ON THE RADIATIVE EQUILIBRIUM OF A STELLAR ATMOSPHERE. II

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ABSTRACT

In this paper a new method is described for solving the various problems of radiative transfer in the theory of stellar atmospheres. The basic idea consists in expressing the integral (proportional to the density of radiation) which occurs in the equation of transfer as a sum according to Gauss's formula for numerical quadratures and replacing the integrodifferential equation by a system of linear equations. General solutions of this linear system can readily be written down, and this enables one to obtain solutions for the various problems to any desired degree of accuracy.

The method has been applied in detail to the problem considered in the earlier paper, and the first four approximations explicitly found. The corresponding laws of darkening have also been determined. An interesting by-product of the investigation is a new and entirely elementary proof of the exact Hopf-Bronstein relation, giving the boundary temperature in terms of the effective temperature.

1. *Introduction.*—In an earlier paper[1] it was shown how successively higher approximations to the solution of the equation of transfer

$$\mu \frac{dI}{d\tau} = I - \tfrac{1}{2} \int_{-1}^{+1} I \, d\mu \tag{1}$$

can be obtained by expanding I in terms of spherical harmonics in the form

$$I(\tau, \mu) = \sum_{l=0}^{\infty} I_l(\tau) P_l(\mu) . \tag{2}$$

More recently an important investigation by G. C. Wick[2] has come to the author's notice in which an alternative method for solving equations similar to (1) has been developed. It is the object of this paper to describe Wick's method in its astrophysical context and to show its particular adaptability for solving the standard problems of radiative transfer in the theory of stellar atmospheres.

2. *The outline of the method.*—Wick's basic idea consists in expressing the integral on the right-hand side of equation (1) as a sum, using for this purpose Gauss's formula for numerical quadratures.[3] Thus, denoting by $\mu_{-n}, \ldots, \mu_{-1}, \mu_1, \ldots, \mu_n$, and $\mu_{-i} = -\mu_i$ the $2n$ real roots of the Legendre polynomial $P_{2n}(\mu)$ of order $2n$, we can write, according to Gauss,

$$\int_{-1}^{+1} I(\tau, \mu) \, d\mu \simeq \sum_{j=-n}^{+n} a_j I(\tau, \mu_j) , \tag{3}$$ [4]

[1] *Ap. J.*, **99**, 180, 1944. Referred to hereafter as "I."

[2] *Zs. f. Phys.*, **120**, 702, 1943.

[3] See, e.g., P. Frank and R. V. Misses, *Differentialgleichungen der Physik*, **1**, 394, New York, 1943.

[4] Note that the summation on the right-hand side does not include the term $j = 0$.

where the a_j's are certain weight factors. It may be further noted that

$$a_j = a_{-j}\,; \qquad \mu_j = -\mu_{-j}\,. \tag{4}$$

It is known that for a given number of subdivisions of the interval $(-1, +1)$ Gauss's choice of the points μ_j and the weights a_j yields the best value for the integral in the sense that for any *arbitrary* polynomial of degree $4n - 1$, the formula (3) is *exact*. In particular,

$$\sum_{j=1}^{n} a_j \mu_j^m = \int_0^1 \mu^m d\mu = \frac{1}{m+1}\,. \tag{5}$$

Accordingly, the representation of the integral as a finite sum of the form (3) can be made as accurate as may be desired by choosing n sufficiently large.

In the "nth approximation" we therefore replace equation (1) by the linear system of ordinary equation of order $2n$:

$$\mu_i \frac{dI_i}{d\tau} = I_i - \tfrac{1}{2}\Sigma a_j I_j \qquad (i = \pm 1, \ldots, \pm n)\,, \tag{6}$$

where, for the sake of brevity, we have written I_i for $I(\tau, \mu_i)$. Further, in equation (6) the summation over j is extended over all the positive and negative values.

3. *The general solution of the system of equations (6)*.—We shall now find the different linearly independent solutions of the system (6) and later, by combining these, obtain the general solution.

First, we shall seek a solution of (6) of the form

$$I_i = g_i e^{-k\tau} \qquad (i = \pm 1, \ldots, \pm n)\,, \tag{7}$$

where the g_i's and k are constants, for the present unspecified. Introducing equation (7) into equation (6), we obtain the relation

$$g_i (1 + \mu_i k) = \tfrac{1}{2}\Sigma a_j g_j\,. \tag{8}$$

Hence,

$$g_i = \frac{\text{constant}}{1 + \mu_i k} \qquad (i = \pm 1, \ldots, \pm n)\,, \tag{9}$$

where the "constant" is independent of i. Substituting the foregoing form for g_i in equation (8), we obtain the following equation for k:

$$1 = \tfrac{1}{2}\Sigma \frac{a_j}{1 + \mu_j k}\,. \tag{10}$$

Remembering that $a_{-j} = a_j$ and $\mu_{-j} = -\mu_j$, we can re-write equation (10) as

$$1 = \sum_{j=1}^{n} \frac{a_j}{1 - \mu_j^2 k^2}\,. \tag{11}$$

It is thus seen that k^2 must satisfy an algebraic equation of order n. However, since

$$\sum_{j=1}^{n} a_j = 1 \tag{12}$$

(cf. eq. [5]), $k^2 = 0$ is a solution of equation (11). Accordingly, equation (10) has only $2n - 2$ distinct roots, which occur in pairs

$$\pm k_a \qquad (a = 1, \ldots, n-1)\,. \tag{13}$$

And, corresponding to these $2n - 2$ roots, we have $2n - 2$ independent solutions of equation (6). To complete the solution we notice that equation (6) admits of a solution of the form

$$I_i = b\,(\tau + q_i) \qquad (i = \pm 1, \ldots, \pm n)\,, \quad (14)$$

where b is an arbitrary constant. For, inserting the form (14) for I_i in equation (6), we find that

$$\mu_i = q_i - \tfrac{1}{2}\Sigma a_j q_j\,; \quad (15)$$

and this can be satisfied by

$$q_i = \mu_i + Q \qquad (i = \pm 1, \ldots, \pm n)\,, \quad (16)$$

where Q is an arbitrary constant. Thus, the system (6) allows the solution

$$I_i = b\,(\tau + Q + \mu_i) \qquad (i = \pm 1, \ldots, \pm n)\,, \quad (17)$$

where b and Q are two arbitrary constants.

Thus, combining solutions of the form (7) and (17), we have the general solution

$$I_i = b\left\{\sum_{a=1}^{n-1}\left[\frac{L_a e^{-k_a\tau}}{1+\mu_i k_a} + \frac{L_{-a} e^{+k_a\tau}}{1-\mu_i k_a}\right] + \mu_i + Q + \tau\right\}, \quad (18)$$

where b, $L_{\pm a}$, $(a = 1, \ldots, n-1)$, and Q are $2n$ arbitrary constants.

4. *The solution of equation (1) satisfying the necessary boundary conditions.*—For the astrophysical case under consideration the boundary conditions are[5] that none of the I_i's increase exponentially as $\tau \to \infty$ and that, further, there is no incident radiation on the surface $\tau = 0$. The first of these conditions implies that in the general solution (18) we omit all the terms in $\exp(+k_a\tau)$. Thus,

$$I_i = b\left\{\sum_{a=1}^{n-1}\frac{L_a e^{-k_a\tau}}{1+\mu_i k_a} + \mu_i + Q + \tau\right\} \quad (i = \pm 1, \ldots, \pm n)\,. \quad (19)$$

Next the nonexistence of any radiation incident on $\tau = 0$ requires that

$$I_{-i} = 0 \quad \text{at} \quad \tau = 0 \quad \text{and for} \quad i = 1, \ldots, n\,, \quad (20)$$

or, according to equations (4) and (19),

$$\sum_{a=1}^{n-1}\frac{L_a}{1-\mu_i k_a} + Q = \mu_i \qquad (i = 1, \ldots, n)\,. \quad (21)$$

These are the n equations which determine the $(n-1)$ constants L_a and the further constant Q. The constant b is left arbitrary; and this, as we shall presently show, is related to the constant net flux in the atmosphere.

Now the net flux is defined in terms of F, where

$$F = 2\int_{-1}^{+1} I\mu\,d\mu\,. \quad (22)$$

Expressing the integral on the right-hand side as a sum over the $I_i\mu_i$'s according to Gauss's formula and using the solution (19) for the I_i's, we find

$$F = 2b\left\{\sum_{a=1}^{n-1} L_a e^{-k_a\tau}\sum_i \frac{a_i\mu_i}{1+\mu_i k_a} + \sum_i a_i\mu_i^2 + (Q+\tau)\sum_i a_i\mu_i\right\}. \quad (23)$$

[5] Cf. I, p. 182.

On the other hand (cf. eq. [5]),

$$\Sigma a_i \mu_i^2 = \tfrac{2}{3} \quad \text{and} \quad \Sigma a_i \mu_i = 0 \, . \tag{24}$$

Moreover,

$$\left.\begin{aligned} \Sigma \frac{a_i \mu_i}{1+\mu_i k_a} &= \frac{1}{k_a} \Sigma a_i \left(1 - \frac{1}{1+\mu_i k_a}\right) \\ &= \frac{1}{k_a}\left(2 - \Sigma \frac{a_i}{1+\mu_i k_a}\right), \end{aligned}\right\} \tag{25}$$

which vanishes according to equation (10). Hence,

$$F = \tfrac{4}{3} b = \text{constant} \; ; \tag{26}$$

i.e., b is related to the constant net flux, as stated.

In terms of our solution for the I_i's we can obtain a convenient formula for J defined by

$$J = \tfrac{1}{2} \int_{-1}^{+1} I \, d\mu = \tfrac{1}{2} \Sigma a_i I_i \tag{27}$$

in our present approximation. We have

$$J = \tfrac{1}{2} b \left\{ \sum_{a=1}^{n-1} L_a e^{-k_a \tau} \sum_i \frac{a_i}{1+\mu_i k_a} + \sum_i a_i \mu_i + (Q+\tau) \sum_i a_i \right\} ; \tag{28}$$

or, according to equations (5), (10), (24), and (26), we have

$$J = \tfrac{3}{4} F \left(\tau + Q + \sum_{a=1}^{n-1} L_a e^{-k_a \tau} \right) . \tag{29}$$

If we express J in its "normal" form (cf. I, eq. [47]),

$$J = \tfrac{3}{4} F \left(\tau + q [\tau] \right) , \tag{30}$$

we have

$$q(\tau) = Q + \sum_{a=1}^{n-1} L_a e^{-k_a \tau} . \tag{31}$$

Finally, we may note that, according to the solution (29) for J, we have the law of darkening (cf. I, eq. [49]):

$$I(0, \mu) = \tfrac{3}{4} F \left(\mu + Q + \sum_{a=1}^{n-1} \frac{L_a}{1 + k_a \mu} \right) . \tag{32}$$

This completes the solution. The clear superiority of our present method over our earlier one of expanding I in terms of spherical harmonics is apparent. It is to be particularly noted that, in contrast to our earlier method, we can now write down the formal solution for any order of approximation quite generally.

5. *The first, second, third, and fourth approximations.*—We shall now obtain in their numerical forms the first four approximations to the solution for J and the corresponding laws of darkening.

i) *The first approximation.*—The first approximation is obtained by choosing $n = 1$, in which case

$$a_1 = a_{-1} = 1 \quad \text{and} \quad \mu_1 = -\mu_{-1} = \frac{1}{\sqrt{3}} . \tag{33}$$

There is no nonzero root for equation (10), and equation (21) now implies that

$$Q = \mu_1 = \frac{1}{\sqrt{3}} . \tag{34}$$

Accordingly,

$$q(\tau) = \frac{1}{\sqrt{3}} \tag{35}$$

and

$$I(0, \mu) = \tfrac{3}{4} F \left(\mu + \frac{1}{\sqrt{3}} \right) . \tag{36}$$

It is remarkable that in the very first approximation our method predicts a boundary value for $q(\tau)$ which is in *exact* agreement with the Hopf-Bronstein value (cf. I, eq. [73]). Actually, as we shall presently show (see § 6 below), this is identically the case in *all* approximations; and, consequently, we have here an essentially new and "elementary" proof of the Hopf-Bronstein relation.

One further remark about this first approximation may be made. The differential equations for I_1 and I_{-1} are

$$\left.\begin{aligned} \frac{1}{\sqrt{3}} \frac{dI_1}{d\tau} &= I_1 - \tfrac{1}{2} (I_1 + I_{-1}) , \\ -\frac{1}{\sqrt{3}} \frac{dI_{-1}}{d\tau} &= I_{-1} - \tfrac{1}{2} (I_1 + I_{-1}) , \end{aligned}\right\} \tag{37}$$

which are essentially the equations of Schwarzschild's first approximation.[6] However, the difference is that on the left-hand sides of the foregoing equations we now have $1/\sqrt{3}$ instead of the usual 1/2. It is interesting to speculate that, had Schwarzschild used our present systematic method, based on Gauss's formula, he might have discovered the exact boundary temperature some twenty-five years before Hopf and Bronstein!

ii) *The second approximation.*—To obtain the second approximation we have to choose for the μ_i's the zeros of $P_4(\mu)$ and the corresponding weight factors. We have[7]

$$\left.\begin{aligned} a_1 = a_{-1} &= 0.652145 ; \quad & \mu_1 = -\mu_{-1} &= 0.339981 , \\ a_2 = a_{-2} &= 0.347855 ; \quad & \mu_2 = -\mu_{-2} &= 0.861136 . \end{aligned}\right\} \tag{38}$$

Equation (11) reduces to

$$\mu_1^2 \mu_2^2 k^2 = a_1 \mu_1^2 + a_2 \mu_2^2 = \tfrac{1}{3} . \tag{39}$$

Hence,

$$k_1 = \frac{1}{\sqrt{3} \mu_1 \mu_2} = 1.972027 . \tag{40}$$

Solving for Q and L_1, we find

$$Q = 0.694025 ; \quad L_1 = -0.116675 . \tag{41}$$

[6] Cf. E. A. Milne, *Handb. d. Ap.*, **3**, No. 1, 114–116, Berlin: Springer, 1930.

[7] A. N. Lowan, N. Davids, and A. Levenson, *Bull. Amer. Math. Soc.*, **48**, 739, 1942.

Accordingly, on this approximation

$$q(\tau) = 0.694025 - 0.116675 e^{-1.97203\tau} \tag{42}$$

and

$$I(0, \mu) = \tfrac{3}{4}F\left(\mu + 0.694025 - \frac{0.116675}{1 + 1.97203\mu}\right). \tag{43}$$

iii) *The third approximation.*—We now have

$$\left.\begin{aligned} a_1 = a_{-1} = 0.467914\,; &\quad \mu_1 = -\mu_{-1} = 0.238619\,, \\ a_2 = a_{-2} = 0.360762\,; &\quad \mu_2 = -\mu_{-2} = 0.661209\,, \\ a_3 = a_{-3} = 0.171324\,; &\quad \mu_3 = -\mu_{-3} = 0.932470\,. \end{aligned}\right\} \tag{44}$$

The equation for k^2 reduces to

$$0.02164502k^4 - 0.254545k^2 + \tfrac{1}{3} = 0\,, \tag{45}$$

the positive roots of which are

$$k_1 = 3.202945 \quad \text{and} \quad k_2 = 1.225211\,. \tag{46}$$

Solving for Q, L_1, and L_2, we find

$$Q = 0.703899\,, \quad L_1 = -0.101245\,, \quad L_2 = -0.02530\,. \tag{47}$$

Hence,

$$q(\tau) = 0.703899 - 0.101245 e^{-3.20295\tau} - 0.02530 e^{-1.22521\tau} \tag{48}$$

and

$$I(0, \mu) = \tfrac{3}{4}F\left(\mu + 0.703899 - \frac{0.101245}{1 + 3.20295\mu} - \frac{0.02530}{1 + 1.22521\mu}\right). \tag{49}$$

iv) *The fourth approximation.*—To obtain the fourth approximation we have to choose for the μ_i's the roots of $P_8(\mu)$ and the corresponding weight factors. We have

$$\left.\begin{aligned} a_1 = a_{-1} = 0.362684\,; &\quad \mu_1 = -\mu_{-1} = 0.183435\,, \\ a_2 = a_{-2} = 0.313707\,; &\quad \mu_2 = -\mu_{-2} = 0.525532\,, \\ a_3 = a_{-3} = 0.222381\,; &\quad \mu_3 = -\mu_{-3} = 0.796666\,, \\ a_4 = a_{-4} = 0.101229\,; &\quad \mu_4 = -\mu_{-4} = 0.960290\,. \end{aligned}\right\} \tag{50}$$

The equation for k^2 reduces to

$$0.00543900k^6 - 0.1284982k^4 + 0.422222k^2 - \tfrac{1}{3} = 0\,, \tag{51}$$

the positive roots of which are

$$k_1 = 4.45808\,; \quad k_2 = 1.59178\,; \quad k_3 = 1.10319\,. \tag{52}$$

The constants Q, L_1, L_2, and L_3 were found to be

$$\left.\begin{aligned} Q = 0.706920\,; &\quad L_1 = -0.083921\,; \\ L_2 = -0.036187\,; &\quad L_3 = -0.009461\,. \end{aligned}\right\} \tag{53}$$

Accordingly,

$$q(\tau) = 0.70692 - 0.08392 e^{-4.45808\tau} - 0.03619 e^{-1.59178\tau} - 0.00946 e^{-1.10319\tau} \tag{54}$$

and

$$I(0, \mu) = \tfrac{3}{4}F\left(\mu + 0.70692 - \frac{0.08392}{1+4.45808\mu} - \frac{0.03619}{1+1.59178\mu} - \frac{0.00946}{1+1.10319\mu}\right). \quad (55)$$

In Tables 1 and 2 we have tabulated the functions $q(\tau)$ and $I(0, \mu)/F$ according to our second, third, and fourth approximations. It is also seen that, in agreement with our earlier remarks, $q(0)$ agrees with the exact value $1/\sqrt{3}$ in all our approximations. (For a proof of this relation see § 6 below.) Moreover, a comparison of the law of darkening on our fourth approximation with the exact values given in I, Table 2, indicates that in this approximation we have reached an over-all accuracy of about one part in two hundred.

TABLE 1

THE FUNCTION $q(\tau)$ DERIVED ON THE BASIS OF THE SECOND, THIRD, AND FOURTH APPROXIMATIONS (EQS. [42], [48], AND [54])

τ	$q(\tau)$			τ	$q(\tau)$		
	Second Approximation	Third Approximation	Fourth Approximation		Second Approximation	Third Approximation	Fourth Approximation
0.00	0.5774	0.5774	0.5774	0.90	0.6743	0.6898	0.6933
.05	.5883	.5938	.5974	1.0	.6778	.6924	.6954
.10	.5982	.6080	.6139	1.2	.6831	.6959	.6987
.15	.6072	.6202	.6274	1.4	.6867	.6982	.7008
.20	.6154	.6307	.6386	1.6	.6891	.6997	.7024
.25	.6228	.6398	.6479	1.8	.6907	.7008	.7035
.30	.6295	.6477	.6557	2.0	.6918	.7016	.7044
.35	.6355	.6544	.6621	2.2	.6925	.7021	.7050
.40	.6410	.6603	.6676	2.4	.6930	.7025	.7055
.50	.6505	.6698	.6761	2.6	.6933	.7028	.7058
.60	.6583	.6770	.6823	2.8	.6936	.7031	.7064
.70	.6647	.6824	.6870	3.0	.6937	.7033	.7065
0.80	0.6699	0.6866	0.6905	∞	0.6940	0.7039	0.7069

TABLE 2

THE LAWS OF DARKENING GIVEN BY THE SECOND, THIRD AND FOURTH APPROXIMATIONS (EQS. [43], [49], AND [55])

μ	$I(0, \mu)/F$		
	Second Approximation	Third Approximation	Fourth Approximation
0.0	0.4330	0.4330	0.4330
0.1	0.5224	0.5285	0.5319
0.2	0.6078	0.6164	0.6205
0.3	0.6905	0.7003	0.7046
0.4	0.7716	0.7819	0.7861
0.5	0.8515	0.8620	0.8660
0.6	0.9304	0.9410	0.9449
0.7	1.0088	1.0193	1.0231
0.8	1.0866	1.0970	1.1007
0.9	1.1640	1.1743	1.1779
1.0	1.2411	1.2513	1.2548

6. *A proof of the Hopf-Bronstein relation* J (0) = ($\sqrt{3}/4$) F.—In the preceding section we have verified that $q(0)$ agrees with the Hopf-Bronstein value $1/\sqrt{3}$ in all the four approximations we have numerically worked out. We shall now show how this result can be demonstrated to be quite generally and rigorously true. In order to do this, we start by considering the function

$$S(\mu) = \sum_{a=1}^{n-1} \frac{L_a}{1 - k_a \mu} + Q - \mu \,. \tag{56}$$

According to the boundary conditions (21),

$$S(\mu_i) = 0 \qquad (i = 1, \ldots, n) \,. \tag{57}$$

This fact enables us to determine $S(\mu)$ explicitly. For, by multiplying equation (56) by the function

$$R(\mu) = (1 - k_1\mu)(1 - k_2\mu)\ldots(1 - k_{n-1}\mu) \,, \tag{58}$$

we obtain a polynomial of degree n in μ which vanishes for $\mu = \mu_i$, $i = 1, \ldots, n$. Accordingly, $S(\mu)R(\mu)$ cannot differ from the polynomial

$$P(\mu) = (\mu - \mu_1)(\mu - \mu_2)\ldots(\mu - \mu_n) \tag{59}$$

by more than a constant factor; and this factor can be determined by comparing the coefficients of the highest power of μ (namely, μ^n) in $P(\mu)$ and $S(\mu)R(\mu)$. In the former it is unity, while in the latter it is

$$(-1)^n k_1 k_2 \ldots k_{n-1} \,. \tag{60}$$

Hence,

$$S(\mu) = (-1)^n k_1 k_2 \ldots k_{n-1} \frac{P(\mu)}{R(\mu)} \,. \tag{61}$$ [8]

From equations (31), (56), and (61) we now conclude that

$$q(0) = \sum_{a=1}^{n-1} L_a + Q = S(0) = (-1)^n k_1 k_2 \ldots k_{n-1} \frac{P(0)}{R(0)} \,. \tag{62}$$

On the other hand, according to our definitions of the functions $P(\mu)$ and $R(\mu)$,

$$P(0) = (-1)^n \mu_1 \mu_2 \ldots \mu_n \,; \qquad R(0) = 1 \,. \tag{63}$$

Hence,

$$q(0) = k_1 k_2 \ldots k_{n-1} \mu_1 \mu_2 \ldots \mu_n \,. \tag{64}$$

We shall now show how the quantity on the right-hand side of equation (64) can be explicitly evaluated from the equation for the roots $k_1, \ldots, k_{n-1}$. We have (eq. [11])

$$1 = \sum_{j=1}^{n} \frac{a_j}{1 - k^2 \mu_j^2} \,. \tag{65}$$

Multiplying this equation by the product of the factors $(1 - k^2\mu_1^2), \ldots, (1 - k^2\mu_n^2)$, we obtain

$$\left.\begin{aligned} (-1)^{n-1}\mu_1^2\mu_2^2 \ldots \mu_n^2 k^{2n} + \ldots - \sum_{i=1}^{n} a_i \left(\sum_{j=1}^{n} \mu_j^2 - \mu_i^2 \right) k^2 + \sum_{i=1}^{n} \mu_i^2 k^2 & \\ + \sum_{i=1}^{n} a_i - 1 = 0 \,; & \end{aligned}\right\} \tag{66}$$

[8] This relation can be used for a direct evaluation of the constants L_a without going through a routine solution of linear equations (21).

or, using equation (5), we have

$$(-1)^{n-1}\mu_1^2\mu_2^2 \ldots \mu_n^2 k^{2n-2} + \ldots + \tfrac{1}{3} = 0 . \tag{67}$$

Hence the product of the roots $k_1^2 \ldots k_{n-1}^2$ is given by

$$k_1^2 k_2^2 \ldots k_{n-1}^2 = \frac{1}{3\mu_1^2\mu_2^2 \ldots \mu_n^2} \tag{68}$$

or

$$k_1 k_2 \ldots k_{n-1} = \frac{1}{\sqrt{3}\,\mu_1\mu_2 \ldots \mu_n} . \tag{69}$$

Combining equations (64) and (69), we have

$$q(0) = \frac{1}{\sqrt{3}} , \tag{70}$$

a result which is thus seen to be true in all orders of approximation. We therefore conclude that equation (70) represents an exact relation.

Finally, it is to be noted that equations (30) and (70) imply that

$$J(0) = \frac{\sqrt{3}}{4} F , \tag{71}$$

which is the well-known relation of Hopf and Bronstein.

7. *Further applications of the method.*—Our analysis in the preceding sections has demonstrated the extreme simplicity with which solutions accurate to any desired extent can be obtained. But the usefulness of the method is by no means limited to the particular problem which has been considered. Indeed, the possible applications of the method are so numerous that it would hardly be possible to consider all of them within the limits of a single paper. We shall therefore content ourselves with a brief consideration of two further standard problems in the theory of radiative transfer, postponing to a later occasion the more detailed discussion of the various solutions.

i) *The radiative equilibrium of a planetary nebula.*[9]—As was first pointed out by Ambarzumian, the equation of transfer for the "ultraviolet" radiation (i.e., radiation beyond the head of the Lyman series) consistent with Zanstra's theory is

$$\mu \frac{dI}{d\tau} = I - \tfrac{1}{2}p \int_{-1}^{+1} I\,d\mu - \tfrac{1}{4}pSe^{-(\tau_1-\tau)} , \tag{72}$$

where p is a certain factor less than unity, τ_1 the optical thickness of the nebula for the ultraviolet radiation, and πS the amount of ultraviolet radiant energy incident on each square centimeter of the inner surface of the nebula (i.e., at $\tau = \tau_1$).

Again, in equation (72), we approximate the integral occurring on the right-hand side by a sum according to Gauss's formula for numerical quadratures. And in this manner we replace equation (72) by the system of $2n$ linear equations

$$\mu_i \frac{dI_i}{d\tau} = I_i - \tfrac{1}{2}p\Sigma a_j I_j - \tfrac{1}{4}pSe^{-(\tau_1-\tau)} \quad (i = \pm 1, \ldots, \pm n) , \tag{73}$$

in the nth approximation. We shall now briefly indicate how the general solution of this linear system of equations can be obtained.

[9] For earlier discussions of this problem see V. A. Ambarzumian, *Pulkovo Obs. Bull.*, No. 13, and *M.N.*, **93**, 50, 1931; also S. Chandrasekhar, *Zs. f. Ap.*, **9**, 266, 1935.

Setting

$$I_i = S\left[g_i e^{-k\tau} + h_i e^{-(\tau_1-\tau)}\right] \qquad (i = \pm 1, \ldots, \pm n) \quad (74)$$

in equation (73) (where g_i, h_i, and k are constants), we find

$$g_i(1 + k\mu_i) = \tfrac{1}{2}p\Sigma a_j g_j \quad (75)$$

and

$$h_i(1 - \mu_i) = \tfrac{1}{2}p\Sigma a_j h_j + \tfrac{1}{4}p\,. \quad (76)$$

Equation (75) implies that

$$g_i = \frac{\text{constant}}{1 + \mu_i k} \qquad (i = \pm 1, \ldots, \pm n) \quad (77)$$

and that k is a root of the equation

$$1 = p\sum_{j=1}^{n} \frac{a_j}{1 - \mu_j^2 k^2}\,. \quad (78)$$

Since $p < 1$, the foregoing equation admits of $2n$ distinct roots, which occur in pairs as

$$\pm k_a \qquad (a = 1, \ldots, n)\,. \quad (79)$$

Considering next equation (76), we observe that h_i must be expressible in the form

$$h_i = \frac{\beta}{1 - \mu_i} \qquad (i = \pm 1, \ldots, \pm n)\,, \quad (80)$$

where β is a constant which must, in turn, be so chosen that

$$\beta = \tfrac{1}{2}p\beta\Sigma \frac{a_j}{1 - \mu_j} + \tfrac{1}{4}p\,. \quad (81)$$

Hence,

$$\beta = \tfrac{1}{4}p \frac{1}{1 - p\displaystyle\sum_{j=1}^{n} \frac{a_j}{1 - \mu_j^2}}\,. \quad (82)$$

Accordingly, the general solution of equation (73) can be written in the form

$$I_i = S\left\{\sum_{a=1}^{n}\left[\frac{L_a e^{-k_a\tau}}{1 + \mu_i k_a} + \frac{L_{-a} e^{+k_a\tau}}{1 - \mu_i k_a}\right] + \frac{p\,e^{-(\tau_1-\tau)}}{4(1 - \mu_i)\left(1 - p\displaystyle\sum_{j=1}^{n}\frac{a_j}{1 - \mu_j^2}\right)}\right\}, \quad (83)$$

where $L_{\pm a}$, $a = 1, \ldots, n$, are $2n$ constants of integration. For the problem under consideration the solution (83) becomes determinate when the boundary conditions at $\tau = \tau_1$ and at $\tau = 0$ are taken into account. These are

$$I_i = I_{-i} \quad \text{at} \quad \tau = \tau_1 \quad \text{for} \quad i = 1, \ldots, n \quad (84)$$

and

$$I_{-i} = 0 \quad \text{at} \quad \tau = 0 \quad \text{for} \quad i = 1, \ldots, n\,. \quad (85)$$

The conditions (84) arise from the geometry of the problem, as was first pointed out by Milne,[10] while the conditions (85) arise from the nonexistence of any radiation incident on $\tau = 0$. The explicit form which these conditions take can be readily written down in

[10] *Zs. f. Ap.*, 1, 98, 1930.

terms of the general solution (83). We shall not continue with this discussion further, but it is apparent how solutions of any desired degree of accuracy can be obtained in this manner.

ii) *The standard case in the theory of the formation of absorption lines.*—In a standard notation[11] the equation of transfer appropriate for this problem is

$$\mu \frac{dI_\nu}{dt_\nu} = I_\nu - \frac{(1-\epsilon)\,\eta_\nu}{2\,(1+\eta_\nu)} \int_{-1}^{+1} I_\nu d\mu - \frac{1+\epsilon\eta_\nu}{1+\eta_\nu}\,(a_{\nu_0} + b_{\nu_0} t_\nu) \; . \tag{86}$$

In the "standard case" the ratio η_ν of the line to the continuous absorption coefficients is assumed to be constant. Suppressing the suffix ν and introducing the quantity

$$\lambda = \frac{1+\epsilon\eta}{1+\eta} \, , \tag{87}$$

we can re-write equation (86) more conveniently in the form

$$\mu \frac{dI}{dt} = I - \tfrac{1}{2}\,(1-\lambda) \int_{-1}^{+1} I d\mu - \lambda\,(a+bt) \; . \tag{88}$$

In the nth approximation we replace the foregoing equation by the system of linear equations

$$\mu_i \frac{dI_i}{dt} = I_i - \tfrac{1}{2}\,(1-\lambda)\,\Sigma a_j I_j - \lambda\,(a+bt) \qquad (i = \pm 1, \ldots, \pm n) \, . \tag{89}$$

Proceeding as before, we verify that the solution of this system appropriate for the problem on hand is

$$I_i = \sum_{a=1}^{n} \frac{L_a e^{-k_a t}}{1+\mu_i k_a} + (\mu_i b + a + bt) \qquad (i = \pm 1, \ldots, \pm n) \, , \tag{90}$$

where the k_a's are the n positive roots of the equation

$$1 = (1-\lambda) \sum_{j=1}^{n} \frac{a_j}{1-\mu_j^2 k^2} \tag{91}$$

and the L_a's ($a = 1, \ldots, n$) are the n constants of integration to be determined by the boundary conditions

$$\sum_{a=1}^{n} \frac{L_a}{1-\mu_i k_a} + a = \mu_i b \qquad (i = 1, \ldots, n) \, . \tag{92}$$

Again solutions to any desired degree of accuracy can be obtained.

In conclusion, we should further like to point out that the methods developed in this paper can readily be applied also to problems in which the scattering does not take place isotropically, as, for example, in the case of scattering by free electrons. But we postpone a discussion of this problem to a later occasion.

I am indebted to Miss Frances Herman, who carried out the numerical work involved in the preparation of Tables 1 and 2.

[11] See, e.g., B. Strömgren, *Ap. J.*, **86**, 1, 1937.