ON THE RADIATIVE EQUILIBRIUM OF A STELLAR ATMOSPHERE

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ABSTRACT

In this paper the equation of transfer

$$\mu \frac{dI}{d\tau} = I - I_0 ; \qquad I_0 = \int_{-1}^{+1} I d\mu$$

is solved by expanding I in terms of spherical harmonics in the form

$$I(\tau, \mu) = \sum_{l=0}^{\infty} I_l(\tau) P_l(\mu) .$$

It is shown how successively higher approximations to I can be obtained by retaining more and more terms in the foregoing expansion; thus, the first, second, and third approximations are obtained by retaining the first three, five, and seven terms, respectively, in the expansion. The first three approximations to $I_0(\tau)$ are explicitly obtained, and the corresponding laws of darkening derived.

The exact function describing the law of darkening and given by Hopf's theory is also numerically evaluated. A comparison of this exact law with those derived on our second and third approximations indicates that the method described in this paper is sufficiently rapidly convergent.

1. *Introduction.*—As is well known, the solution of the equation of transfer

$$\cos \vartheta \frac{dI}{d\tau} = I - J , \tag{1}$$

where

$$J = \frac{1}{2} \int_0^{\pi} I \sin \vartheta d\vartheta , \tag{2}$$

has occupied a central position in the theory of stellar atmospheres and has been the subject of investigations by Schwarzschild, Milne, Eddington, Jeans, Hopf, Bronstein, and others. An account of these investigations and the successive methods of approximation, as developed particularly by Milne and Eddington, will be found in the various treatises on the subject by Eddington, Milne, and Unsöld.[1] However, the methods described by these latter writers, while adequate for most purposes, are not sufficiently systematic, in the sense that no general procedure is outlined which would enable one to obtain solutions of increasing accuracy by retaining, for example, more and more terms in an expansion of some sort for I. More recently, however, an attempt in this direction has been made by L. Gratton[2] in a paper of considerable interest. Gratton's basic idea is to seek a solution of equation (1) of the form

$$I(\tau, \vartheta) = \sum_{l=1}^{\infty} I_l(\tau) P_l(\cos \vartheta) , \tag{3}$$

[1] A. S. Eddington, *The Internal Constitution of the Stars*, chap. xii, Cambridge, England, 1926; E. A. Milne, *Handb. d. Ap.*, **3**, No. 1, 114–126, Berlin: Springer, 1930; A. Unsöld, *Physik der Sternatmosphären*, pp. 89–104, Berlin: Springer, 1938.

[2] *Societa astronomica italiana*, **10**, 309–325, 1937.

in which the P_l's denote the various Legendre polynomials. In other words, we expand I in terms of spherical harmonics with the expectation that by retaining a sufficient number of terms in the expansion we shall be able to obtain solutions with any desired degree of accuracy. This is, of course, an entirely sound procedure; but, unfortunately, Gratton's specific treatment of the problem following this idea is vitiated by a number of errors and oversights which require corrections. Accordingly, it has been thought worth while to re-examine the whole problem *de novo* and obtain what might properly be described as the first, second, and third approximations to the solution of equation (1). We shall, moreover, compare the predictions of these approximate solutions with regard to those features (e.g., the law of darkening) for which the Hopf-Bronstein[3] theory provides exact information.

2. *The outline of the method.*—Denoting $\cos\vartheta$ by μ, we can re-write equation (1) in the form

$$\mu\frac{dI}{d\tau}=I-I_0\,, \tag{4}$$

where I_0 is the first term in the expansion

$$I(\tau,\mu)=\sum_{l=0}^{\infty}I_l(\tau)P_l(\mu)\,. \tag{5}$$

Substituting the foregoing expansion for I in equation (4) and remembering that the P_l's satisfy the recursion formula

$$\mu P_l=\frac{1}{2l+1}\left[(l+1)P_{l+1}+lP_{l-1}\right], \tag{6}$$

we find

$$\sum_{l=0}^{\infty}\frac{1}{2l+1}\left[(l+1)P_{l+1}+lP_{l-1}\right]\frac{dI_l}{d\tau}=\sum_{l=0}^{\infty}I_lP_l-I_0\,. \tag{7}$$

Equating the coefficients of the various Legendre polynomials in equation (7), we obtain

$$\frac{l}{2l-1}\frac{dI_{l-1}}{d\tau}+\frac{l+1}{2l+3}\frac{dI_{l+1}}{d\tau}=I_l\qquad(l=1,2,\ldots.) \tag{8}$$

and

$$\frac{1}{3}\frac{dI_1}{d\tau}=0\qquad\qquad(l=0)\,. \tag{9}$$

Equation (9) leads at once to the integral

$$I_1=\text{constant}=\tfrac{3}{4}F\qquad(\text{say})\,, \tag{10}$$

which clearly insures the constancy of the net integrated flux and determines the effective temperature of the star. Another integral of the equations is obtained by considering equation (8) for the case $l=1$. For, according to this equation, for $l=1$ we have

$$\frac{dI_0}{d\tau}+\frac{2}{5}\frac{dI_2}{d\tau}=I_1\,, \tag{11}$$

[3] E. Hopf, *Mathematical Problems of Radiative Equilibrium* (Cambridge Mathematical Tract No. 31), Cambridge, England, 1934.

or, since I_1 is a constant (eq. [10]),

$$I_0+\tfrac{2}{5}I_2=\tfrac{3}{4}F\tau+a\,, \tag{12}$$

where a is an arbitrary constant at our disposal. This integral is readily verified to be the same as what is more generally known as the "K-integral."[4]

Before we proceed to outline our method of approximation we shall first formulate the boundary conditions of the problem.

For the problem of the radiative equilibrium of a stellar atmorphere the boundary condition at $\tau=0$ is

$$I\,(0,\,\mu)=0 \quad \text{for} \quad -1<\mu<0\,, \tag{13}$$

since no external radiation is assumed to be incident on the star. The implication of the boundary condition (13) for the I_l's as $\tau\to 0$ can be derived in the following manner. From equation (5) we conclude that for $\tau>0$

$$\frac{2}{2l+1}\,I_l(\tau)=\int_{-1}^{+1} I\,(\tau,\,\mu)\,P_l(\mu)\,d\mu\,. \tag{14}$$

Passing now to the limit $\tau=0$ and remembering that, according to equation (13), $I\,(0,\,\mu)$ vanishes for all negative values of μ, we obtain

$$\frac{2}{2l+1}\,I_l(0)=\int_{0}^{1} I\,(0,\,\mu)\,P_l(\mu)\,d\mu\,. \tag{15}$$

On the other hand, for $\tau=0$ (cf. eq. [5])

$$I\,(0,\,\mu)=\sum_{m=0}^{\infty} I_m(0)\,P_m(\mu)\,. \tag{16}$$

Hence, combining equations (15) and (16), we must have

$$\frac{2}{2l+1}\,I_l(0)=\sum_{m=0}^{\infty} I_m(0)\int_{0}^{1} P_l(\mu)\,P_m(\mu)\,d\mu\,. \tag{17}$$

Thus the boundary condition (1) is equivalent to the infinite set of linear relations (17) among the I_l's as $\tau\to 0$.

It is now apparent that we cannot satisfy the entire set of relations (17) when a solution for I is sought by retaining only a finite number of terms in the expansion (5). But it is clear that the more the number of the relations (17) we are able to satisfy, the more, in general, we may expect the derived solution to be accurate. On the other hand, only as many of the relations (17) can be satisfied as there are disposal constants in the solution for the I_l's. It thus appears that the order of the approximation will be indicated by the number of constants of integration (other than F) which the solutions for the I_l's contain. Thus, as we shall presently show (§§ 3, 4, and 5, below), the solutions obtained by retaining only the first three, five, or seven terms in the expansion (5) involve respectively one, two, or three disposable constants of integrations. These, then, will provide the first, second, or third approximations to the solution of equation (1).[5]

[4] E.g., see Milne, *op. cit.*, p. 121, eq. (125), or Unsöld, *op. cit.*, p. 97, eq. (29.13).

[5] One of the oversights of Gratton (*loc. cit.*) consists precisely in his failure to recognize this point. Thus he obtains his "second" approximation by retaining terms up to and including I_3. However, as

For further reference we may note here that the relations (17) for the cases $l = 0, 1$, and 2 explicitly reduce to

$$2I_0(0) = I_0(0) + \tfrac{1}{2}I_1(0) - \tfrac{1}{8}I_3(0) + \tfrac{1}{16}I_5(0) + \ldots , \tag{18}$$

$$\tfrac{2}{3}I_1(0) = \tfrac{1}{2}I_0(0) + \tfrac{1}{3}I_1(0) + \tfrac{1}{8}I_2(0) - \tfrac{1}{48}I_4(0) + \tfrac{1}{128}I_6(0) + \ldots , \tag{19}$$

and

$$\tfrac{2}{5}I_2(0) = \tfrac{1}{8}I_1(0) + \tfrac{1}{5}I_2(0) + \tfrac{1}{8}I_3(0) - \tfrac{5}{128}I_5(0) + \ldots , \tag{20}$$

where on the right-hand sides we have retained all terms up to and including I_6. Remembering that $I_1 = \text{constant} = 3F/4$, we can write more conveniently in the forms

$$\tfrac{1}{2}I_0(0) + \tfrac{1}{16}I_3(0) - \tfrac{1}{32}I_5(0) + \ldots = \tfrac{3}{16}F , \tag{21}$$

$$\tfrac{1}{2}I_0(0) + \tfrac{1}{8}I_2(0) - \tfrac{1}{48}I_4(0) + \tfrac{1}{128}I_6(0) + \ldots = \tfrac{1}{4}F , \tag{22}$$

and

$$\tfrac{2}{5}I_2(0) - \tfrac{1}{4}I_3(0) + \tfrac{5}{64}I_5(0) + \ldots = \tfrac{3}{16}F . \tag{23}$$

So far we have considered only the boundary conditions at $\tau = 0$. Turning next to the boundary conditions at $\tau = \infty$, it is evident that[6]

$$I_0 \sim \tfrac{3}{4}F\tau \quad \text{and} \quad I_l(\tau) \text{ for } l > 0 \text{ bounded as } \tau \to \infty . \tag{24}$$

3. *The first approximation.*—The first approximation is obtained by retaining the terms I_0, I_1, and I_2 in the expansion for I and considering only the first three equations (namely, for $l = 0$, 1, and 2) which result from equation (7). The equations for $l = 0$ and $l = 1$ lead, as we have already seen, to the integrals (10) and (12). The equation for $l = 2$ is

$$\frac{2}{3}\frac{dI_1}{d\tau} + \frac{3}{7}\frac{dI_3}{d\tau} = I_2 . \tag{25}$$

Remembering that I_1 is a constant and that we have put $I_3 = 0$, it follows that

$$I_2 = 0 . \tag{26}$$

Accordingly, our solution is

$$I_0 = \tfrac{3}{4}F\tau + a ; \qquad I_1 = \tfrac{3}{4}F \tag{27}$$

and involves, as we see, the one disposable constant a. To determine this constant, we should use one or the other of the relations (17). Now the relation for $l = 1$ is given preference over all the others, since (as was first pointed out by Milne) this insures that the

we have indicated, the genuine second approximation is obtained only when the term I_4 is also included. A consequence of this is that Gratton's solution (after correcting some further arithmetical errors in his evaluation of the constants of integration) fails to satisfy certain other necessary conditions of the problem (see n. 7, below).

[6] Cf. Hopf, *op. cit.*

emergent flux is the same as the constant net flux in the interior. Accordingly, we use equation (22) to determine a. We readily find that

$$a = \tfrac{1}{2}F\ . \tag{28}$$

Thus,

$$I_0(\tau) = \tfrac{3}{4}F\left(\tau + \tfrac{2}{3}\right)\,, \tag{29}$$

and the corresponding law of darkening is

$$I(0,\ \mu) = \tfrac{1}{2}F + \tfrac{3}{4}F\mu\ . \tag{30}$$

We therefore see that the first approximation on our present method reduces simply to the solution in the form first derived by Milne.

4. *The second approximation.*—As we shall now show, the second approximation for I is obtained by retaining the terms up to and including I_4 in the expansion (5). This requires the consideration of the first five equations which result from equation (7). Thus, in addition to the integrals (10) and (12), we have now to consider the following equations:

$$\left.\begin{aligned} \frac{3}{7}\frac{dI_3}{d\tau} &= I_2\,, \\ \frac{3}{5}\frac{dI_2}{d\tau} + \frac{4}{9}\frac{dI_4}{d\tau} &= I_3\,, \\ \frac{4}{7}\frac{dI_3}{d\tau} \qquad\qquad &= I_4\,. \end{aligned}\right\} \tag{31}$$

From the first and the last of the foregoing equations we obtain the integral

$$\tfrac{4}{3}I_2 = I_4\,. \tag{32}$$ [7]

Equations (31) also lead to the following differential equation for I_3:

$$\frac{16}{63}\frac{d^2 I_3}{d\tau^2} = \frac{4}{9}\frac{dI_4}{d\tau} = I_3 - \frac{3}{5}\frac{dI_2}{d\tau} = I_3 - \frac{9}{35}\frac{d^2 I_3}{d\tau^2}\,, \tag{33}$$

or

$$\frac{23}{45}\frac{d^2 I_3}{d\tau^2} = I_3\,. \tag{34}$$

The solution of this equation appropriate for our present purposes is (cf. eq. [24])

$$I_3 = A\,e^{-\alpha\tau}\,, \tag{35}$$

where A is an arbitrary constant and

$$\alpha = \sqrt{\tfrac{45}{23}} = 1.399\ . \tag{36}$$

From equations (31) and (32) we now obtain

$$I_2 = -\tfrac{3}{7}A\,\alpha\,e^{-\alpha\tau}\,; \qquad I_4 = -\tfrac{4}{7}A\,\alpha\,e^{-\alpha\tau}\,. \tag{37}$$

[7] The existence of this integral reveals the inconsistency of an approximation in which I_3 is included but I_4 is not (see n. 5). A consequence of this inconsistency is that when the solution for I, including only terms up to I_3, is obtained, it is found that $I(0, 0) \neq I_0(0)$, contrary to what should be expected to be identically true (since at the limb we necessarily see only the surface layers).

The solutions for the I's are thus seen to involve only two disposable constants (namely, A and a) and therefore lead, as already stated, to the second approximation.

To determine the constants A and a, we use equations (21) and (22) as our boundary conditions. Under our present assumptions (namely, that I_5, I_6, etc., can all be ignored) these equations reduce to

$$\tfrac{1}{2}I_0(0)+\tfrac{1}{16}I_3(0)=\tfrac{3}{16}F \tag{38}$$

and

$$\tfrac{1}{2}I_0(0)+\tfrac{1}{8}I_2(0)-\tfrac{1}{48}I_4(0)=\tfrac{1}{4}F. \tag{39}$$

Eliminating I_0 (0) between the foregoing two equations, we obtain

$$\tfrac{1}{8}I_2(0)-\tfrac{1}{16}I_3(0)-\tfrac{1}{48}I_4(0)=\tfrac{1}{16}F. \tag{40}$$

This equation directly determines A. We find

$$A=-\frac{F}{1+\frac{2}{3}a}=-0.5175F. \tag{41}$$

Accordingly, our solutions for I_2, I_3, and I_4 are

$$I_2=0.3102Fe^{-a\tau}\,;\qquad I_3=-0.5175Fe^{-a\tau}\,;\qquad I_4=0.4136Fe^{-a\tau}. \tag{42}$$

Equation (38) or (39) now determines I_0 (0). We find

$$I_0(0)=0.4397F. \tag{43}$$

On the other hand, since (cf. eq. [12])

$$I_0(0)=a-\tfrac{2}{5}I_2(0)\,, \tag{44}$$

we have

$$a=0.5638F. \tag{45}$$

Thus, our solution for I_0 is

$$I_0(\tau)=\tfrac{3}{4}F\tau+0.5638F-0.1241Fe^{-a\tau}\,, \tag{46}$$

which for later comparisons (§ 6) we write in the form

$$I_0(\tau)=\tfrac{3}{4}F[\tau+q(\tau)]\,, \tag{47}$$

where now

$$q(\tau)=0.7517-0.1654e^{-1.399\tau}. \tag{48}$$

To obtain the corresponding law of darkening, we start from the exact relation

$$I(0,\mu)=\int_0^\infty I_0(\tau)\,e^{-\tau/\mu}\frac{d\tau}{\mu} \tag{49}$$

and substitute for I_0 (τ) the solution (46). We thus find that

$$I(0,\mu)=F\left(0.5638+0.75\mu-\frac{0.1241}{1+1.399\mu}\right). \tag{50}$$

5. *The third approximation.*—To obtain the third approximation for I, we retain all the terms up to and including I_6 in the expansion for I and consider the first seven equations which result from equation (7). Accordingly, in addition to the two equations

which lead to the integrals (10) and (12), we have to consider the following five further equations:

$$\left.\begin{aligned} \frac{3}{7}\frac{dI_3}{d\tau} &= I_2\,, \\ \frac{3}{5}\frac{dI_2}{d\tau}+\frac{4}{9}\frac{dI_4}{d\tau} &= I_3\,, \\ \frac{4}{7}\frac{dI_3}{d\tau}+\frac{5}{11}\frac{dI_5}{d\tau} &= I_4\,, \\ \frac{5}{9}\frac{dI_4}{d\tau}+\frac{6}{13}\frac{dI_6}{d\tau} &= I_5\,, \\ \frac{6}{11}\frac{dI_5}{d\tau} &= I_6\,. \end{aligned}\right\} \tag{51}$$

From the first, third, and fifth of the foregoing equations we obtain the integral

$$\tfrac{4}{3}I_2+\tfrac{5}{6}I_6=I_4\,, \tag{52}$$

which is our present analogue of the integral (32) we had in the second approximation.

Again, from equations (51) we obtain

$$\left.\begin{aligned} \frac{36}{143}\frac{d^2I_5}{d\tau^2} &= \frac{6}{13}\frac{dI_6}{d\tau} \\ &= I_5-\frac{5}{9}\frac{dI_4}{d\tau} \\ &= I_5-\frac{5}{9}\left(\frac{4}{7}\frac{d^2I_3}{d\tau^2}+\frac{5}{11}\frac{d^2I_5}{d\tau^2}\right). \end{aligned}\right\} \tag{53}$$

Similarly

$$\left.\begin{aligned} \frac{20}{99}\frac{d^2I_5}{d\tau^2} &= \frac{4}{9}\left(\frac{dI_4}{d\tau}-\frac{4}{7}\frac{d^2I_3}{d\tau^2}\right) \\ &= -\frac{16}{63}\frac{d^2I_3}{d\tau^2}+I_3-\frac{3}{5}\frac{dI_2}{d\tau} \\ &= -\frac{16}{63}\frac{d^2I_3}{d\tau^2}+I_3-\frac{9}{35}\frac{d^2I_3}{d\tau^2}\,. \end{aligned}\right\} \tag{54}$$

Equations (53) and (54) can be re-written alternatively in the forms

$$\left.\begin{aligned} \frac{59}{117}\frac{d^2I_5}{d\tau^2}-I_5 &= -\frac{20}{63}\frac{d^2I_3}{d\tau^2}\,, \\ \frac{23}{45}\frac{d^2I_3}{d\tau^2}-I_3 &= -\frac{20}{99}\frac{d^2I_5}{d\tau^2}\,. \end{aligned}\right\} \tag{55}$$

To solve the foregoing equations we make the substitutions

$$I_3=A\,e^{-a\tau} \quad \text{and} \quad I_5=Be^{-a\tau}\,, \tag{56}$$

where A, B, and a are constants, for the present unspecified. We obtain

$$\left.\begin{aligned}\left(\frac{59}{117}\,a^2-1\right)B&=-\frac{20}{63}\,a^2 A\,,\\ \left(\frac{23}{45}\,a^2-1\right)A&=-\frac{20}{99}\,a^2 B\,.\end{aligned}\right\}\tag{57}$$

Hence a^2 must satisfy the equation

$$\left(\frac{59}{117}\,a^2-1\right)\left(\frac{23}{45}\,a^2-1\right)=\frac{400}{63\times 99}\,a^4\,.\tag{58}$$

Solving this equation, we find that a can have either of the two values

$$a_1=1.9825 \quad \text{or} \quad a_2=1.1464\,.\tag{59}$$

Corresponding to these two roots for a, we must have

$$B_1=-1.2706A_1 \quad \text{and} \quad B_2=1.2368A_2\,.\tag{60}$$

Accordingly, the solutions for I_3 and I_5 appropriate for our present problem are

$$I_3=A_1e^{-a_1\tau}+A_2e^{-a_2\tau}\tag{61}$$

and

$$I_5=-1.2706A_1e^{-a_1\tau}+1.2368A_2e^{-a_2\tau}\,.\tag{62}$$

Using equations (51) and (52), we can now evaluate the remaining I's. We find

$$\left.\begin{aligned}I_2&=-0.8497A_1e^{-a_1\tau}-0.4913A_2e^{-a_2\tau}\,,\\ I_4&=+0.0121A_1e^{-a_1\tau}-1.2995A_2e^{-a_2\tau}\,,\\ I_6&=+1.3740A_1e^{-a_1\tau}-0.7733A_2e^{-a_2\tau}\,.\end{aligned}\right\}\tag{63}$$

We thus see that our solution for the I's involves three disposable constants (namely, A_1, A_2, and a) and leads, therefore, as we have already stated, to the third approximation for I.

To determine next the constants A_1, A_2, and a we use equations (21), (22), and (23) as our boundary conditions. Eliminating $I_0\,(0)$ between equations (21) and (22), we have

$$\tfrac{1}{8}I_2\,(0)-\tfrac{1}{16}I_3\,(0)-\tfrac{1}{48}I_4\,(0)+\tfrac{1}{32}I_5\,(0)+\tfrac{1}{128}I_6\,(0)=\tfrac{1}{16}F\,.\tag{64}$$

Equations (23) and (64) directly determine A_1 and A_2. We find

$$A_1=-0.3931F \quad \text{and} \quad A_2=0.2384F\,.\tag{65}$$

Substituting these values for A_1 and A_2 in equations (61), (62), and (63), we obtain

$$\left.\begin{aligned}I_2&=(+0.3340\,e^{-a_1\tau}-0.1171\,e^{-a_2\tau})\,F\,,\\ I_3&=(-0.3931\,e^{-a_1\tau}+0.2384\,e^{-a_2\tau})\,F\,,\\ I_4&=(-0.0048\,e^{-a_1\tau}-0.3098\,e^{-a_2\tau})\,F\,,\\ I_5&=(+0.4995\,e^{-a_1\tau}+0.2949\,e^{-a_2\tau})\,F\,,\\ I_6&=(-0.5402\,e^{-a_1\tau}-0.1844\,e^{-a_2\tau})\,F\,.\end{aligned}\right\}\tag{66}$$

The value of I_0 (0) can now be determined from equation (64). We find

$$I_0(0) = 0.4440F . \tag{67}$$

Equations (12), (66), and (67) now determine a. We find

$$a = 0.5307F . \tag{68}$$

Thus the solution for I_0 is given by

$$I_0(\tau) = \tfrac{3}{4}F\tau + 0.5307F - 0.1336Fe^{-a_1\tau} + 0.0469Fe^{-a_2\tau} ; \tag{69}$$

or, writing I_0 (τ) in the form (47), we now have

$$q(\tau) = 0.7077 - 0.1781\,e^{-1.9825\tau} + 0.0625\,e^{-1.1464\tau} . \tag{70}$$

Finally, to determine the law of darkening we again start with the equation (49) but substitute for I_0 our present solution (69). In this manner we find

$$I(0, \mu) = F\left(0.5307 + 0.75\mu - \frac{0.1336}{1 + 1.19825\mu} + \frac{0.0469}{1 + 1.1464\mu}\right). \tag{71}$$

6. *A comparison of the second and the third approximations in their predictions regarding* q (τ) *and the law of darkening: the exact law of darkening.*—As is well known, Hopf and Bronstein have succeeded in deriving certain exact results concerning the solution of the equation of transfer (1) satisfying the boundary condition (13). Thus it has been shown (see Hopf, *op. cit.*) that, if the solution for I_0 be written in the form

$$I_0(\tau) = \tfrac{3}{4}F[\tau + q(\tau)] , \tag{72}$$

then q (τ) is a monotonic increasing function of τ and that, moreover,

$$q(0) = \frac{1}{\sqrt{3}} = 0.57735 \tag{73}$$

and

$$q(\infty) = \frac{1}{\pi}\int_0^{\pi/2}\left(\frac{3}{\sin^2\phi} + \frac{\tan\phi}{\phi - \tan\phi}\right)d\phi . \tag{74}$$

The integral on the right-hand side of equation (74) was evaluated numerically, and it was found that

$$q(\infty) = 0.710447 . \tag{75}$$

These exact values for q (0) and q (∞) should be compared with those given by equations (48) and (70) on our second and third approximations, respectively. We have

$$\left.\begin{array}{ll} q(0) = 0.5862 ; & q(\infty) = 0.7517 \text{ (second approximation)} , \\ q(0) = 0.5920 ; & q(\infty) = 0.7077 \text{ (third approximation)} . \end{array}\right\} \tag{76}$$

It is seen that, while the third approximation effects a marked improvement over the second in the agreement of q (∞) with the exact value (75), it worsens somewhat the agreement of q (0) with its exact value (73). However, it appears that the third approximation actually represents a substantial improvement over the second over the entire range of τ except in the very immediate vicinity of the boundary $\tau = 0$. That this is so is indicated, for example, by the comparison made in Table 1 of the functions q (τ) given by our second and third approximations with that given by Eddington's second approximation.[8] This conclusion is further strengthened when we compare the laws of darkening

TABLE 1

A COMPARISON OF THE FUNCTION $q(\tau)$ DERIVED ON THE BASIS OF THE SECOND AND THE THIRD APPROXIMATION (EQS. [48] AND [70]) WITH THAT DERIVED ON EDDINGTON'S SECOND APPROXIMATION

τ	$q(\tau)$ Second Approximation	$q(\tau)$ Third Approximation	$q(\tau)$ Eddington's Second Approximation	τ	$q(\tau)$ Second Approximation	$q(\tau)$ Third Approximation	$q(\tau)$ Eddington's Second Approximation
0.00	0.5862	0.5920	0.583	0.90	0.7047	0.7000	0.692
.05	.5974	.6053	.620	1.0	.7108	.7030	.694
.10	.6078	.6172	.635	1.2	.7208	.7070	.697
.15	.6175	.6280	.645	1.4	.7283	.7091	.699
.20	.6266	.6375	.653	1.6	.7340	.7102	.701
.25	.6351	.6460	.660	1.8	.7383	.7106	.702
.30	.6429	.6537	.665	2.0	.7416	.7106	.703
.35	.6503	.6605	.670	2.2	.7441	.7104	.704
.40	.6571	.6666	.673	2.4	.7459	.7101	.704
.50	.6695	.6768	.679	2.6	.7473	.7098	.705
.60	.6802	.6848	.683	2.8	.7484	.7095	.705
.70	.6895	.6912	.686	3.0	.7492	.7092	.706
0.80	0.6977	0.6962	0.689	∞	0.7517	0.7077	0.710

TABLE 2

A COMPARISON OF THE LAWS OF DARKENING GIVEN BY THE SECOND AND THE THIRD APPROXIMATION (EQS. [50] AND [71]) WITH THAT GIVEN BY THE EXACT FORMULA (EQ. [77])

μ	$I(0, \mu)/F$ Second Approximation	$I(0, \mu)/F$ Third Approximation	$I(0, \mu)/F$ Exact	$I(0, \mu)/I(0, 1)$ Second Approximation	$I(0, \mu)/I(0, 1)$ Third Approximation	$I(0, \mu)/I(0, 1)$ Exact
0	0.4397	0.4440	0.43301	0.3484	0.3530	0.34390
0.1	0.5299	0.5363	0.54011	0.4199	0.4264	0 42896
0.2	0.6168	0.6232	0.62802	0.4887	0.4955	0.49878
0.3	0.7014	0.7068	0.71123	0.5557	0.5620	0.56487
0.4	0.7842	0.7884	0.79210	0.6214	0.6268	0.62909
0.5	0.8657	0.8684	0.87156	0.6860	0.6904	0.69220
0.6	0.9463	0.9475	0.95009	0.7498	0.7533	0.75457
0.7	1.0261	1.0258	1.02796	0.8130	0.8156	0.81642
0.8	1.1052	1.1035	1.10536	0.8757	0.8774	0.87788
0.9	1.1838	1.1808	1.18238	0.9380	0.9388	0.93906
1.0	1.2620	1.2578	1.25912	1.0000	1.0000	1.00000

given by our two approximations with the exact law of darkening. This comparison is made in the following paragraph.

[8] Milne, *op. cit.*, pp. 122–124, and Unsöld, *op. cit.*, p. 102.

It has been shown by Hopf (*op. cit.*) that for the case under discussion the problem of darkening admits of an exact solution. Thus, expressing the emergent intensity $I(0, \mu)$ in the form

$$I(0, \mu) = \frac{\sqrt{3}}{4} F\Phi(\mu) , \tag{77}$$

Hopf has shown (*op. cit.*, p. 105) that

$$\Phi(\mu) = (1+\mu) \exp \left\{ -\frac{\mu}{\pi} \int_0^{\pi/2} \frac{\log [\,(1-\phi \cot \phi)\,/\sin^2 \phi]}{\cos^2 \phi + \mu^2 \sin^2 \phi} d\phi \right\} . \tag{78}$$

Although the foregoing solution has been known for over ten years, it does not appear that the integral defining $\Phi(\mu)$ has been evaluated. Accordingly, we have evaluated the function $\Phi(\mu)$ numerically for the values $\mu = 0, 0.1, 0.2, \ldots, 1.0$ and tabulated the resulting values of $I(0, \mu)$ in Table 2. In Table 2 we have also tabulated the functions $I(0, \mu)$ as given by the equations (50) and (71), i.e., by our second and third approximations, respectively. It is seen that, except for μ very close to zero, equation (71) for $I(0, \mu)$ provides an appreciably better approximation to the exact law of darkening than does equation (50). And, moreover, the agreement of either (50) or (71) with the exact law of darkening over the whole range of μ is entirely satisfactory.

In conclusion I wish to record my indebtedness to Miss Frances Herman, who carried out most of the numerical work connected with the preparation of Tables 1 and 2.