ON THE RADIATIVE EQUILIBRIUM OF A STELLAR ATMOSPHERE. VI

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ABSTRACT

In this paper the solutions to the problem of line formation in stellar atmospheres, in the third approximation, are tabulated for various values of the parameters which are involved and at close enough intervals to allow interpolation.

Numerical forms of the solution for the problem of the radiative equilibrium of a planetary nebula are also obtained.

This paper is of the nature of a supplement to two earlier papers[1] of this series and provides some further information relating to the problems of the formation of absorption lines in stellar atmospheres and the radiative equilibrium of a planetary nebula.

I. THE FORMATION OF ABSORPTION LINES; THE TABULATION OF THE SOLUTIONS IN THE THIRD APPROXIMATION

As we have seen in papers II (§ 7) and IV, the solution to the problem of line formation under certain "standard" circumstances and in the nth approximation depends on the roots k_a, $(a = 1, \ldots, n)$, of the characteristic equation (II, eq. [91]) and on the solutions of the linear equations

$$\sum_{a=1}^{n} \frac{L_a}{1-\mu_i k_a} = \mu_i - \frac{a_{\nu_0}}{b_\nu} \qquad (i = 1, \ldots, n), \quad (1)$$

which determine the constants of integration L_a, $(a = 1, \ldots, n)$.

It is perhaps of some interest to note that the solution of the foregoing system of equations can be written down explicitly. Thus, in terms of the matrix

$$G = (G_{ij}) = \left(\frac{1}{1-\mu_i k_j}\right) \quad (2)$$

and the vectors

$$L = (L_a) \quad \text{and} \quad M = (M_i) = \left(\mu_i - \frac{a_{\nu_0}}{b_\nu}\right), \quad (3)$$

the system of equations which equation (1) represents is equivalent to

$$GL = M, \quad (4)$$

of which the formal solution is

$$L = G^{-1}M. \quad (5)$$

That we can express the solution for L in this form in terms of the inverse of the matrix G is, of course, obvious. But what is noteworthy in our present context is that for the

[1] *Ap. J.*, **100**, 76, 355, 1944. These papers will be referred to as "II" and "IV," respectively.

particular matrix (2) it is possible to write the inverse quite simply. For, considering the determinant $|\boldsymbol{G}|$ of $\boldsymbol{G}$, it is evident that it vanishes when any two of the μ's are set equal; similarly, it also vanishes when any two of the k's are set equal. Accordingly, $|\boldsymbol{G}|$ cannot differ from

$$\frac{1}{\prod_{i,\,j}^{1,\,n}(1-\mu_i k_j)}\prod_{i\neq j}(\mu_i-\mu_j)\prod_{\alpha\neq\beta}(k_\alpha-k_\beta) \tag{6}$$

by more than a constant factor if we include in the two products in the numerator all distinct combinations $(\mu_i - \mu_j)$ and $(k_\alpha - k_\beta)$. (There are thus $n(n-1)/2$ terms in each of the two products in the numerator, while there are n^2 terms in the product in the denominator.) A simple inspection now shows that we have, in fact,

$$|\boldsymbol{G}| = \frac{1}{\prod_{i,\,j}^{1,\,n}(1-\mu_i k_j)}\prod_{i\neq j}(\mu_i-\mu_j)\prod_{\alpha\neq\beta}(k_\alpha-k_\beta) \tag{7}$$

if the cyclical order among the μ_i's and the k_α's is maintained.[2] And, since the matrix of every minor of $\boldsymbol{G}$ is also of the same form as $\boldsymbol{G}$ itself but with fewer rows and columns, it is apparent that the inverse of $\boldsymbol{G}$ can explicitly be written down. Thus, for the case $n = 3$, we clearly have

[2] For $n > 3$ the cyclical order has to be understood in the sense that if 1, ... , and n are regarded as the verticies of an inscribed polygon we enumerate the various sides $(i, i + 1)$ and the diagonals $(i, i + 2)$, $(i, i + 3)$, etc., in their cyclical orders. Thus, for $n = 6$ the required order is maintained in the sequence (1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (6, 1), (1, 3), (2, 4), (3, 5), (4, 6), (5,1), (6, 2), (1, 4), (2, 5), and (3, 6).

$$
\left.\begin{aligned}
G^{-1}=&\frac{\prod_{i,\,j}^{1,2,3}(1-\mu_i k_j)}{(\mu_1-\mu_2)(\mu_2-\mu_3)(\mu_3-\mu_1)(k_1-k_2)(k_2-k_3)(k_3-k_1)}\\
&\times\begin{pmatrix}
\frac{(\mu_2-\mu_3)(k_2-k_3)}{(1-\mu_2k_2)(1-\mu_2k_3)(1-\mu_3k_2)(1-\mu_3k_3)} & \frac{(\mu_3-\mu_1)(k_2-k_3)}{(1-\mu_1k_2)(1-\mu_1k_3)(1-\mu_3k_2)(1-\mu_3k_3)} & \frac{(\mu_1-\mu_2)(k_2-k_3)}{(1-\mu_1k_2)(1-\mu_1k_3)(1-\mu_2k_2)(1-\mu_2k_3)}\\
\frac{(\mu_2-\mu_3)(k_3-k_1)}{(1-\mu_2k_1)(1-\mu_2k_3)(1-\mu_3k_1)(1-\mu_3k_3)} & \frac{(\mu_3-\mu_1)(k_3-k_1)}{(1-\mu_1k_1)(1-\mu_1k_3)(1-\mu_3k_1)(1-\mu_3k_3)} & \frac{(\mu_1-\mu_2)(k_3-k_1)}{(1-\mu_1k_1)(1-\mu_1k_3)(1-\mu_2k_1)(1-\mu_2k_3)}\\
\frac{(\mu_2-\mu_3)(k_1-k_2)}{(1-\mu_2k_1)(1-\mu_2k_2)(1-\mu_3k_1)(1-\mu_3k_2)} & \frac{(\mu_3-\mu_1)(k_1-k_2)}{(1-\mu_1k_1)(1-\mu_1k_2)(1-\mu_3k_1)(1-\mu_3k_2)} & \frac{(\mu_1-\mu_2)(k_1-k_2)}{(1-\mu_1k_1)(1-\mu_1k_2)(1-\mu_2k_1)(1-\mu_2k_2)}
\end{pmatrix}.
\end{aligned}\right\}\quad(8)
$$

For $n > 3$ we have similar representations.

With G^{-1} as given by equation (8), the constants L_1, L_2, and L_3 of the third approximation can be found according to equation (5) and are

$$\left.\begin{aligned} L_1 &= \frac{(1-\mu_1 k_1)(1-\mu_2 k_1)(1-\mu_3 k_1)}{(k_1-k_2)(k_3-k_1)}\left[(k_2+k_3)-k_2k_3(\mu_1+\mu_2+\mu_3)+\frac{a_{\nu 0}}{b_\nu}k_2k_3\right], \\ L_2 &= \frac{(1-\mu_1 k_2)(1-\mu_2 k_2)(1-\mu_3 k_2)}{(k_2-k_3)(k_1-k_2)}\left[(k_3+k_1)-k_3k_1(\mu_1+\mu_2+\mu_3)+\frac{a_{\nu 0}}{b_\nu}k_3k_1\right], \\ L_3 &= \frac{(1-\mu_1 k_3)(1-\mu_2 k_3)(1-\mu_3 k_3)}{(k_3-k_1)(k_2-k_3)}\left[(k_1+k_2)-k_1k_2(\mu_1+\mu_2+\mu_3)+\frac{a_{\nu 0}}{b_\nu}k_1k_2\right]. \end{aligned}\right\} \quad (9)$$

Substituting the foregoing expressions for the L's in the equation for the emergent flux (IV, eq. [9]), we find that

$$\left.\begin{aligned} F_\nu(0) = \tfrac{4}{3}b_\nu\Bigg[1-\frac{3\lambda}{1-\lambda}\Bigg\{&\frac{k_1k_2+k_2k_3+k_3k_1}{k_1k_2k_3}(\mu_1+\mu_2+\mu_3)-\frac{k_1+k_2+k_3}{k_1k_2k_3} \\ &-(\mu_1+\mu_2+\mu_3)^2+(\mu_1\mu_2+\mu_2\mu_3+\mu_3\mu_1)+\frac{a_{\nu 0}}{b_\nu}\Bigg(\mu_1+\mu_2+\mu_3 \\ &-\frac{k_1k_2+k_2k_3+k_3k_1}{k_1k_2k_3}\Bigg)\Bigg\}\Bigg]. \end{aligned}\right\} \quad (10)$$

The roots k_1, k_2, and k_3 for various values of λ have already been tabulated in IV (Table 1). With these values for the k's, the constants L_1, L_2, and L_3 were determined for various values of λ and x (IV, eq. [18]). The results of the calculations are given in Table 1. In Table 2 the computed residual intensities are given.

It may finally be noted that in the second approximation the results analogous to equations (9) and (10) are

$$\left.\begin{aligned} L_1 &= \frac{(1-\mu_1k_1)(1-\mu_2k_1)}{k_1-k_2}\left[1-k_2(\mu_1+\mu_2)+\frac{a_{\nu 0}}{b_\nu}k_2\right], \\ L_2 &= \frac{(1-\mu_2k_2)(1-\mu_1k_2)}{k_2-k_1}\left[1-k_1(\mu_1+\mu_2)+\frac{a_{\nu 0}}{b_\nu}k_1\right], \end{aligned}\right\} \quad (11)$$

and

$$\left.\begin{aligned} F_\nu(0) = \tfrac{4}{3}b_\nu\Bigg[1-\frac{3\lambda}{1-\lambda}\Bigg\{&\frac{k_1+k_2}{k_1k_2}(\mu_1+\mu_2)-\frac{1}{k_1k_2}-(\mu_1+\mu_2)^2+\mu_1\mu_2 \\ &+\frac{a_{\nu 0}}{b_\nu}\Bigg(\mu_1+\mu_2-\frac{k_1+k_2}{k_1k_2}\Bigg)\Bigg\}\Bigg]. \end{aligned}\right\} \quad (12)$$

II. THE FIELD OF THE ULTRAVIOLET RADIATION IN A PLANETARY NEBULA

The equation of transfer for the ultraviolet radiation beyond the head of the Lyman series consistent with Zanstra's theory of the hydrogen-line emission in planetary nebulae has been considered in II, and the solution found in the form (II, eq. [83])

$$\left.\begin{aligned} I_i = S\sum_{a=1}^{n}&\left\{\frac{L_a e^{-k_a\tau}}{1+\mu_i k_a}+\frac{L_{-a}e^{+k_a\tau}}{1-\mu_i k_a}\right\} \\ &+pSe^{-(\tau_1-\tau)}\frac{1}{4(1-\mu_i)\left(1-p\sum_{j=1}^{n}\frac{a_j}{1-\mu_j^2}\right)} \quad (i=\pm 1,\ldots,\pm n), \end{aligned}\right\} \quad (13)$$

TABLE 1

VALUES OF L_{α} IN THE THIRD APPROXIMATION

λ	$x=0.1$			$x=0.125$			$x=0.15$			$x=1/6$			$x=0.175$		
	L_1	L_2	L_3	L_1	L_2	L_3	L_1	L_2	L_3	L_1	L_2	L_3	L_1	L_2	L_3
0.05	−0.014701	−0.16635	−0.0045033	−0.017522	−0.21435	−0.0054351	−0.020344	−0.26236	−0.0063669	−0.022224	−0.29436	−0.0069881	−0.023165	−0.31037	−0.0072987
.10	− .021170	− .11674	− .0071086	− .025153	− .15679	− .0086670	− .029136	− .19683	− .010225	− .031792	− .22352	− .011264	− .033119	− .23687	− .011784
.20	− .027302	− .050707	− .010099	− .032650	− .079414	− .012897	− .037998	− .10812	− .015694	− .041563	− .12726	− .017559	− .043346	− .13683	− .018492
.30	− .027914	− .010976	− .0097769	− .033877	− .031208	− .013733	− .039840	− .051441	− .017690	− .043816	− .064929	− .020327	− .045803	− .071674	− .021646
.40	− .025144	+ .010466	− .0060659	− .031186	− .0032326	− .010829	− .037227	− .016931	− .015592	− .041255	− .026063	− .018767	− .043269	− .030629	− .020355
.50	− .020449	+ .019086	− .00031817	− .026146	+ .010223	− .0052845	− .031843	+ .0013602	− .010251	− .035641	− .0045484	− .013562	− .037540	− .0075027	− .015217
.60	− .014957	+ .019835	+ .0051726	− .019968	+ .014353	+ .00063841	− .024978	+ .0088707	− .0038958	− .028318	+ .0052157	− .0069186	− .029988	+ .0033883	− .0084300
.70	− .0095545	+ .016514	+ .0084948	− .013602	+ .013305	+ .0048624	− .017649	+ .010095	+ .0012299	− .020347	+ .0079552	− .0011917	− .021696	+ .0068853	− .0024025
.80	− .0049302	+ .011430	+ .0087436	− .0077926	+ .0097291	+ .0062681	− .010655	+ .0080290	+ .0037925	− .012563	+ .0068955	+ .0021421	− .013517	+ .0063288	+ .0013169
.90	− .0016131	+ .0057508	+ .0058139	− .0031139	+ .0050618	+ .0045830	− .0046148	+ .0043721	+ .0033524	− .0056153	+ .0039123	+ .0025320	− .0061156	+ .0036824	+ .0021218
0.95	−0.00057312	+0.0028670	+0.0032419	−0.0013389	+0.0025546	+0.0026324	−0.0021046	+0.0022415	+0.0020232	−0.0026152	+0.0020328	+0.0016170	−0.0028704	+0.0019284	+0.0014139

λ	$x=0.20$			$x=0.225$			$x=0.25$			$x=0.275$			$x=0.3$		
	L_1	L_2	L_3	L_1	L_2	L_3	L_1	L_2	L_3	L_1	L_2	L_3	L_1	L_2	L_3
0.05	−0.025986	−0.35837	−0.0082305	−0.028808	−0.40638	−0.0091623	−0.031629	−0.45439	−0.010094	−0.034450	−0.50239	−0.011026	−0.037271	−0.55040	−0.011958
.10	− .037102	− .27692	− .013342	− .041085	− .31696	− .014901	− .045068	− .35700	− .016459	− .049051	− .39705	− .018018	− .053034	− .43709	− .019576
.20	− .048694	− .16553	− .021289	− .054042	− .19424	− .024087	− .059391	− .22295	− .026885	− .064739	− .25165	− .029682	− .070087	− .28036	− .032480
.30	− .051766	− .091906	− .025602	− .057729	− .11214	− .029559	− .063692	− .13237	− .033515	− .069655	− .15260	− .037471	− .075618	− .17284	− .041428
.40	− .049311	− .044327	− .025118	− .055352	− .058026	− .029881	− .061394	− .071724	− .034644	− .067435	− .085422	− .039407	− .073477	− .099121	− .044170
.50	− .043237	− .016366	− .020183	− .048934	− .025229	− .025150	− .054631	− .034092	− .030116	− .060328	− .042955	− .035082	− .066025	− .051818	− .040049
.60	− .034999	− .0020942	− .012964	− .040009	− .0075766	− .017498	− .045019	− .013059	− .022033	− .050029	− .018541	− .026567	− .055040	− .024024	− .031101
.70	− .025744	+ .0036758	− .0060349	− .029791	+ .00046618	− .0096674	− .033838	− .0027434	− .013300	− .037886	− .0059530	− .016932	− .041933	− .0091626	− .020565
.80	− .016380	+ .0046286	− .0011587	− .019242	+ .0029285	− .0036343	− .022105	+ .0012284	− .0061098	− .024967	− .00047182	− .0085855	− .027830	− .0021720	− .011061
.90	− .0076164	+ .0029927	+ .00089124	− .0091173	+ .0023030	− .00033935	− .010618	+ .0016131	− .0015700	− .012119	+ .00092360	− .0028006	− .013620	+ .00023391	− .0040311
0.95	−0.0036362	+0.0016153	+0.00080470	−0.0044019	+0.0013023	+0.00019547	−0.0051677	+0.00098897	−0.00041378	−0.0059334	+0.00067612	−0.0010230	−0.0066992	+0.00036305	−0.0016322

λ	$x=0.325$			$x=1/3$			$x=0.35$		
	L_1	L_2	L_3	L_1	L_2	L_3	L_1	L_2	L_3
0.05....	−0.040093	−0.59840	−0.012889	−0.041033	−0.61441	−0.013200	−0.042914	−0.64641	−0.013821
.10....	− .057017	− .47713	− .021134	− .058345	− .49048	− .021654	− .061000	− .51718	− .022693
.20....	− .075435	− .30907	− .035278	− .077218	− .31864	− .036210	− .080783	− .33777	− .038075
.30....	− .081581	− .19307	− .045384	− .083568	− .19981	− .046703	− .087544	− .21330	− .049341
.40....	− .079519	− .11282	− .048933	− .081533	− .11739	− .050521	− .085560	− .12652	− .053696
.50....	− .071722	− .060681	− .045015	− .073621	− .063635	− .046670	− .077419	− .069544	− .049981
.60....	− .060050	− .029506	− .035635	− .061720	− .031334	− .037146	− .065060	− .034988	− .040169
.70....	− .045980	− .012372	− .024197	− .047329	− .013442	− .025408	− .050028	− .015582	− .027830
.80....	− .030692	− .0038721	− .013537	− .031646	− .0044388	− .014362	− .033554	− .0055725	− .016012
.90....	− .015121	− .00045579	− .0052617	− .015621	− .00068569	− .0056719	− .016621	− .0011454	− .0064926
0.95....	−0.0074650	+0.000049971	−0.0022415	−0.0077202	−0.000054387	−0.0024445	−0.0082307	−0.00026305	−0.0028509

where $L_{\pm a}$, ($a = 1, \ldots, n$), are $2n$ constants of integration, where k_a, ($a = 1, \ldots, n$), are the n positive roots of the equation,

$$1 = p \sum_{j=1}^{n} \frac{a_j}{1 - \mu_j^2 k^2}, \tag{14}$$

and where the rest of the symbols have the same meanings as in paper II.

The boundary conditions which determine the constants of integrations are (cf. II, eqs. [84] and [85]).

$$I_i = I_{-i} \quad \text{at} \quad \tau = \tau_1 \quad \text{for} \quad i = 1, \ldots, n \tag{15}$$

and

$$I_{-i} = 0 \quad \text{at} \quad \tau = 0 \quad \text{for} \quad i = 1, \ldots, n\,. \tag{16}$$

TABLE 2

VALUES OF THE RESIDUAL INTENSITY GIVEN BY THE THIRD APPROXIMATION

λ	x												
	0.1	0.125	0.15	1/6	0.175	0.20	0.225	0.25	0.275	0.3	0.325	1/3	0.35
0.05	0.1722	0.1876	0.2012	0.2093	0.2132	0.2238	0.2333	0.2419	0.2496	0.2567	0.2631	0.2651	0.2690
0.10	0.2544	0.2724	0.2882	0.2977	0.3021	0.3145	0.3255	0.3355	0.3445	0.3526	0.3601	0.3625	0.3670
0.20	0.3794	0.3983	0.4148	0.4248	0.4294	0.4424	0.4540	0.4645	0.4739	0.4825	0.4904	0.4928	0.4976
0.30	0.4815	0.4995	0.5152	0.5246	0.5290	0.5413	0.5523	0.5622	0.5712	0.5794	0.5868	0.5891	0.5936
0.40	0.5717	0.5879	0.6021	0.6106	0.6146	0.6257	0.6356	0.6446	0.6526	0.6600	0.6667	0.6688	0.6729
0.50	0.6542	0.6683	0.6805	0.6879	0.6914	0.7010	0.7096	0.7174	0.7244	0.7307	0.7366	0.7384	0.7419
0.60	0.7313	0.7429	0.7531	0.7592	0.7621	0.7701	0.7772	0.7836	0.7894	0.7947	0.7996	0.8011	0.8040
0.70	0.8042	0.8133	0.8213	0.8260	0.8283	0.8345	0.8401	0.8451	0.8496	0.8538	0.8575	0.8587	0.8610
0.80	0.8739	0.8803	0.8860	0.8893	0.8909	0.8953	0.8993	0.9028	0.9060	0.9090	0.9116	0.9125	0.9141
0.90	0.9409	0.9446	0.9479	0.9498	0.9507	0.9533	0.9556	0.9576	0.9595	0.9611	0.9627	0.9632	0.9641
0.95	0.9736	0.9759	0.9779	0.9791	0.9797	0.9813	0.9828	0.9841	0.9852	0.9863	0.9872	0.9876	0.9881
1.00	1.0057	1.0066	1.0075	1.0030	1.0082	1.0088	1.0094	1.0100	1.0104	1.0109	1.0112	1.0114	1.0116

With the solution for the I_i's given by equation (13), these conditions require that

$$\sum_{a=1}^{n} \frac{k_a}{1 - \mu_i^2 k_a^2} \left(L_a e^{-k_a \tau_1} - L_{-a} e^{+k_a \tau_1} \right) = \frac{p}{4\,(1 - \mu_i^2) \left(1 - p \sum_{j=1}^{n} \dfrac{a_j}{1 - \mu_j^2} \right)} \qquad (i = 1, \ldots, n) \tag{17}$$

and

$$\sum_{a=1}^{n} \left(\frac{L_a}{1 - \mu_i k_a} + \frac{L_{-a}}{1 + \mu_i k_a} \right) = -\frac{p\, e^{-\tau_1}}{4\,(1 + \mu_i) \left(1 - p \sum_{j=1}^{n} \dfrac{a_j}{1 - \mu_j^2} \right)} \qquad (i = 1, \ldots, n)\,. \tag{18}$$

The field of the diffuse ultraviolet radiation in the planetary nebula is best described in terms of its net flux, proportional to F, and its density, proportional to J. In our present scheme of approximation

$$F = 2 \Sigma a_i \mu_i I_i \quad \text{and} \quad J = \tfrac{1}{2} \Sigma a_i I_i \,. \tag{19}$$

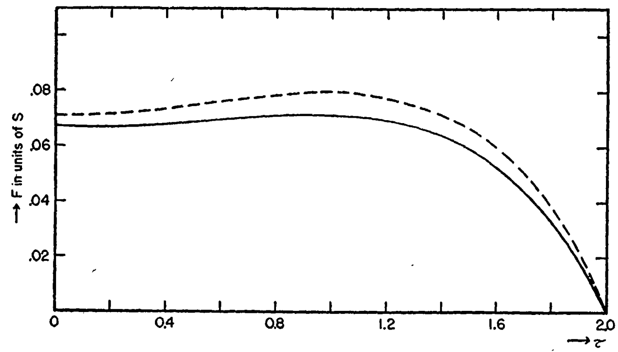

FIG. 1

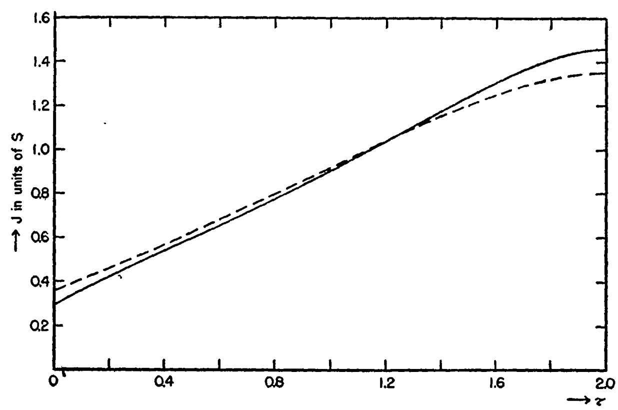

FIG. 2

TABLE 3

THE FIELD OF THE DIFFUSE ULTRAVIOLET RADIATION IN A PLANETARY NEBULA ($p=0.5$ AND $\tau_1=1$)

τ	F/S	J/S	τ	F/S	J/S
0.	0.1122	0.0533	0.625.	0.0621	0.1156
0.125.	.1031	.0692	0.750.	.0463	.1236
0.250.	.0944	.0829	0.875.	.0261	.1293
0.375.	.0852	.0951	1.000.	0.0000	0.1315
0.500.	0.0747	0.1060			

TABLE 4

THE FIELD OF THE DIFFUSE ULTRAVIOLET RADIATION IN A PLANETARY NEBULA ($p=0.5$ AND $\tau_1=2$)

τ	F/S	J/S	τ	F/S	J/S
0.	0.0677	0.0295	1.000.	0.0713	0.0905
0.125.	.0671	.0377	1.250.	.0684	.1072
0.250.	.0673	.0452	1.500.	.0592	.1241
0.375.	.0679	.0525	1.750.	.0389	.1388
0.500.	.0689	.0598	1.875.	.0225	.1436
0.750.	0.0707	0.0746	2.000.	0.0000	0.1459

or, substituting for I_i according to equation (13), we find, after some minor reductions, that

$$F = 4S \frac{1-p}{p} \sum_{a=1}^{n} \frac{1}{k_a} \left(-L_a e^{-k_a \tau} + L_{-a} e^{+k_a \tau} \right) - pS e^{-(\tau_1 - \tau)} \frac{1 - \sum_{j=1}^{n} \frac{a_j}{1-\mu_j^2}}{1 - p \sum_{j=1}^{n} \frac{a_j}{1-\mu_j^2}} \qquad (20)$$

and

$$J = S \frac{1}{p} \sum_{a=1}^{n} \left(L_a e^{-k_a \tau} + L_{-a} e^{+k_a \tau} \right) + \tfrac{1}{4} pS e^{-(\tau_1 - \tau)} \frac{\sum_{j=1}^{n} \frac{a_j}{1-\mu_j^2}}{1 - p \sum_{j=1}^{n} \frac{a_j}{1-\mu_j^2}} . \qquad (21)$$

As an illustration of the foregoing solutions, we shall consider the case $p = \frac{1}{2}$ and assume for τ_1 the values 1 and 2. For $p = \frac{1}{2}$ the characteristic roots are (cf. IV, Table 1, the entry for $\lambda = \frac{1}{2}$)

$$k_1 = 3.69160 , \quad k_2 = 0.964672 , \quad \text{and} \quad k_3 = 1.34962 . \qquad (22)$$

For $\tau_1 = 1$, respectively 2, it was found that

$$\left.\begin{array}{ll} \tau_1 = 1 ; & \tau_1 = 2 , \\ L_1 = -0.0071701 ; & L_1 = -0.0030984 , \\ L_2 = -0.0072975 ; & L_2 = -0.0031239 , \\ L_3 = -0.0047223 ; & L_3 = -0.0020265 , \\ L_{-1} = -0.00025276 ; & L_{-1} = -0.00000619 , \\ L_{-2} = +0.310465 ; & L_{-2} = +0.118659 , \\ L_{-3} = -0.014003 ; & L_{-3} = -0.0035584 . \end{array}\right\} \qquad (23)$$

With these values for the constants, F and J were computed according to equations (20) and (21) for various values of τ ($\leq \tau_1$). The results of the calculations are summarized in Tables 3 and 4, and the solutions for the case $\tau_1 = 2$ are illustrated in Figures 1 and 2. In these figures we have further compared our present solutions on the third approximation (full-line curves) with those given earlier by Chandrasekhar[3] on the Milne-Eddington type of approximation (dashed-line curves).

[3] *Zs.f.Ap.*, **9**, 266, 1935.