0.993	0.985	0.979	0.972	0.966	0.960	0.954	0.948	0.942	0.936
0.4	0.992	0.984	0.976	0.968	0.961	0.954	0.947	0.940	0.933	0.927
0.5	0.991	0.982	0.973	0.965	0.957	0.949	0.941	0.934	0.927	0.920
0.6	0.990	0.980	0.971	0.962	0.954	0.945	0.937	0.929	0.921	0.914
0.7	0.989	0.979	0.970	0.960	0.951	0.942	0.934	0.925	0.917	0.909
0.8	0.989	0.978	0.968	0.958	0.949	0.940	0.931	0.922	0.914	0.906
0.9	0.989	0.978	0.967	0.957	0.947	0.938	0.928	0.920	0.911	0.903
1.0	0.988	0.977	0.966	0.956	0.946	0.936	0.926	0.917	0.908	0.900

II. DIFFUSE REFLECTION IN ACCORDANCE WITH RAYLEIGH'S PHASE FUNCTION

7. *The reduction of the equation of transfer.*—For Rayleigh's form of the phase function the equation of transfer appropriate for the problem of diffuse reflection is (cf. eq. [2])

$$\left.\begin{aligned}\cos\vartheta\,\frac{dI(\tau,\vartheta,\varphi)}{d\tau} &= I(\tau,\vartheta,\varphi)-\frac{3}{16\pi}\int_0^{\pi}\int_0^{2\pi} I(\tau,\vartheta',\varphi')\,[1+\cos^2\vartheta\cos^2\vartheta' \\ &\quad+\tfrac{1}{2}\sin^2\vartheta\sin^2\vartheta'+2\cos\vartheta\sin\vartheta\cos\vartheta'\sin\vartheta'\cos(\varphi-\varphi') \\ &\quad+\tfrac{1}{2}\sin^2\vartheta\sin^2\vartheta'\cos 2(\varphi-\varphi')\,]\sin\vartheta' d\vartheta' d\varphi'-\tfrac{3}{16}Fe^{-\tau\sec\beta} \\ &\quad\times[1+\cos^2\vartheta\cos^2\beta+\tfrac{1}{2}\sin^2\vartheta\sin^2\beta-2\sin\vartheta\cos\vartheta\sin\beta\cos\beta\cos\varphi \\ &\quad+\tfrac{1}{2}\sin^2\vartheta\sin^2\beta\cos 2\varphi]\,.\end{aligned}\right\}\qquad(151)$$

The form of equation (151) suggests that we seek a solution in the form

$$I(\tau,\vartheta,\varphi)=I^{(0)}(\tau,\vartheta)+I^{(1)}(\tau,\vartheta)\cos\varphi+I^{(2)}(\tau,\vartheta)\cos 2\varphi\,. \qquad (152)$$

Inserting the foregoing form for $I(\tau, \vartheta, \varphi)$ in equation (151), we find that it breaks up into three equations, one for each of the functions $I^{(0)}$, $I^{(1)}$, and $I^{(2)}$. These are

$$\left.\begin{aligned}\mu \frac{dI^{(0)}}{d\tau} = I^{(0)} - \frac{3}{16}\Big[\int_{-1}^{+1} I^{(0)}(\tau, \mu')(3-\mu'^2)\,d\mu' \\ + \mu^2 \int_{-1}^{+1} I^{(0)}(\tau, \mu')(3\mu'^2 - 1)\,d\mu'\Big] - \tfrac{3}{32} F e^{-\tau \sec\beta}[(3-\cos^2\beta) \\ + \mu^2(3\cos^2\beta - 1)],\end{aligned}\right\} \quad (153)$$

$$\left.\begin{aligned}\mu \frac{dI^{(1)}}{d\tau} = I^{(1)} - \tfrac{3}{8}\mu(1-\mu^2)^{\frac{1}{2}} \int_{-1}^{+1} I^{(1)}(\tau, \mu')\mu'(1-\mu'^2)^{\frac{1}{2}} d\mu' \\ + \tfrac{3}{8} F \sin\beta \cos\beta\, e^{-\tau\sec\beta}\mu(1-\mu^2)^{\frac{1}{2}},\end{aligned}\right\} \quad (154)$$

and

$$\left.\begin{aligned}\mu \frac{dI^{(2)}}{d\tau} = I^{(2)} - \tfrac{3}{32}(1-\mu^2) \int_{-1}^{+1} I^{(2)}(\tau, \mu')(1-\mu'^2)\,d\mu' \\ - \tfrac{3}{32} F \sin^2\beta\, e^{-\tau\sec\beta}(1-\mu^2).\end{aligned}\right\} \quad (155)$$

8. *The solution of equations (153), (154), and (155) in the* n*th approximation.*—Considering equation (153) first, the equivalent system of linear equations in the nth approximation is

$$\left.\begin{aligned}\mu_i \frac{dI_i^{(0)}}{d\tau} = I_i^{(0)} - \tfrac{3}{16}[\Sigma a_j(3-\mu_j^2) I_j^{(0)} + \mu_i^2 \Sigma a_j (3\mu_j^2 - 1) I_j^{(0)}] \\ - \tfrac{3}{32} F e^{-\tau\sec\beta}[(3-\cos^2\beta) + \mu_i^2(3\cos^2\beta - 1)] \quad (i = \pm 1, \ldots, \pm n),\end{aligned}\right\} \quad (156)$$

where the various symbols have their usual meanings.

It is seen that the homogeneous system associated with equation (156) is the same as that considered in paper III, §§ 3–5. Accordingly, the complementary function of equation (156) is the same as the general solution (III, eq. [36]) of the homogeneous system. Accordingly, to complete the solution we need only find a particular integral. This can be found in the following manner:

Setting

$$I_i^{(0)} = \tfrac{3}{32} F h_i e^{-\tau\sec\beta} \quad (i = \pm 1, \ldots, \pm n) \quad (157)$$

in equation (156) (the h_i's are certain constants unspecified for the present), we verify that we must have

$$\left.\begin{aligned}h_i(1 + \mu_i \sec\beta) = [\tfrac{3}{16}\Sigma a_j h_j (3-\mu_j^2) + 3 - \cos^2\beta] \\ + \mu_i^2[\tfrac{3}{16}\Sigma a_j h_j (3\mu_j^2 - 1) + 3\cos^2\beta - 1].\end{aligned}\right\} \quad (158)$$

The foregoing equation implies that the constants h_i must be expressible in the form

$$h_i = \frac{\delta - \mu_i^2 \gamma}{1 + \mu_i \sec\beta} \quad (i = \pm 1, \ldots, \pm n), \quad (159)$$

where the constants γ and δ have to be determined in accordance with the relation

$$\left.\begin{aligned}\delta - \mu_i^2 \gamma = \{\tfrac{3}{16}[\delta(3E_0 - E_2) - \gamma(3E_2 - E_4)] + 3 - \cos^2\beta\} \\ + \mu_i^2\{\tfrac{3}{16}[\delta(3E_2 - E_0) - \gamma(3E_4 - E_2)] + 3\cos^2\beta - 1\}\end{aligned}\right\} \quad (160)$$

and where the E's have the same meaning as in equation (37). From equation (160) we conclude that the equations which determine γ and δ are

$$\delta = \tfrac{3}{16} \left[\delta \left(3E_0 - E_2 \right) - \gamma \left(3E_2 - E_4 \right) \right] + 3 - \cos^2 \beta \tag{161}$$

and

$$\gamma = \tfrac{3}{16} \left[\gamma \left(3E_4 - E_2 \right) - \delta \left(3E_2 - E_0 \right) \right] - \left(3 \cos^2 \beta - 1 \right) . \tag{162}$$

Solving these equations, we find that

$$\gamma = \frac{1}{1 - \dfrac{3}{8} \displaystyle\sum_{j=1}^{n} \frac{a_j \left(3 - \mu_j^2 \right)}{1 - \mu_j^2 \sec^2 \beta}} \tag{163}$$

and

$$\delta = 3 \gamma . \tag{164}$$

In reducing the solutions for γ and δ to the foregoing forms, repeated use has been made of the recursion formula (42) which the E's satisfy.

It is seen that the formula (163) for γ bears the same relation to the characteristic equation (III, eq. [30])

$$1 = \frac{3}{8} \sum_{j=1}^{n} \frac{a_j \left(3 - \mu_j^2 \right)}{1 - \mu_j^2 k^2} \tag{165}$$

as the γ defined in paper VIII, equation (40), bears to the corresponding characteristic equation (10) of the same paper. We can, accordingly, express γ in the form (cf. eqs. [52] and [123])

$$\gamma = \frac{1}{\mu_1^2 \ldots \mu_n^2} \frac{P \left(\cos \beta \right) P \left(- \cos \beta \right)}{R \left(\cos \beta \right) R \left(- \cos \beta \right)} , \tag{166}$$

where

$$R \left(\mu \right) = \prod_{a=1}^{n-1} \left(1 - k_a \mu \right) \tag{167}$$

has naturally to be evaluated in terms of the $(n - 1)$ nonvanishing positive roots of the equation (165).

Returning to equation (157), we now express the particular integral in the form

$$I_i^{(0)} = \tfrac{3}{32} F e^{-\tau \sec \beta} \gamma \frac{3 - \mu_i^2}{1 + \mu_i \sec \beta} \qquad \left(i = \pm 1 , \ldots , \pm n \right) . \tag{168}$$

Adding to this particular integral the general solution of the homogeneous system which is compatible with the boundedness of the solution for $\tau \to \infty$, we have

$$\left. \begin{gathered} I_i^{(0)} = \tfrac{3}{32} F \left[\left(3 - \mu_i^2 \right) \sum_{a=1}^{n-1} \frac{M_a e^{-k_a \tau}}{1 + \mu_i k_a} + X + \gamma \frac{3 - \mu_i^2}{1 + \mu_i \sec \beta} e^{-\tau \sec \beta} \right] \\ \left(i = \pm 1 , \ldots , \pm n \right) , \end{gathered} \right\} \tag{169}$$

where the constants M_a ($a = 1, \ldots, n - 1$) and X have to be determined from the boundary conditions at $\tau = 0$.

At $\tau = 0$ we must require that

$$I_i^{(0)} = 0 \qquad \left(i = 1 , \ldots , n , \tau = 0 \right) . \tag{170}$$

In terms of the function

$$G(\mu) = (3-\mu^2)\sum_{a=1}^{n-1}\frac{M_a}{1-\mu k_a}+X+(3-\mu^2)\frac{\gamma}{1-\mu\sec\beta} \tag{171}$$

the boundary conditions are

$$G(\mu_i)=0 \qquad (i=1,\ldots,n). \tag{172}$$

The angular distribution of the reflected radiation corresponding to the part $I^{(0)}$ of I (cf. eq. [152]) is also expressible in terms of $G(\mu)$. We have

$$I^{(0)}(0,\mu) = \tfrac{3}{32}FG(-\mu). \tag{173}$$

As in the earlier cases, we shall now show how an explicit formula for $G(\mu)$ can be found without having the necessity to solve for the constants M_a and X.

Consider the function

$$(1-\mu\sec\beta)R(\mu)G(\mu) = (1-\mu\sec\beta)\prod_{a=1}^{n-1}(1-k_a\mu)G(\mu). \tag{174}$$

This is a polynomial of degree $n+1$ in μ which vanishes for $\mu=\mu_i$, $i=1,\ldots,n$. Consequently, there must exist a proportionality of the form

$$(1-\mu\sec\beta)R(\mu)G(\mu)\propto P(\mu)(\mu+c), \tag{175}$$

where c is some constant. The constant of proportionality can be found from a comparison of the coefficients of the highest powers of μ on either side. In this manner we obtain

$$G(\mu) = (-1)^n k_1\ldots k_{n-1}\left[\sec\beta\sum_{a=1}^{n-1}\frac{M_a}{k_a}+\gamma\right]\frac{P(\mu)(\mu+c)}{R(\mu)(1-\mu\sec\beta)}, \tag{176}$$

On the other hand, since

$$\gamma(3-\cos^2\beta) = \lim_{\mu\to\cos\beta}(1-\mu\sec\beta)G(\mu), \tag{177}$$

we have, according to equations (166) and (176),

$$\left.\begin{aligned}\frac{3-\cos^2\beta}{\mu_1^2\ldots\mu_n^2}\frac{P(\cos\beta)P(-\cos\beta)}{R(\cos\beta)R(-\cos\beta)} &= (-1)^n k_1\ldots k_{n-1}\left[\sec\beta\sum_{a=1}^{n-1}\frac{M_a}{k_a}+\gamma\right]\\ &\quad\times\frac{P(\cos\beta)}{R(\cos\beta)}(\cos\beta+c).\end{aligned}\right\} \tag{178}$$

In other words,

$$(-1)^n k_1\ldots k_{n-1}\left[\sec\beta\sum_{a=1}^{n-1}\frac{M_a}{k_a}+\gamma\right] = \frac{3-\cos^2\beta}{\mu_1^2\ldots\mu_n^2}\frac{P(-\cos\beta)}{R(-\cos\beta)}\frac{1}{\cos\beta+c}. \tag{179}$$

In virtue of this relation, equation (176) becomes

$$G(\mu) = \frac{1}{\mu_1^2\ldots\mu_n^2}\frac{P(\mu)P(-\cos\beta)}{R(\mu)R(-\cos\beta)}\frac{(3-\cos^2\beta)(\mu+c)}{(1-\mu\sec\beta)(\cos\beta+c)}. \tag{180}$$

It remains only to determine c.

From equation (171) it follows that

$$X = G(+\sqrt{3}) = G(-\sqrt{3}). \tag{181}$$

With $G(\mu)$ given by equation (180), the foregoing equation reduces to

$$\frac{P(+\sqrt{3})}{R(+\sqrt{3})}\,\frac{c+\sqrt{3}}{\cos\beta-\sqrt{3}}=\frac{P(-\sqrt{3})}{R(-\sqrt{3})}\,\frac{c-\sqrt{3}}{\cos\beta+\sqrt{3}}. \tag{182}$$

Solving for c, we find

$$c=-\sqrt{3}\,\frac{A(\sqrt{3}+\cos\beta)-B(\sqrt{3}-\cos\beta)}{A(\sqrt{3}+\cos\beta)+B(\sqrt{3}-\cos\beta)}, \tag{183}$$

where we have written

$$A=\frac{P(+\sqrt{3})}{R(+\sqrt{3})}\quad\text{and}\quad B=\frac{P(-\sqrt{3})}{R(-\sqrt{3})}. \tag{184}$$

With this value of c we verify that

$$\cos\beta+c=-\frac{(A-B)(3-\cos^2\beta)}{A(\sqrt{3}+\cos\beta)+B(\sqrt{3}-\cos\beta)} \tag{185}$$

and

$$\left.\begin{aligned}(3-\cos^2\beta)\frac{\mu+c}{\cos\beta+c}=\Big[\frac{A}{A-B}(\sqrt{3}+\cos\beta)(\sqrt{3}-\mu)\\ -\frac{B}{A-B}(\sqrt{3}-\cos\beta)(\sqrt{3}+\mu)\Big].\end{aligned}\right\} \tag{186}$$

The formula (180) for $G(\mu)$ becomes

$$\left.\begin{aligned}G(\mu)&=\frac{1}{\mu_1^2\cdots\mu_n^2}[\xi(\sqrt{3}+\cos\beta)(\sqrt{3}-\mu)\\ &+(1-\xi)(\sqrt{3}-\cos\beta)(\sqrt{3}+\mu)]\times\frac{P(-\cos\beta)P(\mu)}{R(-\cos\beta)R(\mu)}\,\frac{1}{1-\mu\sec\beta},\end{aligned}\right\} \tag{187}$$

where

$$\xi=\frac{A}{A-B}. \tag{188}$$

With $G(\mu)$ given by equation (187), the angular distribution of the reflected radiation corresponding to $I^{(0)}$ can be written in the form

$$\left.\begin{aligned}I^{(0)}(0,\mu)&=\tfrac{3}{32}FH^{(0)}(\mu)\,H^{(0)}(\cos\beta)\frac{\cos\beta}{\cos\beta+\mu}\\ &\times[\xi(\sqrt{3}+\cos\beta)(\sqrt{3}+\mu)+(1-\xi)(\sqrt{3}-\cos\beta)(\sqrt{3}-\mu)].\end{aligned}\right\} \tag{189}$$

This completes the solution of equation (153).

Turning our attention next to equations (154) and (155), we observe that both these equations belong to the general type considered in § 4. We can, therefore, write down at once the expressions governing the angular distribution of the emergent radiations $I^{(1)}(0,\mu)$ and $I^{(2)}(0,\mu)$. We have (cf. eq. [137])

$$I^{(1)}(0,\mu)=-\tfrac{3}{8}F\sin\beta\cos\beta\sin\vartheta\cos\vartheta H^{(1)}(\mu)\,H^{(1)}(\cos\beta)\frac{\cos\beta}{\cos\beta+\mu} \tag{190}$$

and

$$I^{(2)}(0,\mu)=\tfrac{3}{32}F\sin^2\beta\sin^2\vartheta H^{(2)}(\mu)\,H^{(2)}(\cos\beta)\frac{\cos\beta}{\cos\beta+\mu}, \tag{191}$$

where $H^{(1)}(\mu)$ and $H^{(2)}(\mu)$ are to be evaluated in terms of the positive nonvanishing roots of the equations

$$1=\frac{3}{4}\sum_{j=1}^{n}\frac{a_j(1-\mu_j^2)\mu_j^2}{1-k^2\mu_j^2} \tag{192}$$

and

$$1=\frac{3}{16}\sum_{j=1}^{n}\frac{a_j(1-\mu_j^2)^2}{1-k^2\mu_j^2}, \tag{193}$$

respectively.[10]

9. *The angular distribution of the reflected radiation: numerical results.*—Combining the solutions (189), (190), and (191) in accordance with equation (152), we obtain the angular distribution of the reflected radiation. We have

$$\left.\begin{aligned} I(0,\mu) = \tfrac{3}{32}F\,[&H^{(0)}(\mu)H^{(0)}(\mu')\{\xi(\sqrt{3}+\mu)(\sqrt{3}+\mu')+(1-\xi) \\ &\times(\sqrt{3}-\mu)(\sqrt{3}-\mu')\}-4\mu\mu'(1-\mu^2)^{\frac{1}{2}}(1-\mu'^2)^{\frac{1}{2}}H^{(1)}(\mu)H^{(1)}(\mu')\cos\varphi \\ &+(1-\mu^2)(1-\mu'^2)H^{(2)}(\mu)H^{(2)}(\mu')\cos 2\varphi]\frac{\mu'}{\mu+\mu'}, \end{aligned}\right\} \tag{194}$$

where we have written μ' for $\cos\beta$.

TABLE 9

THE CONSTANTS OCCURRING IN THE FORMULA FOR THE ANGULAR DISTRIBUTION OF THE RADIATION REFLECTED FROM A SEMI-INFINITE PLANE-PARALLEL ATMOSPHERE IN ACCORDANCE WITH RAYLEIGH'S LAW IN VARIOUS APPROXIMATIONS

Second Approximation	Third Approximation	Fourth Approximation
$k_1^{(0)}=1.870829$	$k_1^{(0)}=3.08624$ $k_2^{(0)}=1.20629$	$k_1^{(0)}=4.337235$ $k_2^{(0)}=1.562180$ $k_3^{(0)}=1.096117$
$\xi=0.29926$	$\xi=0.29646$	$\xi=0.29561$
$k_1^{(1)}=2.86760$ $k_2^{(1)}=1.13000$	$k_1^{(1)}=4.15155$ $k_2^{(1)}=1.46094$ $k_3^{(1)}=1.06316$	
$k_1^{(2)}=2.79728$ $k_2^{(2)}=1.15840$	$k_1^{(2)}=4.02457$ $k_2^{(2)}=1.49449$ $k_3^{(2)}=1.07209$	

[10] In terms of the quantities

$$\Delta_{2m}^{(1)}=\sum_{j=1}^{n}\frac{a_j(1-\mu_j^2)\mu_j^{2m+2}}{1-k^2\mu_j^2}\quad\text{and}\quad\Delta_{2m}^{(2)}=\sum_{j=1}^{n}\frac{a_j(1-\mu_j^2)^2\mu_j^{2m}}{1-k^2\mu_j^2}$$

the equations (192) and (193) can be reduced to the form (cf. eq. [150])

$$\sum_{m=0}^{n}\Delta_{2m}^{(i)}p_{2m}=0,$$

where the p_{2m}'s have the same meanings as in eq. (150). The Δ's themselves can be evaluated simply from the recurrence formulae

$$\Delta_{2m}^{(1)}=\frac{1}{k^2}\left[\Delta_{2m-2}^{(1)}-\frac{2}{(2m+1)(2m+3)}\right],$$

$$\Delta_{2m}^{(2)}=\frac{1}{k^2}\left[\Delta_{2m-2}^{(2)}-\frac{8}{(2m-1)(2m+1)(2m+3)}\right],$$

and the relations

$$\Delta_0^{(1)}=\tfrac{4}{3}\quad\text{and}\quad\Delta_0^{(2)}=\tfrac{16}{3}.$$

In Table 9 the various constants which occur in the foregoing solution in the first four approximations are collected together. In Table 10 the function $H^{(0)}(\mu)$ in the second, third, and fourth approximations is tabulated, while in Table 11 the functions $H^{(1)}(\mu)$ and $H^{(2)}(\mu)$ are tabulated in the second and the third approximations. An inspection of these tables reveals that the accuracy of the third approximation is within 1 per cent over the entire range in which the functions are defined.

TABLE 10

THE FUNCTION $H^{(0)}(\mu)$ IN SECOND, THIRD, AND FOURTH APPROXIMATIONS (Rayleigh's Case)

μ	Second Approximation	Third Approximation	Fourth Approximation
0	1.000	1.000	1.000
0.1	1.217	1.233	1.242
0.2	1.424	1.448	1.460
0.3	1.626	1.653	1.665
0.4	1.823	1.853	1.864
0.5	2.018	2.048	2.060
0.6	2.210	2.241	2.252
0.7	2.401	2.432	2.443
0.8	2.591	2.622	2.632
0.9	2.779	2.810	2.820
1.0	2.967	2.998	3.008

TABLE 11

THE FUNCTION $H^{(1)}(\mu)$ AND $H^{(2)}(\mu)$ IN THE SECOND AND THIRD APPROXIMATIONS (Rayleigh's Case)

μ	$H^{(1)}(\mu)$		$H^{(2)}(\mu)$	
	Second Approximation	Third Approximation	Second Approximation	Third Approximation
0	1.000	1.000	1.000	1.000
0.1	1.008	1.008	1.011	1.013
0.2	1.014	1.014	1.019	1.021
0.3	1.019	1.018	1.024	1.026
0.4	1.022	1.022	1.028	1.030
0.5	1.025	1.024	1.031	1.033
0.6	1.028	1.027	1.033	1.035
0.7	1.029	1.029	1.035	1.037
0.8	1.031	1.030	1.037	1.038
0.9	1.033	1.032	1.038	1.040
1.0	1.034	1.033	1.039	1.041

As stated in the introduction, applications of the solutions obtained in this paper to the problem of planetary illumination will be found in a forthcoming paper.

I wish to record my indebtedness to Mrs. Frances H. Breen for assistance with the numerical work.

APPENDIX

The methods used in §8 enable us to complete the explicit solution of the constants of integration and the law of darkening for the problem considered in paper III. Thus, considering the function (III, eq. [55])

$$S(\mu) = (3-\mu^2)\sum_{a=1}^{n-1}\frac{L_a}{1-\mu k_a}+Q-\mu\,, \tag{1}$$

we have already shown that (cf. III, eqs. [59] and [60])

$$S(\mu) = \sigma\frac{P(\mu)}{R(\mu)}\,, \tag{2}$$

where

$$\sigma = (-1)^n k_1 \ldots\ldots k_{n-1}\left(1-\sum_{a=1}^{n-1}\frac{L_a}{k_a}\right) \tag{3}$$

is a constant and $P(\mu)$ and $R(\mu)$ are defined as in III, equations (57) and (58). In paper III the constant σ was left undetermined. We shall now show how σ can also be determined.

Setting $\mu = +\sqrt{3}$, respectively, $-\sqrt{3}$ in equations (1) and (2), we have (cf. eq. [186])

$$Q-\sqrt{3} = \sigma\frac{P(+\sqrt{3})}{R(+\sqrt{3})} = \sigma A\,, \tag{4}$$

and

$$Q+\sqrt{3} = \sigma\frac{P(-\sqrt{3})}{R(-\sqrt{3})} = \sigma B\,. \tag{5}$$

Solving these equations, we find

$$Q = \sqrt{3}\,\frac{B+A}{B-A}\,, \tag{6}$$

and

$$\sigma = \frac{2\sqrt{3}}{B-A}\,. \tag{7}$$

With σ thus determined, the constants L_a also become determinate according to III, equation (63).

The law of darkening takes the form

$$I(0,\mu) = \tfrac{3}{4}FS(-\mu) = \frac{3\sqrt{3}}{2(B-A)}F\frac{P(-\mu)}{R(-\mu)}\,, \tag{8}$$

or, introducing the function $H^{(0)}(\mu)$ as in equation (189), we can write

$$I(0,\mu) = \lambda F H^{(0)}(\mu)\,, \tag{9}$$

where

$$\lambda = (-1)^n \mu_1 \ldots\ldots \mu_n \frac{3\sqrt{3}}{2(B-A)}\,. \tag{10}$$

In the second, third, and fourth approximations λ takes the values

$$\left.\begin{aligned} \lambda &= 0.42064 \quad \text{(second approximation)}\,,\\ &= 0.41950 \quad \text{(third approximation)}\,,\\ &= 0.41916 \quad \text{(fourth approximation)}\,. \end{aligned}\right\} \tag{11}$$