ON THE RADIATIVE EQUILIBRIUM OF A STELLAR ATMOSPHERE. X

S. CHANDRASEKHAR
Yerkes Observatory
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ABSTRACT

In this paper a detailed theory of the radiative equilibrium of an atmosphere in which the Thomson scattering by free electrons governs the transfer of radiation is developed. In particular, allowance has been made for the polarization of the scattered radiation; and the equations of transfer for the intensities I_l and I_r, referring, respectively, to the two states of polarization in which the electric vector vibrates in the meridian plane and at right angles to it, are separately formulated. The equations of transfer are found to be

$$\mu \frac{dI_l}{d\tau} = I_l - \frac{3}{8}\left\{2\int_{-1}^{+1} I_l(\tau, \mu')(1-\mu'^2)d\mu' + \mu^2\int_{-1}^{+1} I_l(\tau, \mu')(3\mu'^2 - 2)d\mu' + \mu^2\int_{-1}^{+1} I_r(\tau, \mu')d\mu'\right\}$$

and

$$\mu \frac{dI_r}{d\tau} = I_r - \frac{3}{8}\left\{\int_{-1}^{+1} I_r(\tau, \mu')d\mu' + \int_{-1}^{+1} I_l(\tau, \mu')\mu'^2 d\mu'\right\}.$$

These equations have been solved in a general nth approximation, and their explicit numerical forms have been found in the second and the third approximations.

It is found that the theory predicts different laws of darkening for the two states of polarization distinguished by I_l and I_r. The emergent radiation is therefore polarized, and it is further predicted that the degree of polarization must vary from zero at the center of the disk to 11 per cent at the limb.

1. *Introduction.*—It is now generally recognized that the Thomson scattering by free electrons must play an important role in the transfer of radiation in the atmospheres of the early-type stars.[1] Thus, it has been suggested by J. L. Greenstein[2] that the absolute-magnitude effect among the early-types stars shown by the discontinuity at the head of the Balmer series is probably to be attributed to the increased importance of electron scattering as we go to the more luminous stars. But it does not seem to have been observed so far that, if electron scattering is as major a factor as there appears to be evidence for, then there is an associated effect which should be detectable, namely, the polarization of the continuous radiation. It is the object of this paper to analyze this effect theoretically and to show that it is within the possibilities of detection. The detailed theory of radiative transfer in an atmosphere in which electron scattering plays the principal part, developed in this paper, predicts a degree of polarization to the extent of 11 per cent at the limb. In practice, this effect may be partially masked by other sources of continuous opacity; but it would seem, from all the available evidence, that the effect is probably present in detectable amounts in the early-type stars. Moreover, it would appear that the most favorable conditions under which the phenomenon could be observed are during the phases close to the primary minimum in an eclipsing binary, one component of which is an early-type star.

On the theoretical side, the problem discussed in this paper provides the first example in which the equations of transfer for the two states of polarization have been explicitly formulated and solved.[3]

[1] A. Unsöld, *Zs. f. Ap.*, **21**, 229, 1942; M. Rudkjøbing, *Zs. f. Ap.*, **21**, 254, 1942.

[2] *Ap. J.*, **95**, 299, 1942.

[3] The method of solution adopted is those of the earlier papers of this series. Familiarity with papers II and III (*Ap. J.*, **100**, 76, 117, 1944) is necessary to follow the analysis of this paper.

2. *The equations of transfer for the two components of polarization in an atmosphere in which the scattering by free electrons governs the transfer of radiation.*—We shall consider the radiative equilibrium of a semi-infinite plane-parallel atmosphere with a constant net flux of radiation and in which the transfer of radiation takes place in accordance with Thomson's laws of scattering by free electrons. It is apparent that under these conditions we can characterize the field of radiation by the intensities $I_l(z, \vartheta)$ and $I_r(z, \vartheta)$ at height z and inclined at an angle ϑ to the positive normal and referring to the states of polarization in which the electric vectors vibrate along the principal meridian and at right angles to it, respectively. And we require to write down the equations of transfer for the two components separately. For this purpose we shall first formulate the laws of scattering in the form we shall need them.

According to Thomson's classical theory[4] of the scattering by free electrons, the amount of radiation (initially unpolarized) which is scattered (per unit time) in a direction inclined at an angle Θ to the direction of incidence, and confined to an element of solid angle $d\omega'$ and per free electron, is given by

$$\frac{8\pi}{3}\frac{e^4}{m^2c^4}I\left\{\tfrac{3}{4}\left(1+\cos^2\Theta\right)\frac{d\omega'}{4\pi}\right\}, \tag{1}$$

where I denotes the intensity[5] of the incident radiation, e the charge of the electron, m its mass, and c the velocity of light. Moreover, the scattered radiation is polarized with the direction of the electric vector perpendicular to the *plane of scattering*.[6] For our purposes, however, we need a more detailed formulation of the law of scattering which will allow us to take into account a partial polarization of the incident light. We shall formulate these more detailed laws in the following manner:

Let $I_\perp$ denote the intensity of radiation plane-polarized with the electric vector perpendicular to the plane of scattering. Then the amount of radiation scattered in a direction inclined at an angle Θ to direction of $I_\perp$ and confined to an element of solid angle $d\omega'$ and per free electron is

$$\frac{8\pi}{3}\frac{e^4}{m^2c^4}I_\perp\left(\frac{3}{2}\frac{d\omega'}{4\pi}\right). \tag{2}$$

The scattered radiation is also polarized with the electric vector perpendicular to the plane of scattering. On the other hand, if the incident light is polarized with the electric vector parallel to the plane of scattering, then the scattered radiation is also polarized in the same way, but its amount is now given by (cf. eq.[2])

$$\frac{8\pi}{3}\frac{e^4}{m^2c^4}I_{||}\left(\frac{3}{2}\cos^2\Theta\,\frac{d\omega'}{4\pi}\right), \tag{3}$$

where $I_{||}$ denotes the intensity of the incident polarized radiation. More generally, if the incident light is plane-polarized with its electric vector inclined to the plane of scattering by an angle α, then the amount of scattered radiation (under the same circumstances to which equations [2] and [3] refer) is given by

$$\frac{8\pi}{3}\frac{e^4}{m^2c^4}I\left\{\tfrac{3}{2}\left(\sin\alpha+\cos\alpha\,\cos\Theta\right)^2\frac{d\omega'}{4\pi}\right\}, \tag{4}$$

[4] Cf. A. H. Compton and S. K. Allison, *X-Rays in Theory and Experiment*, pp. 117–119, New York: D. Van Nostrand, 1935.

[5] Since the Thomson scattering coefficient is independent of wave length, we can directly consider the intensity I integrated over all the frequencies.

[6] This is the plane which contains the directions of the incident and the scattered light.

while its plane of polarization is such that the electric vector is inclined to the plane of scattering by an angle

$$\beta = \tan^{-1}(\tan \alpha \sec \Theta) . \tag{5}$$

Thus there is, in general, a "turning" of the plane of polarization. We shall now show how, with the laws of scattering formulated in this manner, we can obtain the equations of transfer for the two components of polarization distinguished by I_l and I_r.

It is first evident that the equations of transfer must be expressible in the forms

$$\cos\vartheta \frac{dI_l}{d\tau} = I_l - \mathfrak{J}_{l,l}(\tau, \vartheta) - \mathfrak{J}_{r,l}(\tau, \vartheta) \tag{6}$$

and

$$\cos\vartheta \frac{dI_r}{d\tau} = I_r - \mathfrak{J}_{l,r}(\tau, \vartheta) - \mathfrak{J}_{r,r}(\tau, \vartheta) , \tag{7}$$

where τ denotes the optical depth, measured in terms of the Thomson scattering coefficient

$$\sigma = \frac{8\pi}{3} \frac{e^4}{m^2 c^4} N_e . \tag{8}$$

with N_e denoting the number of electrons per unit mass. Further, in equation (6), $\mathfrak{J}_{l,l}$ and $\mathfrak{J}_{r,l}$ are the contributions to the source function for the radiation in a particular direction and polarized with the electric vector in the meridian plane, arising from the scattering from all other directions of radiations polarized with the electric vectors parallel, respectively, perpendicular to the appropriate meridian planes. Similarly, in equation (7), $\mathfrak{J}_{l,r}$ and $\mathfrak{J}_{r,r}$ are the contributions to the source function for the radiation in a particular direction polarized with the electric vector perpendicular to the meridian plane, arising from the scattering from all other directions of radiations polarized with the electric vectors parallel, respectively, perpendicular to the appropriate meridian planes.

To evaluate $\mathfrak{J}_{l,l}(\tau, \vartheta)$ and $\mathfrak{J}_{l,r}(\tau, \vartheta)$, we consider the contributions to these source functions for the radiation in the direction $(\vartheta, 0)$, say, arising from the scattering of radiation of intensity $I_l(\tau, \vartheta')$ in the direction (ϑ', φ') and polarized with the electric vector in the meridian plane through (ϑ', φ'). Let ξ_l denote a quantity proportional to the amplitude of the electric vector such that

$$I_l = \xi_l^2 . \tag{9}$$

(We shall refer, quite generally, to ξ's defined in this manner as simply the *amplitude*.) The components of the amplitude that are parallel, respectively, perpendicular to the plane of scattering, are

$$\xi_l \cos i_1 \qquad \text{and} \qquad \xi_l \sin i_1 , \tag{10}$$

where i_1 denotes the angle between the meridian plane OZP_1 through $P_1 = (\vartheta', \varphi')$ and the plane of scattering OP_1P_2 ($P_2 = [\vartheta, 0]$) (see Fig. 1). When this radiation is scattered into the direction OP_2, the components of the scattered amplitude that are parallel, respectively, perpendicular to the plane OP_1P_2, are proportional to

$$\xi_l \cos i_1 \cos \Theta \qquad \text{and} \qquad \xi_l \sin i_1 , \tag{11}$$

while the amplitudes in the meridian plane OZX through P_2, and at right angles to it, are, respectively, proportional to

$$\xi_l (\sin i_1 \sin i_2 - \cos i_1 \cos i_2 \cos \Theta) \tag{12}$$

and

$$-\xi_l (\sin i_1 \cos i_2 + \sin i_2 \cos i_1 \cos \Theta) . \tag{13}$$

where i_2 denotes the angle between the planes OZX and OP_1P_2. The contributions to the source functions $\mathfrak{I}_{l,l}(z, \vartheta, 0)$ $(= \mathfrak{I}_{l,l}(z, \vartheta))$ and $\mathfrak{I}_{l,r}(z, \vartheta, 0)$ $(= \mathfrak{I}_{l,r}(z, \vartheta))$, arising from the scattering of the radiation $I_l(z, \vartheta', \varphi')$ $(= I_l(z, \vartheta'))$ in the direction (ϑ', φ') and confined to an element of solid angle $d\omega'$ are, therefore,

$$d\mathfrak{I}_{l,l} = \tfrac{3}{2} I_l(\tau, \vartheta')(\sin i_1 \sin i_2 - \cos i_1 \cos i_2 \cos \Theta)^2 \frac{d\omega'}{4\pi} \tag{14}$$

and

$$d\mathfrak{I}_{l,r} = \tfrac{3}{2} I_l(\tau, \vartheta')(\sin i_1 \cos i_2 + \sin i_2 \cos i_1 \cos \Theta)^2 \frac{d\omega'}{4\pi}. \tag{15}$$

Hence,

$$\mathfrak{I}_{l,l} = \frac{3}{8\pi}\int_0^{\pi}\int_0^{2\pi} I_l(\tau, \vartheta')(\sin i_1 \sin i_2 - \cos i_1 \cos i_2 \cos \Theta)^2 \sin \vartheta' d\vartheta' d\varphi' \tag{16}$$

and

$$\mathfrak{I}_{l,r} = \frac{3}{8\pi}\int_0^{\pi}\int_0^{2\pi} I_l(\tau, \vartheta')(\sin i_1 \cos i_2 + \sin i_2 \cos i_1 \cos \Theta)^2 \sin \vartheta' d\vartheta' d\varphi'. \tag{17}$$

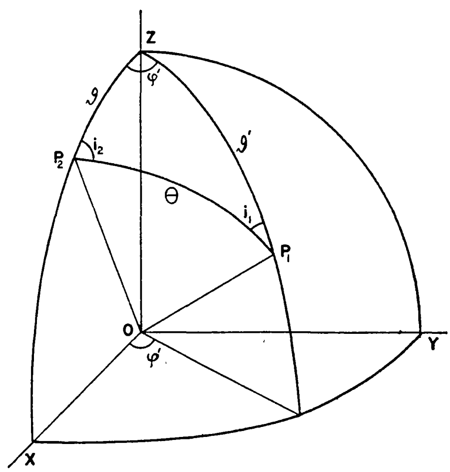

FIG. 1

On the other hand, from the spherical triangle ZP_1P_2 we have

$$\sin \varphi' \cos \vartheta' = \cos i_2 \sin i_1 + \sin i_2 \cos i_1 \cos \Theta . \tag{18}$$

Equation (17) accordingly reduces to the form

$$\mathfrak{I}_{l,r} = \frac{3}{8\pi}\int_0^{\pi}\int_0^{2\pi} I_l(\tau, \vartheta') \sin^2 \varphi' \cos^2 \vartheta' \sin \vartheta' d\vartheta' d\varphi'; \tag{19}$$

or, in view of the axial symmetry of the radiation field about the z-axis, in our problem, we can write

$$\mathfrak{I}_{l,r}(\tau, \mu) = \frac{3}{8}\int_{-1}^{+1} I_l(\tau, \mu') \mu'^2 d\mu', \tag{20}$$

where we have used μ and μ' to denote $\cos \vartheta$ and $\cos \vartheta'$, respectively.

From equations (16) and (17) we have

$$\mathfrak{I}_{l,l} + \mathfrak{I}_{l,r} = \frac{3}{8\pi}\int_0^{\pi}\int_0^{2\pi} I_l(\tau, \vartheta')(\sin^2 i_1 + \cos^2 i_1 \cos^2 \Theta) \sin \vartheta' d\vartheta' d\varphi'; \tag{21}$$

or, since

$$\left.\begin{aligned}\sin^2 i_1+\cos^2 i_1\cos^2\Theta &= 1-\cos^2 i_1\sin^2\Theta\\ &= 1-(\cos\vartheta\sin\vartheta'-\sin\vartheta\cos\vartheta'\cos\varphi')^2\,,\end{aligned}\right\}\tag{22}$$

we have

$$\mathfrak{J}_{l,l}+\mathfrak{J}_{l,r}=\frac{3}{8}\int_{-1}^{+1}I_l(\tau,\mu')\left[2-\mu'^2+\mu^2(3\mu'^2-2)\right]d\mu'.\tag{23}$$

From equations (20) and (23) we now obtain

$$\mathfrak{J}_{l,l}(\tau,\mu)=\frac{3}{8}\int_{-1}^{+1}I_l(\tau,\mu')\left[2(1-\mu'^2)+\mu^2(3\mu'^2-2)\right]d\mu'.\tag{24}$$

To determine the source functions $\mathfrak{J}_{r,l}$ and $\mathfrak{J}_{r,r}$, we proceed along similar lines, considering radiation of intensity $I_r(\tau,\vartheta',\varphi')$ $(=I_r(\tau,\vartheta'))$ in the direction (ϑ',φ') and polarized with the electric vector perpendicular to the meridian plane OZP_1 and evaluating its contribution to the source function for the radiation in the direction $(\vartheta, 0)$. Let ξ_r denote the amplitude corresponding to the intensity $I_r(\tau,\vartheta',\varphi')$. Its components, parallel, respectively, perpendicular to the scattering plane, are

$$\xi_r\sin i_1\qquad\text{and}\qquad\xi_r\cos i_1\,.\tag{25}$$

When this radiation is scattered into the direction OP_2, the components of the scattered amplitude, parallel, respectively, perpendicular to the plane OP_1P_2, are proportional to

$$\xi_r\sin i_1\cos\Theta\qquad\text{and}\qquad\xi_r\cos i_1\,.\tag{26}$$

The amplitudes in the meridian plane OZX and at right angles to it are, respectively, proportional to

$$\xi_r(\sin i_1\cos i_2\cos\Theta+\cos i_1\sin i_2)=\xi_r\sin\varphi'\cos\vartheta\tag{27}$$

and

$$\xi_r(\sin i_1\sin i_2\cos\Theta-\cos i_1\cos i_2)=\xi_r\cos\varphi'\,.\tag{28}$$

Accordingly, the contributions to the source functions $\mathfrak{J}_{r,l}(\tau,\vartheta,0)$ and $\mathfrak{J}_{r,r}(\tau,\vartheta,0)$, arising from the scattering of the radiation $I_r(\tau,\vartheta',\varphi')$ in the direction (ϑ',φ') and confined to an element of solid angle $d\omega'$, are given by

$$d\mathfrak{J}_{r,l}=\tfrac{3}{2}I_r(\tau,\vartheta')(\sin i_1\cos i_2\cos\Theta+\cos i_1\sin i_2)^2\frac{d\omega'}{4\pi}\tag{29}$$

and

$$d\mathfrak{J}_{r,r}=\tfrac{3}{2}I_r(\tau,\vartheta')\cos^2\varphi'\frac{d\omega'}{4\pi}.\tag{30}$$

Hence,

$$\mathfrak{J}_{r,r}=\frac{3}{8\pi}\int_0^{\pi}\int_0^{2\pi}I_r(\tau,\vartheta')\cos^2\varphi'\sin\vartheta'd\vartheta'd\varphi',\tag{31}$$

or

$$\mathfrak{J}_{r,r}(\tau,\mu)=\frac{3}{8}\int_{-1}^{+1}I_r(\tau,\mu')\,d\mu'.\tag{32}$$

On the other hand (cf. eqs. [27] and [28]),

$$\mathfrak{J}_{r,r}+\mathfrak{J}_{r,l}=\frac{3}{8\pi}\int_0^{\pi}\int_0^{2\pi}I_r(\tau,\vartheta')(\sin^2 i_1\cos^2\Theta+\cos^2 i_1)\sin\vartheta'd\vartheta'd\varphi';\tag{33}$$

or, since

$$\sin^2 i_1\cos^2\Theta+\cos^2 i_1=1-\sin^2 i_1\sin^2\Theta=1-\sin^2\vartheta\sin^2\varphi'\,,\tag{34}$$

we have

$$\mathfrak{J}_{r,r}+\mathfrak{J}_{r,l}=\frac{3}{8}\int_{-1}^{+1}I_r(\tau,\mu')(1+\mu^2)\,d\mu'. \tag{35}$$

From equations (32) and (35) we now obtain

$$\mathfrak{J}_{r,l}(\tau,\mu)=\tfrac{3}{8}\mu^2\int_{-1}^{+1}I_r(\tau,\mu')\,d\mu'. \tag{36}$$

Combining equations (6), (24), and (36) and similarly equations (7), (20), and (32), we obtain

$$\left.\begin{aligned}\mu\frac{dI_l}{d\tau}=I_l-\tfrac{3}{8}\Big\{2\int_{-1}^{+1}I_l(\tau,\mu')(1-\mu'^2)\,d\mu'+\mu^2\int_{-1}^{+1}I_l(\tau,\mu')(3\mu'^2-2)\,d\mu'\\+\mu^2\int_{-1}^{+1}I_r(\tau,\mu')\,d\mu'\Big\}\end{aligned}\right\} \tag{37}$$

and

$$\mu\frac{dI_r}{d\tau}=I_r-\tfrac{3}{8}\Big\{\int_{-1}^{+1}I_r(\tau,\mu')\,d\mu'+\int_{-1}^{+1}I_l(\tau,\mu')\,\mu'^2d\mu'\Big\}, \tag{38}$$

which are the required equations of transfer for I_l and I_r.

We may note here that, for radiation initially unpolarized, the source functions for radiations polarized in the two ways can be obtained from equations (20), (24), (32), and (36) by setting $I_l = I_r = \frac{1}{2}I$. Thus,

$$\mathfrak{J}_l=\frac{3}{16}\Big\{2\int_{-1}^{+1}I(\tau,\mu')(1-\mu'^2)\,d\mu'+\mu^2\int_{-1}^{+1}I(\tau,\mu')(3\mu'^2-1)\,d\mu'\Big\} \tag{39}$$

and

$$\mathfrak{J}_r=\frac{3}{16}\int_{-1}^{+1}I(\tau,\mu')(1+\mu'^2)\,d\mu', \tag{40}$$

which agree with A. Schuster's well-known formulae.[7]

In terms of the quantities J and K, defined in the usual manner (cf. paper III, eq. [6]), we can re-write the equations of transfer (37) and (38) in the following forms:

$$\mu\frac{dI_l}{d\tau}=I_l-\tfrac{3}{4}\Big\{2(J_l-K_l)+\mu^2(3K_l-2J_l+J_r)\Big\} \tag{41}$$

and

$$\mu\frac{dI_r}{d\tau}=I_r-\tfrac{3}{4}(J_r+K_l). \tag{42}$$

From the equations of transfer in the foregoing forms we can readily establish the flux integral

$$F_l+F_r=F=2\int_{-1}^{+1}[I_l(\tau,\mu)+I_r(\tau,\mu)]\mu d\mu=\text{constant} \tag{43}$$

and the "K-integral"

$$K_l+K_r=\tfrac{1}{4}F(\tau+Q) \tag{44}$$

where Q is a constant.

3. *The general solution of the equations of transfer in the* n*th approximation.*—In solving equations (37) and (38) we shall follow the method developed in the earlier papers of this series and replace the various integrals which occur on the right-hand sides of the equations by sums according to Gauss's formula for numerical quadratures. Thus, in

[7] *M.N.*, **40**, 35, 1879; see also M. Minnaert, *Zs. f. Ap.*, **1**, 209, 1930, and H. Zanstra, *M.N.*, **101**, 250, 1941.

the nth approximation, equations (37) and (38) are replaced by the following system of $4n$ linear equations:

$$\left.\begin{aligned}\mu_i \frac{dI_{l,i}}{d\tau} = I_{l,i} - \tfrac{3}{8}\left[2\Sigma a_j I_{l,j}(1-\mu_j^2) + \mu_i^2\{\Sigma a_j I_{l,j}(3\mu_j^2-2) + \Sigma a_j I_{r,j}\}\right] \\ (i=\pm 1, \ldots, \pm n)\end{aligned}\right\} \quad (45)$$

and

$$\mu_i \frac{dI_{r,i}}{d\tau} = I_{r,i} - \tfrac{3}{8}(\Sigma a_j I_{r,j} + \Sigma a_j I_{l,j}\mu_j^2) \qquad (i=\pm 1, \ldots, \pm n), \quad (46)$$

where the μ_i's ($i = \pm 1, \ldots, \pm n$) are the zeros of the Legendre polynomial of order $2n$ and the a_i's are the appropriate Gaussian weights. Further, in equations (45) and (46) we have written $I_{l,i}$ and $I_{r,i}$ for $I_l(\tau, \mu_i)$ and $I_r(\tau, \mu_i)$, respectively.

We shall now find the different linearly independent solutions of equations (45) and (46) and later, by combining these, obtain the general solution.

First, we seek a solution of equations (45) and (46) of the form

$$I_{l,i} = g_i e^{-k\tau} \quad \text{and} \quad I_{r,i} = h_i e^{-k\tau} \qquad (i=\pm 1, \ldots, \pm n), \quad (47)$$

where the g_i's, h_i's, and k are constants, for the present unspecified. Substituting the foregoing forms for $I_{l,i}$ and $I_{r,i}$ in equations (45) and (46), we obtain

$$g_i(1+\mu_i k) = \tfrac{3}{8}\left[2\Sigma a_j g_j(1-\mu_j^2) + \mu_i^2\{\Sigma a_j g_j(3\mu_j^2-2) + \Sigma a_j h_j\}\right] \quad (48)$$

and

$$h_i(1+\mu_i k) = \tfrac{3}{8}\left[\Sigma a_j g_j \mu_j^2 + \Sigma a_j h_j\right]. \quad (49)$$

Equations (48) and (49) imply that g_i and h_i must be expressible in the forms

$$g_i = \frac{\alpha\mu_i^2 + \beta}{1+\mu_i k} \quad \text{and} \quad h_i = \frac{\gamma}{1+\mu_i k} \qquad (i=\pm 1, \ldots, \pm n), \quad (50)$$

where α, β, and γ are certain constants, independent of i. Inserting the solution (50) back into equations (48) and (49), we find

$$\left.\begin{aligned}\alpha\mu_i^2 + \beta = \tfrac{3}{8}\left[2\{\alpha(D_2-D_4) + \beta(D_0-D_2)\} + \mu_i^2\{\alpha(3D_4-2D_2)\right. \\ \left. + \beta(3D_2-2D_0) + \gamma D_0\}\right]\end{aligned}\right\} \quad (51)$$

and

$$\gamma = \tfrac{3}{8}(\alpha D_4 + \beta D_2 + \gamma D_0), \quad (52)$$

where we have introduced the quantities D_0, D_2, and D_4, defined according to the formula

$$D_m = \Sigma \frac{a_j \mu_j^m}{1+\mu_j k}. \quad (53)$$

Since equations (51) and (52) are valid for all i, we must require that

$$\tfrac{8}{3}\alpha = \alpha(3D_4 - 2D_2) + \beta(3D_2 - 2D_0) + \gamma D_0, \quad (54)$$

$$\tfrac{4}{3}\beta = \alpha(D_2 - D_4) + \beta(D_0 - D_2), \quad (55)$$

and

$$\tfrac{8}{3}\gamma = \alpha D_4 + \beta D_2 + \gamma D_0. \quad (56)$$

It is seen that equations (54), (55), and (56), together, represent a system of homogeneous linear equations for α, β, and γ. The determinant of this system must, therefore, be required to vanish. Thus,

$$\begin{vmatrix} 3D_4 - 2D_2 - \frac{8}{3} & 3D_2 - 2D_0 & D_0 \\ D_2 - D_4 & D_0 - D_2 - \frac{4}{3} & 0 \\ D_4 & D_2 & D_0 - \frac{8}{3} \end{vmatrix} = 0 . \tag{57}$$

Expanding this determinant by the elements of the last column, we find, after some minor reductions, that

$$D_0\left(\tfrac{4}{3} - D_0 + 2D_2 - D_4\right) - \left(\tfrac{32}{9} - \tfrac{8}{3}D_0 + \tfrac{16}{3}D_2 - 4D_4 + D_0D_4 - D_2^2\right) = 0 . \tag{58}$$

To simplify equation (58) still further, we must make use of certain relations which can be derived from the recursion formulae[8]

$$D_{2m} = \frac{1}{k^2}\left(D_{2m-2} - \frac{2}{2m-1}\right) \tag{59}$$

and

$$D_{2m-1} = -k D_{2m} , \tag{60}$$

which the D's satisfy. From equation (59) we infer, in particular, that

$$D_2 = \frac{1}{k^2}(D_0 - 2) \tag{61}$$

and

$$D_4 = \frac{1}{k^2}\left(D_2 - \tfrac{2}{3}\right) = \frac{1}{k^4}(D_0 - 2) - \frac{2}{3k^2} . \tag{62}$$

A further relation which follows from equations (61) and (62) is

$$D_2\left(D_2 - \tfrac{2}{3}\right) = D_4(D_0 - 2) , \tag{63}$$

or

$$D_0D_4 - D_2^2 = 2D_4 - \tfrac{2}{3}D_2 . \tag{64}$$

By using equation (64), equation (58) reduces to

$$D_0\left(\tfrac{4}{3} - D_0 + 2D_2 - D_4\right) - \left(\tfrac{32}{9} - \tfrac{8}{3}D_0 + \tfrac{14}{3}D_2 - 2D_4\right) = 0 . \tag{65}$$

Now, substituting for D_2 and D_4 according to equations (61) and (62) in terms of D_0 in equation (65), we find, after some simplification, that

$$\frac{1}{k^4}(D_0 - 2)^2 - \frac{2}{k^2}(D_0 - 2)^2 + D_0^2 - 4D_0 + \tfrac{32}{9} = 0 . \tag{66}$$

Again using equations (61) and (62), we can re-write the foregoing equation in the form

$$D_2^2 - 2D_2(D_0 - 2) + D_0^2 - 4D_0 + \tfrac{32}{9} = 0 \tag{67}$$

or

$$(D_0 - D_2)^2 - 4(D_0 - D_2) + \tfrac{32}{9} = 0 . \tag{68}$$

[8] Cf. *Ap. J.*, **101**, 328, 1945 (eqs. [54] and [56]).

Equation (68) is equivalent to

$$(D_0 - D_2 - \tfrac{8}{3})(D_0 - D_2 - \tfrac{4}{3}) = 0 . \tag{69}$$

In other words, either

$$D_0 - D_2 = \Sigma \frac{a_j(1-\mu_j^2)}{1+\mu_j k} = \frac{8}{3} \qquad \text{(case 1)} \tag{70}$$

or

$$D_0 - D_2 = \Sigma \frac{a_j(1-\mu_j^2)}{1+\mu_j k} = \frac{4}{3} \qquad \text{(case 2)} . \tag{71}$$

Equations (70) and (71) can be written alternatively in the forms

$$\sum_{j=1}^{n} \frac{a_j(1-\mu_j^2)}{1-\mu_j^2 k^2} = \frac{4}{3} \qquad \text{(case 1)} \tag{72}$$

and

$$\sum_{j=1}^{n} \frac{a_j(1-\mu_j^2)}{1-\mu_j^2 k^2} = \frac{2}{3} \qquad \text{(case 2)} . \tag{73}$$

And k^2 must be a root of either of the two foregoing equations.

Equation (72) is of order n in k^2 and admits $2n$ distinct nonvanishing roots for k which occur in pairs as

$$\pm k_\alpha \qquad (\alpha = 1, \ldots, n) . \tag{74}$$

On the other hand, equation (73), though of order $2n$ in k^2, admits of only $(2n - 2)$ distinct nonvanishing roots for k, since $k^2 = 0$ is a root.[9] However, these $2n - 2$ roots also occur in pairs, which we shall denote by

$$\pm \kappa_\beta \qquad (\beta = 1 \ldots, n-1) , \tag{75}$$

to distinguish them from the roots of equation (72).

Case 1: k^2 *a root of equation* (72).—In this case, $D_0 - D_2 = 8/3$; and from equations (61) and (62) we readily find that

$$D_0 = \frac{2}{3}\frac{4k^2-3}{k^2-1}; \qquad D_2 = \frac{2}{3(k^2-1)}; \qquad D_4 = \frac{2}{3k^2}\frac{2-k^2}{k^2-1} . \tag{76}$$

With these values for D_0, D_2, and D_4, equations (55) and (56) lead to

$$\alpha = -k^2 \beta \tag{77}$$

and

$$\gamma = -(k^2-1)\beta . \tag{78}$$

Accordingly, equations (45) and (46) allow $2n$ linearly independent integrals of the form

$$\left.\begin{aligned} I_{l,i} &= \text{constant}\,(1 \mp k_\alpha \mu_i)\, e^{\mp k_\alpha \tau} \\ I_{r,i} &= -\,\text{constant}\,\frac{k_\alpha^2 - 1}{1 \pm k_\alpha \mu_i}\, e^{\mp k_\alpha \tau} \end{aligned}\right\} , \qquad \left\{\begin{matrix} (i = \pm 1, \ldots, \pm n) \\ (\alpha = 1, \ldots, n) \end{matrix}\right\} . \tag{79}$$

[9] Note that $\sum_{j=1}^{n} a_j(1-\mu_j^2) = \frac{4}{3}$.

Case 2: κ^2 a root of equation (73).—In this case, $D_0 - D_2 = 4/3$; and equations (61) and (62) now give

$$D_0 = \frac{2}{3}\frac{2\kappa^2 - 3}{\kappa^2 - 1}; \qquad D_2 = D_4 = -\frac{2}{3(\kappa^2 - 1)}. \tag{80}$$

With these values of D_0, D_2, and D_4 it is seen that equation (55) is satisfied identically, while the consideration of equations (54) and (56) leads to the result that

$$\alpha = -\beta \quad \text{and} \quad \gamma = 0. \tag{81}$$

Accordingly, equations (45) and (46) admit of $(2n - 2)$ linearly independent integrals of the form

$$I_{l,i} = \text{constant}\, \frac{1 - \mu_i^2}{1 \pm \mu_i \kappa_\beta}\, e^{\mp \kappa_\beta \tau} \qquad \left\{ \begin{array}{l} (i = \pm 1, \ldots, \pm n) \\ (\beta = 1, \ldots, n - 1) \end{array} \right\} \tag{82}$$

and

$$I_{r,i} \equiv 0 \qquad (i = \pm 1, \ldots, \pm n). \tag{83}$$

To complete the solution, we verify that equations (45) and (46) also admit the solution

$$I_{l,i} = I_{r,i} = b(\tau + \mu_i + Q) \qquad (i = \pm 1, \ldots, \pm n), \tag{84}$$

where b and Q are arbitrary constants.

Combining the solutions (79), (82), (83), and (84), we observe that the general solution of the system of equations (45) and (46) can be written in the forms

$$\left. \begin{aligned} I_{l,i} = b\Big\{ \tau + \mu_i + Q + (1 - \mu_i^2) \sum_{\beta=1}^{n-1} \left(\frac{L_\beta e^{-\kappa_\beta \tau}}{1 + \mu_i \kappa_\beta} + \frac{L_{-\beta} e^{+\kappa_\beta \tau}}{1 - \mu_i \kappa_\beta} \right) \\ + \sum_{\alpha=1}^{n} M_\alpha (1 - k_\alpha \mu_i)\, e^{-k_\alpha \tau} + \sum_{\alpha=1}^{n-1} M_{-\alpha}(1 + k_\alpha \mu_i)\, e^{+k_\alpha \tau} \Big\} \\ (i = \pm 1, \ldots, \pm n) \end{aligned} \right\} \tag{85}$$

and

$$\left. \begin{aligned} I_{r,i} = b\Big\{ \tau + \mu_i + Q - \sum_{\alpha=1}^{n} M_\alpha \frac{(k_\alpha^2 - 1)}{1 + \mu_i k_\alpha}\, e^{-k_\alpha \tau} - \sum_{\alpha=1}^{n} M_{-\alpha} \frac{(k_\alpha^2 - 1)}{1 - \mu_i k_\alpha}\, e^{+k_\alpha \tau} \Big\} \\ (i = \pm 1, \ldots, \pm n), \end{aligned} \right\} \tag{86}$$

where $L_{\pm\beta}$ $(\beta = 1, \ldots, n - 1)$, $M_{\pm\alpha}$ $(\alpha = 1, \ldots, n)$, b, and Q are the $4n$ constants of integration.

4. *The solution satisfying the necessary boundary conditions.*—The boundary conditions for the astrophysical problem on hand are that none of the I_i's tend to infinity exponentially as $\tau \to \infty$ and that there is no radiation incident on the surface $\tau = 0$. The first of these conditions implies that in the general solution (85) and (86) we omit all terms in $\exp(+k_\alpha \tau)$ and $\exp(+\kappa_\beta \tau)$. We are thus left with

$$\left. \begin{aligned} I_{l,i} = b\Big\{ \tau + \mu_i + Q + (1 - \mu_i^2) \sum_{\beta=1}^{n-1} \frac{L_\beta e^{-\kappa_\beta \tau}}{1 + \mu_i \kappa_\beta} + \sum_{\alpha=1}^{n} M_\alpha (1 - k_\alpha \mu_i)\, e^{-k_\alpha \tau} \Big\} \\ (i = \pm 1, \ldots, \pm n) \end{aligned} \right\} \tag{87}$$

and

$$I_{r,i} = b\Big\{ \tau + \mu_i + Q - \sum_{\alpha=1}^{n} \frac{M_\alpha (k_\alpha^2 - 1)}{1 + \mu_i k_\alpha}\, e^{-k_\alpha \tau} \Big\} \qquad (i = \pm 1, \ldots, \pm n). \tag{88}$$

Next, the nonexistence of any radiation incident on $\tau = 0$ requires that

$$I_{l,i} = I_{r,i} = 0 \quad \text{at} \quad \tau = 0 \quad \text{and for} \quad i = -1, \ldots, -n; \tag{89}$$

or, according to equations (86) and (87),

$$(1-\mu_i^2)\sum_{\beta=1}^{n-1}\frac{L_\beta}{1-\mu_i\kappa_\beta}+\sum_{a=1}^{n}M_a(1+k_a\mu_i)-\mu_i+Q=0 \qquad (i=1,\ldots,n) \tag{90}$$

and

$$\sum_{a=1}^{n}\frac{M_a(k_a^2-1)}{1-\mu_i k_a}+\mu_i-Q=0 \qquad (i=1,\ldots,n). \tag{91}$$

These are the $2n$ equations which determine the $2n$ constants L_β ($\beta = 1, \ldots, n-1$), M_a ($a = 1, \ldots = n$), and Q.[10] The constant b is left arbitrary and is related to the constant net flux of radiation in the atmosphere.

For, defining the net flux in terms of F_l and F_r where

$$F_q = 2\int_{-1}^{+1} I_q \mu d\mu \simeq 2\Sigma a_i I_{q,i}\mu_i \qquad (q = l, r), \tag{92}$$

we have, according to equations (87) and (88),

$$F_l = 2b\left\{\frac{2}{3}+\sum_{\beta=1}^{n-1}L_\beta[D_1(\kappa_\beta)-D_3(\kappa_\beta)]e^{-\kappa_\beta\tau}-\frac{2}{3}\sum_{a=1}^{n}M_a k_a e^{-k_a\tau}\right\} \tag{93}$$

and

$$F_r = 2b\left\{\frac{2}{3}-\sum_{a=1}^{n}M_a(k_a^2-1)D_1(k_a)e^{-k_a\tau}\right\}. \tag{94}$$

On the other hand, from equations (60), (76), and (80) we conclude that

$$D_1(k_a) = -k_a D_2(k_a) = -\frac{2}{3}\frac{k_a}{k_a^2-1} \tag{95}$$

and

$$D_1(\kappa_\beta) - D_3(\kappa_\beta) = -\kappa_\beta[D_2(\kappa_\beta)-D_4(\kappa_\beta)] = 0. \tag{96}$$

Hence,

$$F_l = \tfrac{4}{3}b\left(1-\sum_{a=1}^{n}M_a k_a e^{-k_a\tau}\right) \tag{97}$$

and

$$F_r = \tfrac{4}{3}b\left(1+\sum_{a=1}^{n}M_a k_a e^{-k_a\tau}\right). \tag{98}$$

From the two preceding equations we infer the constancy of the net flux. More particularly,

$$F = F_l + F_r = \tfrac{8}{3}b = \text{constant}. \tag{99}$$

[10] Adding equations (90) and (91), we obtain the equation

$$\sum_{a=1}^{n}\frac{M_a k_a^2}{1-\mu_i k_a}+\sum_{\beta=1}^{n}\frac{L_\beta}{1-\mu_i\kappa_\beta}=0 \qquad (i=1,\ldots,n),$$

which together with equation (91) is more convenient for the practical determination of these constants.

We can, therefore, re-write the solution (85) and (86) in the form

$$I_{l,i} = \tfrac{3}{8}F\left\{\tau + \mu_i + Q + (1-\mu_i^2)\sum_{\beta=1}^{n-1}\frac{L_\beta e^{-\kappa_\beta \tau}}{1+\mu_i \kappa_\beta} + \sum_{\alpha=1}^{n} M_\alpha (1 - k_\alpha \mu_i)\, e^{-k_\alpha \tau}\right\} \quad (i = \pm 1, \ldots, \pm n) \tag{100}$$

and

$$I_{r,i} = \tfrac{3}{8}F\left\{\tau + \mu_i + Q - \sum_{\alpha=1}^{n}\frac{M_\alpha (k_\alpha^2 - 1)}{1+\mu_i k_\alpha}\, e^{-k_\alpha \tau}\right\} \quad (i = \pm 1, \ldots, \pm n). \tag{101}$$

In terms of the foregoing solutions for $I_{l,i}$ and $I_{r,i}$ we can readily establish the following formulae

$$J_l = \tfrac{1}{2}\Sigma a_j I_{l,j} = \tfrac{3}{8}F\left(\tau + Q + \frac{2}{3}\sum_{\beta=1}^{n-1} L_\beta e^{-\kappa_\beta \tau} + \sum_{\alpha=1}^{n} M_\alpha e^{-k_\alpha \tau}\right), \tag{102}$$

$$K_l = \tfrac{1}{2}\Sigma a_j I_{l,j}\mu_j^2 = \tfrac{1}{8}F\left(\tau + Q + \sum_{\alpha=1}^{n} M_\alpha e^{-k_\alpha \tau}\right), \tag{103}$$

$$J_r = \tfrac{3}{8}F\left(\tau + Q - \frac{1}{3}\sum_{\alpha=1}^{n} M_\alpha [4k_\alpha^2 - 3]\, e^{-k_\alpha \tau}\right), \tag{104}$$

and

$$K_r = \tfrac{1}{8}F\left(\tau + Q - \sum_{\alpha=1}^{n} M_\alpha e^{-k_\alpha \tau}\right). \tag{105}$$

Now the source functions $\mathfrak{J}_l(\tau, \mu)$ and $\mathfrak{J}_r(\tau, \mu)$ for I_l and I_r are (cf., eqs. [41] and [42])

$$\mathfrak{J}_l(\tau, \mu) = \tfrac{3}{4}[2(J_l - K_l) + \mu^2(3K_l - 2J_l + J_r)] \tag{106}$$

and

$$\mathfrak{J}_r(\tau, \mu) = \tfrac{3}{4}(J_r + K_l); \tag{107}$$

or, substituting for J_l, K_l, J_r, and K_r from equations (102)–(105), we find

$$\mathfrak{J}_l(\tau, \mu) = \tfrac{3}{8}F\left\{\tau + Q + (1-\mu^2)\sum_{\beta=1}^{n-1} L_\beta e^{-\kappa_\beta \tau} + \sum_{\alpha=1}^{n} M_\alpha (1 - k_\alpha^2 \mu^2)\, e^{-k_\alpha \tau}\right\} \tag{108}$$

and

$$\mathfrak{J}_r(\tau, \mu) = \tfrac{3}{8}F\left\{\tau + Q - \sum_{\alpha=1}^{n} M_\alpha (k_\alpha^2 - 1)\, e^{-k_\alpha \tau}\right\}. \tag{109}$$

With the explicit forms for the source functions now found, we can readily obtain formulae for the intensity distribution of the emergent radiation. For, since quite generally

$$I(0, \mu) = \int_0^\infty \mathfrak{J}(\tau, \mu)\, e^{-\tau/\mu}\frac{d\tau}{\mu}, \tag{110}$$

we have, in our present case,

$$I_l(0, \mu) = \tfrac{3}{8}F\left\{\mu + Q + (1-\mu^2)\sum_{\beta=1}^{n-1}\frac{L_\beta}{1+\kappa_\beta \mu} + \sum_{\alpha=1}^{n} M_\alpha (1 - k_\alpha \mu)\right\} \tag{111}$$

and

$$I_r(0, \mu) = \tfrac{3}{8}F\left\{\mu + Q - \sum_{a=1}^{n} \frac{M_a(k_a^2 - 1)}{1 + k_a\mu}\right\}. \tag{112}$$

It is to be particularly noted that the foregoing expressions for $I_l(0, \mu)$ and $I_r(0, \mu)$ agree with the solution (100) and (101) for $\tau = 0$ at the points of the Gaussian division.

This completes the solution of the problem in the nth approximation.

5. *The numerical forms of the solution in the second and the third approximations; the degree of polarization of the emergent radiation.*—The consideration of the solution obtained in the preceding section in the first approximation is of no special interest except possibly to emphasize the importance of having a method which gives solutions to any desired degree of accuracy. For, in the first approximation ($n = 1$) there are no L's; and, though there is an M, this is also seen to be zero; and the only nonvanishing constant of integration is Q, which has the value $1/\sqrt{3}$. Accordingly, in this approximation (cf. II, eq. [35], and III eq. [72])

$$I_l(0, \mu) = I_r(0, \mu) = \tfrac{3}{8}F\left(\mu + \frac{1}{\sqrt{3}}\right). \tag{113}$$

It is, therefore, seen that the first approximation is far too crude to disclose the essentially finer features of our problem. We therefore proceed to the higher approximations.

i) *Second approximation.*—In this approximation it is found that

$$\kappa_1 = \sqrt{\tfrac{7}{3}}\ ; \qquad k_1 = \sqrt{5}; \qquad \text{and} \qquad k_2 = \sqrt{\tfrac{7}{6}}\ . \tag{114}$$

Further, the constants L_1, M_1, M_2, and Q have the values

$$\left.\begin{aligned} L_1 = -0.19265\ ; \quad M_1 = +0.021830\ ; \quad M_2 = -0.029516\ ; \\ Q = +0.69638\ ; \end{aligned}\right\} \tag{115}$$

and the laws of darkening for the emergent radiation in the two states of polarization take the forms

$$\left.\begin{aligned} I_l(0, \mu) = \tfrac{3}{8}F\Big\{\mu + 0.69638 - (1 - \mu^2)\frac{-0.19265}{1 + 1.5275\mu} \\ + 0.021830(1 - 2.23607\mu) - 0.029516(1 - 1.08012\mu)\Big\} \end{aligned}\right\} \tag{116}$$

and

$$I_r(0, \mu) = \tfrac{3}{8}F\left\{\mu + 0.69638 - \frac{0.0873215}{1 + 2.23607\mu} + \frac{0.0049193}{1 + 1.08012\mu}\right\}. \tag{117}$$

Values of $I_l(0, \mu)$ and $I_r(0, \mu)$ obtained[11] from the preceding formulae are given in Table 1.

ii) *Third approximation.*—In this approximation the characteristic equations for κ^2 and k^2 are

$$\kappa^4 - 8.64\kappa^2 + 9.24 = 0 \tag{118}$$

and

$$k^6 - 14.82k^4 + 36.12k^2 - 23.1 = 0\ . \tag{119}$$

The characteristic roots are

$$\left.\begin{aligned} \kappa_1 = 2.718381\ ; \quad \kappa_2 = 1.118216\ ; \quad k_1 = 3.458589\ ; \\ k_2 = 1.327570\ ; \quad \text{and} \quad k_3 = 1.046766\ . \end{aligned}\right\} \tag{120}$$

[11] I am indebted to Mrs. Frances H. Breen for assistance with these calculations.

The constants of integration L_1, L_2, Q, M_1, M_2, and M_3 are found to have the values

$$\left.\begin{aligned} L_1 &= -0.1402646\,; & L_2 &= -0.06791696\,; & Q &= +0.705927\,, \\ M_1 &= +0.00718392\,; & M_2 &= +0.01861255\,; & M_3 &= -0.0328664\,; \end{aligned}\right\} \tag{121}$$

and the laws of darkening for the emergent radiation in the two states of polarization take the forms

$$\left.\begin{aligned} I_l(0,\mu) = \tfrac{3}{8}F\Big\{\mu + 0.705927 - (1-\mu^2)\Big[\frac{0.1402646}{1+2.718381\mu} \\ + \frac{0.06791696}{1+1.118216\mu}\Big] + 0.00718392(1-3.458589\mu) \\ + 0.01861255(1-1.327570\mu) - 0.0328664(1-1.046766\mu)\Big\} \end{aligned}\right\} \tag{122}$$

TABLE 1

THE LAW OF DARKENING IN THE EMERGENT RADIATION IN THE TWO STATES OF POLARIZATION GIVEN BY THE SECOND APPROXIMATION

μ	$I_l(0, \mu)/F$	$I_r(0, \mu)/F$	$I_l(0, \mu)/I_l(0, 1)$	$I_r(0, \mu)/I_r(0, 1)$
0	0.1860	0.2302	0.2967	0.3673
0.1	.2331	.2735	0.3718	0.4363
0.2	.2789	.3150	0.4448	0.5025
0.3	.3238	.3554	0.5165	0.5670
0.4	.3681	.3951	0.5871	0.6303
0.5	.4119	.4344	0.6570	0.6929
0.6	.4553	.4733	0.7263	0.7549
0.7	.4985	.5119	0.7952	0.8166
0.8	.5415	.5504	0.8637	0.8779
0.9	.5843	.5887	0.9320	0.9391
1.0	0.6269	0.6269	1.0000	1.0000

and

$$\left.\begin{aligned} I_r(0,\mu) = \tfrac{3}{8}F\Big\{\mu + 0.705927 - \frac{0.0787490}{1+3.458589\mu} - \frac{0.01419099}{1+1.327570\mu} \\ + \frac{0.00314593}{1+1.0467659\mu}\Big\}. \end{aligned}\right\} \tag{123}$$

Values of $I_l(0, \mu)$ and $I_r(0, \mu)$ obtained from the foregoing formulae are given in Table 2. Comparison with the values given in Table 1 indicates that the solution obtained in the third approximation is probably accurate to within 1 per cent over the entire range of the variables.

In Figure 2 we have illustrated the laws of darkening on the third approximation for the intensities $I_l(0, \mu)$ and $I_r(0, \mu)$. It is seen that, while they are equal at the center of the disk ($\mu = 1$), they differ by about 25 per cent at the limit ($\mu = 0$). The theory, therefore, predicts a polarization of the emergent radiation. And the degree of polarization $\delta(\mu)$, defined by

$$\delta(\mu) = \frac{I_r(0,\mu) - I_l(0,\mu)}{I_r(0,\mu) + I_l(0,\mu)}, \tag{124}$$

varies from 0 at $\mu = 1$ to 11 per cent at $\mu = 0$ (see Table 2). It is not impossible that this predicted polarization of the radiation of the early-type stars (in which scattering by free electrons is believed to play an important part in the transfer of radiation) could be detected under suitably favorable conditions.

One further comparison is of interest. In an earlier paper (paper III) we have worked out the theory of radiative transfer in an atmosphere in which radiation is scattered in accordance with Rayleigh's phase function. This is not a strictly correct procedure, inasmuch as no allowance is made for the polarization of the existing radiation field. However, a comparison (see Table 2) of the emergent intensity $I(0, \mu)/F$ derived on the theory of transfer incorporating Rayleigh's phase function with the total emergent in-

TABLE 2

THE LAWS OF DARKENING IN THE TWO STATES OF POLARIZATION GIVEN BY THE THIRD APPROXIMATION; THE DEGREE OF POLARIZATION OF THE EMERGENT RADIATION; COMPARISON OF THE TOTAL INTENSITIES GIVEN BY THE THEORY IGNORING THE POLARIZATION OF THE EXISTING FIELD OF RADIATION BUT INCORPORATING RAYLEIGH'S PHASE FUNCTION

μ	$\frac{I_l(0, \mu)}{F}$	$\frac{I_r(0, \mu)}{F}$	$\frac{I_l(0, \mu)}{I_l(0, 1)}$	$\frac{I_r(0, \mu)}{I_r(0, 1)}$	$\frac{I_r(0, \mu)}{I_l(0, \mu)}$	$\delta(\mu)$	$\frac{I_l(0, \mu)+I_r(0, \mu)}{F}$	$\frac{I(0, \mu)}{F}$ for Rayleigh's Phase Function
0.	0.1840	0.2310	0.2914	0.3659	1.2557	0.1134	0.4151	0.4195
0.1.	.2354	.2767	0.3728	0.4382	1.1753	.0806	0.5120	0.5175
0.2.	.2832	.3190	0.4486	0.5053	1.1264	.0594	0.6023	0.6076
0.3.	.3291	.3598	0.5213	0.5699	1.0932	.0445	0.6890	0.6937
0.4.	.3738	.3997	0.5921	0.6330	1.0691	.0334	0.7735	0.7773
0.5.	.4178	.4390	0.6616	0.6953	1.0508	.0248	0.8567	0.8593
0.6.	.4611	.4779	0.7303	0.7569	1.0364	.0179	0.9390	0.9402
0.7.	.5041	.5165	0.7983	0.8181	1.0247	.0122	1.0206	1.0203
0.8.	.5467	.5549	0.8659	0.8789	1.0150	.0075	1.1017	1.0998
0.9.	.5891	.5932	0.9331	0.9396	1.0069	0.0034	1.1824	1.1789
1.0.	0.6314	0.6314	1.0000	1.0000	1.0000	0	1.2628	1.2576

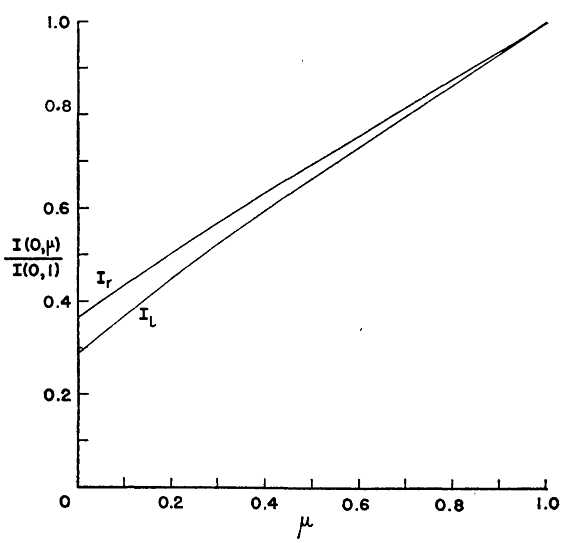

FIG. 2.—The laws of darkening in the two states of polarization. The symbol I_l refers to the component polarized with the electric vector in the meridian plane, while I_r refers to the component polarized with the electric vector at right angles to the meridian plane.

tensity $[I_l(0, \mu) + I_r(0, \mu)]/F$ given by our present more exact theory shows that the errors made in the *total intensities* by ignoring the polarization of the radiation field are small.

6. *The reduction of the laws of darkening in the two states of polarization to certain closed forms.*—In § 4 we derived expressions for the angular distribution of the emergent radiation in the two states of polarization (eqs. [111] and [112]). These expressions involve certain constants of integration ($2n$ of them in the nth approximation), and it would appear that these have to be evaluated before the solutions can be brought to their numerical forms. However, we shall now show how, for the purposes of characterizing the emergent radiation, we can avoid the necessity of solving explicitly for these constants by expressing the laws of darkening in forms in which they require only a knowledge of characteristic roots k_a and κ_β.

First, we may observe that equations (90) and (91), which determine the constants of integration, can be re-written as

$$S_l(\mu_i) = 0 \quad \text{and} \quad S_r(\mu_i) = 0 \qquad (i = 1, \ldots, n), \tag{125}$$

where

$$S_l(\mu) = (1-\mu^2)\sum_{\beta=1}^{n-1}\frac{L_\beta}{1-\mu\kappa_\beta} + \sum_{a=1}^{n} M_a(1+k_a\mu) - \mu + Q \tag{126}$$

and

$$S_r(\mu) = -\mu + Q - \sum_{a=1}^{n}\frac{M_a(k_a^2-1)}{1-\mu k_a}. \tag{127}$$

In terms of these same functions the angular distribution of the emergent radiation in the two states of polarization can also be expressed. For, according to equations (111) and (112), we can write

$$I_l(0, \mu) = \tfrac{3}{8}FS_l(-\mu) \quad \text{and} \quad I_r(0, \mu) = \tfrac{3}{8}FS_r(-\mu). \tag{128}$$

We shall now show how explicit expressions for the functions $S_l(\mu)$ and $S_r(\mu)$ can be obtained.

First, we shall define the functions $R(\mu)$, $R_a(\mu)$, $\rho(\mu)$ and $\rho_\beta(\mu)$ according to the formulae

$$R(\mu) = \prod_{a=1}^{n}(1-k_a\mu); \qquad R_a(\mu) = \prod_{a\neq a}(1-k_a\mu), \tag{129}$$

and

$$\rho(\mu) = \prod_{\beta=1}^{n-1}(1-\kappa_\beta\mu); \qquad \rho_\beta(\mu) = \prod_{b\neq\beta}(1-\kappa_b\mu). \tag{130}$$

Considering, now, the function $S_l(\mu)$, we see that $R(\mu)S_l(\mu)$ is a polynomial of degree n in μ which vanishes for $\mu = \mu_i$ ($i = 1, \ldots, n$). We must, accordingly, have a relation of the form

$$S_l(\mu) = \sigma\frac{P(\mu)}{\rho(\mu)}, \tag{131}$$

where σ is a constant and

$$P(\mu) = \prod_{i=1}^{n}(\mu-\mu_i). \tag{132}$$

The function $S_l(\mu)$ is therefore determinate apart from a constant of proportionality. Apart from this same constant of proportionality, the constants L_β can also be determined. For, according to equations (126) and (131),

$$\left(1-\frac{1}{\kappa_\beta^2}\right)L_\beta=\sigma\frac{P(1/\kappa_\beta)}{\rho_\beta(1/\kappa_\beta)} \qquad (\beta=1,\ldots,n-1). \quad (133)$$

Moreover, setting $\mu = +1$, respectively, -1, in equations (126) and (131) we obtain

$$S_l(+1)=\sum_{a=1}^{n}M_a(1+k_a)-1+Q=\sigma\frac{P(1)}{\rho(1)}, \quad (134)$$

and

$$S_l(-1)=\sum_{a=1}^{n}M_a(1-k_a)+1+Q=\sigma\frac{P(-1)}{\rho(-1)}. \quad (135)$$

Adding and subtracting these two equations, we have

$$\sum_{a=1}^{n}M_a=a\sigma-Q \quad (136)$$

and

$$\sum_{a=1}^{n}M_a k_a=\beta\sigma+1, \quad (137)$$

where we have written

$$a=\frac{1}{2}\left[\frac{P(1)}{\rho(1)}+\frac{P(-1)}{\rho(-1)}\right] \quad \text{and} \quad \beta=\frac{1}{2}\left[\frac{P(1)}{\rho(1)}-\frac{P(-1)}{\rho(-1)}\right]. \quad (138)$$

Considering, next, the function $S_r(\mu)$, we observe that we must have a proportionality of the form

$$R(\mu)S_r(\mu)\propto P(\mu)(\mu+c), \quad (139)$$

where c is a constant, since the quantity on the right-hand side is a polynomial of degree $n+1$ in μ and has the zeros $\mu=\mu_i$ $(i=1,\ldots,n)$. The constant of proportionality in equation (139) can be found from a comparison of the coefficients of the highest powers of μ on either side. In this manner we find that

$$S_r(\mu)=(-1)^{n+1}k_1\ldots k_n\frac{P(\mu)}{R(\mu)}(\mu+c). \quad (140)$$

From equations (127) and (140) we now conclude that

$$M_a=(-1)^n k_1\ldots k_n\frac{P(1/k_a)}{R_a(1/k_a)(k_a^2-1)}\left(\frac{1}{k_a}+c\right) \qquad (a=1,\ldots,n). \quad (141)$$

Also, setting $\mu = 0$ in equations (127) and (140), we have

$$\sum_{a=1}^{n}M_a(k_a^2-1)-Q=k_1\ldots k_n\mu_1\ldots\mu_n c. \quad (142)$$

On the other hand, from the characteristic equation (72) for k we infer that

$$k_1 \ldots . k_n \mu_1 \ldots . \mu_n = \frac{1}{\sqrt{2}} . \qquad (143)[12]$$

Hence,

$$\sum_{a=1}^{n} M_a (k_a^2 - 1) = Q + \frac{c}{\sqrt{2}} . \qquad (144)$$

Adding equations (136) and (144), we have

$$\sum_{a=1}^{n} M_a k_a^2 = a\sigma + \frac{c}{\sqrt{2}} . \qquad (145)$$

Finally, substituting for M_a according to equation (141), we can re-write equations (136) and (137) in the forms

$$\xi_{-1} + c\,\xi_0 = a\sigma - Q , \qquad (146)$$

$$\xi_0 + c\,\xi_1 = \beta\sigma + 1 , \qquad (147)$$

and

$$\xi_1 + c\,\xi_2 = a\sigma + \frac{c}{\sqrt{2}} , \qquad (148)$$

where

$$\xi_m = (-1)^n k_1 \ldots . k_n \sum_{a=1}^{n} \frac{P(1/k_a)\, k_a^m}{R_a(1/k_a)(k_a^2 - 1)} \qquad (m = -1, 0, 1, 2) . \qquad (149)$$

To evaluate the sum occurring on the right-hand side of equation (149), we introduce the function

$$f_m(x) = \sum_{a=1}^{n} \frac{P(1/k_a)\, k_a^m}{R_a(1/k_a)(k_a^2 - 1)} R_a(x) \qquad (m = -1, 0, 1, 2) , \qquad (149a)$$

and express ξ_m in terms of it. Thus,

$$\xi_m = (-1)^n k_1 \ldots . k_n f_m(0) . \qquad (150)$$

[12] This relation follows most readily from the characteristic equation written in the form

$$\sum_{j=0}^{n} p_{2j} \Delta_{2j} = 0 ,$$

where the p_{2j}'s are the coefficients of μ^{2j} in the Legendre polynomial $P_{2n}(\mu)$ and

$$\Delta_{2j} = \sum_{i=1}^{n} \frac{a_i (1 - \mu_i^2) \mu_i^{2j}}{1 - k^2 \mu_i^2} .$$

The Δ's defined in this manner satisfy the recursion formula

$$\Delta_{2j} = \frac{1}{k^2} \left(\Delta_{2j-2} - \frac{2}{4j^2 - 1} \right) .$$

For the characteristic equation (72), $\Delta_0 = \frac{4}{3}$ and $\Delta_2 = 2/3k^2$. From the recursion formula we therefore conclude that Δ_{2n} starts with $2/3k^{2n}$. The equation for k must accordingly have the form

$$\tfrac{4}{3} p_0 k^{2n} + \ldots . + \tfrac{2}{3} p_{2n} = 0 .$$

Hence,

$$k_1^2 \ldots . k_n^2 = \frac{(-1)^n p_{2n}}{2 p_0} = \frac{1}{2 \mu_1^2 \ldots . \mu_n^2} .$$

Now, $f_m(x)$ defined as in equation (149) is a polynomial of degree $(n-1)$ in x, which takes the values

$$\frac{P(1/k_a)\,k_a^m}{k_a^2-1} \tag{151}$$

for $x = 1/k_a$, $(a = 1, \ldots, n)$. In other words,

$$(1-x^2)\,f_m(x) - x^{2-m}P(x) = 0 \quad \text{for} \quad x = \frac{1}{k_a} \quad \text{and} \quad a = 1, \ldots\ldots n. \tag{152}$$

The polynomial on the right-hand side of equation (152) must therefore divide $R(x)$. There must, accordingly, exist a relation of the form

$$(1-x^2)\,f_m(x) - x^{2-m}P(x) = R(x)\Psi(x), \tag{153}$$

where $\Psi(x)$ is a polynomial of degree 3, 2, 1, or 1 in x for $m = -1, 0, 1$, or 2, respectively. To determine $\Psi(x)$ more explicitly we must consider each case separately. We shall illustrate the procedure for the case $m = -1$.

For $m = -1$, equation (153) becomes

$$(1-x^2)\,f_{-1}(x) - x^3P(x) = R(x)\,(A_{-1}x^3 + B_{-1}x^2 + C_{-1}x + D_{-1}), \tag{154}$$

where A_{-1}, B_{-1}, C_{-1}, and D_{-1} are certain constants to be determined. The constants A_{-1} and B_{-1} readily follow from a comparison of the coefficients of x^{n+3} and x^{n+2} on either side of equation (154). In this manner we find

$$A_{-1} = \frac{(-1)^{n+1}}{k_1 \ldots\ldots k_n} \quad \text{and} \quad B_{-1} = \frac{(-1)^n}{k_1 \ldots\ldots k_n}\left(\sum_{j=1}^{n}\mu_j - \sum_{a=1}^{n}\frac{1}{k_a}\right). \tag{155}$$

Next, putting $x = +1$, respectively -1, in equation (154), we have

$$A_{-1} + B_{-1} + C_{-1} + D_{-1} = -\frac{P(1)}{R(1)}, \tag{156}$$

and

$$-A_{-1} + B_{-1} - C_{-1} + D_{-1} = +\frac{P(-1)}{R(-1)}. \tag{157}$$

These equations determine the remaining constants C_{-1} and D_{-1}. In particular,

$$D_{-1} = f_{-1}(0) = -b + \frac{(-1)^{n+1}}{k_1 \ldots\ldots k_n}\left(\sum_{j=1}^{n}\mu_j - \sum_{a=1}^{n}\frac{1}{k_a}\right), \tag{158}$$

where (cf. eq. [138])

$$b = \frac{1}{2}\left[\frac{P(1)}{R(1)} - \frac{P(-1)}{R(-1)}\right]. \tag{159}$$

From equation (150) we now conclude that

$$\xi_{-1} = (-1)^{n+1}k_1 \ldots\ldots k_n b - \left(\sum_{j=1}^{n}\mu_j - \sum_{a=1}^{n}\frac{1}{k_a}\right). \tag{160}$$

The evaluation of ξ_0, ξ_1, and ξ_2 proceeds along similar lines. We find

$$\xi_0 = (-1)^{n+1}k_1 \ldots\ldots k_n a + 1, \tag{161}$$

$$\xi_1 = (-1)^{n+1}k_1 \ldots\ldots k_n b, \tag{162}$$

and

$$\xi_2 = (-1)^{n+1}k_1 \ldots\ldots k_n a + \frac{1}{\sqrt{2}}. \tag{163}$$

In equations (161) and (163) a stands for (cf. eq. [138])

$$a=\frac{1}{2}\left[\frac{P(1)}{R(1)}+\frac{P(-1)}{R(-1)}\right]. \tag{164}$$

With the foregoing expressions for ξ's, equations (147) and (148) become

$$a+bc=\frac{(-1)^{n+1}}{k_1\ldots k_n}\beta\sigma, \tag{165}$$

and

$$b+ac=\frac{(-1)^{n+1}}{k_1\ldots k_n}\alpha\sigma. \tag{166}$$

These equations determine c and σ. We find

$$c=-\frac{\alpha a-\beta b}{\alpha b-\beta a}, \tag{167}$$

and

$$\sigma=(-1)^n k_1\ldots k_n\frac{a^2-b^2}{\alpha b-\beta a}. \tag{168}$$

Equation (146) now determines Q. After some minor reductions we find that

$$Q=-c+\sum_{j=1}^{n}\mu_j-\sum_{\alpha=1}^{n}\frac{1}{k_\alpha}. \tag{169}$$

Finally, substituting for c and σ according to equations (167) and (168) in equations (131) and (140), we obtain

$$S_l(\mu)=\frac{1}{\sqrt{2}}\frac{a^2-b^2}{\alpha b-a\beta}\frac{(-1)^n}{\mu_1\ldots\mu_n}\frac{P(\mu)}{\rho(\mu)}, \tag{170}$$

and

$$S_r(\mu)=\frac{1}{\sqrt{2}}\frac{(-1)^{n+1}P(\mu)}{\mu_1\ldots\mu_n R(\mu)}\left(\mu-\frac{\alpha a-\beta b}{\alpha b-\beta a}\right). \tag{171}$$

The laws of darkening now follow according to equation (128).

We may note here that in the third approximation

$$\sigma=-3.3351 \quad \text{and} \quad c=-0.87134\,.$$

7. *Concluding remarks.*—The successful solution of a specific problem in theory of radiative transfer distinguishing the different states of polarization justifies the hope that it will be possible to solve other astrophysical problems in which polarization plays a significant role. Thus it may be expected that a theory of diffuse reflection along the lines of an earlier paper of this series[13] but incorporating the polarization of the existing field of diffuse radiation will account, in a general way, for the remarkable observations of Lyot[14] on the polarization of the reflected light from Venus. We hope to return to these and similar essentially more difficult problems in the theory of radiative transfer in the near future.

I am indebted to Dr. G. Herzberg for helpful discussions on some of the physical aspects of the problem considered in this paper.

Note added May 6: The problem of diffuse reflection from a semi-infinite plane-parallel atmosphere, allowing for the partial polarization of the diffuse radiation has now been solved. It is hoped to publish the results of this investigation in the near future.

[13] *Ap. J.*, **103**, 165, 1946.

[14] *Ann. Obs. Meudon* (Paris), **8**, 66, 1929.