DYNAMICAL FRICTION

II. THE RATE OF ESCAPE OF STARS FROM CLUSTERS AND THE EVIDENCE FOR THE OPERATION OF DYNAMICAL FRICTION

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ABSTRACT

In this paper a general method is described for determining the rate of escape of stars from galactic and globular clusters which is based on certain general statistical principles. Essentially the method consists in reducing the problem to a boundary-value problem in partial differential equations and in making use of the interpretation of the stochastic process in the velocity space as a diffusion process of a rather general type.

The rate of escape has been evaluated, first, ignoring dynamical friction, and, second, making due allowance for it. It appears that the rate of escape of stars predicted on the first basis is too rapid to be compatible with a life for galactic clusters even of the order of 5×10^8 years. However, the rates of escape are drastically reduced when dynamical friction is allowed for and permits a time scale of the order of 3×10^9 years. It is concluded that in the very existence of galactic clusters like the Pleiades we can look for direct evidence for the operation of dynamical friction which was predicted on theoretical grounds in the preceding paper.

1. *Introduction.*—In the preceding paper[1] we have shown that stars must experience dynamical friction during their motion. This conclusion, first reached on the basis of certain very general considerations, was later confirmed by a more direct analysis of the fluctuating force acting on a star in terms of the two-body approximation for stellar encounters. In this paper we propose to draw attention to certain facts of stellar dynamics which provide direct evidence for the operation of dynamical friction.

Since the coefficient of dynamical friction is of the order of the reciprocal of the time of relaxation of the system (cf. I, eq. [14]), it is evident that it is only during times of the order of the time of relaxation itself that dynamical friction will have a chance to become an effective agent. Consequently, the effects of dynamical friction will be apparent only in stellar systems with relatively short times of relaxation. Such systems are provided by galactic clusters like the Pleiades, which are characterized by times of relaxation of the order of 6×10^7 years.[2] Since the times of relaxation of the galactic clusters are of this order of magnitude, it is clear that an important factor in their evolution must be the escape of stars from them.[3] For, in times of the order of the time of relaxation, the probability that a star will, on account of accidental fluctuations, acquire a velocity equal to or greater than the velocity of escape must be appreciable. And, if this should happen, we can reasonably expect the star to escape from the cluster. The question now arises as to the rate at which stars will thus leave the cluster. In this paper we shall show how this rate can be evaluated on the basis of certain general statistical principles and how

[1] Referred to hereafter as "I."

[2] Cf. S. Chandrasekhar, *Principles of Stellar Dynamics*, chap. v, University of Chicago Press, 1942. This monograph will be referred to hereafter as "*Stellar Dynamics*."

In *Stellar Dynamics* (p. 202) the time of relaxation of the Pleiades is given as 2.9×10^7 years. However, in view of the fact that in a sufficient approximation η^{-1} is equal to *twice* the time of relaxation as defined in *Stellar Dynamics* (cf. the remarks in I following eq. [14]), and since for our present purposes η^{-1} provides a better unit for measuring time, we have quoted in the text a value which is twice that given in *Stellar Dynamics.*

[3] This fact was first clearly recognized by Ambarzumian and Spitzer. For references to these papers and for a general discussion of the related ideas see *Stellar Dynamics*, chap v, §§ 5.2–5.4.

precisely in this rate of escape we can look for evidence for the operation of dynamical friction.

2. *A general method for estimating the rate of escape of stars from galactic and globular clusters.*—In the preceding paper we have shown that, when the diffusion coefficient, q, and the coefficient of dynamical friction, η, are functions of $\boldsymbol{u}$, the equation which governs the distribution $W(\boldsymbol{u}, t)$ of $\boldsymbol{u}$ at time t is

$$\frac{\partial W}{\partial t} = \text{div}_{\boldsymbol{u}} \left(q \, \text{grad}_{\boldsymbol{u}} W + \eta W \boldsymbol{u} \right) , \tag{1}$$

where q and η are further related according to

$$\frac{q}{\eta} = \tfrac{1}{3} \overline{|\boldsymbol{u}|^2} = \text{constant} . \tag{2}$$

This differential equation for W leads to an important interpretation of the stochastic process which takes place in the velocity space. For, according to equation (1), we can visualize the motion of the representative points in the velocity space as a *process of diffusion* in which the rate of flow across an element of surface $d\sigma$ is given by

$$- \left(q \, \text{grad}_{\boldsymbol{u}} W + \eta W \boldsymbol{u} \right) \cdot \mathbf{1}_{d\sigma} d\sigma , \tag{3}$$

where $\mathbf{1}_{d\sigma}$ is a unit vector which is normal to the element of surface considered. With this interpretation of the stochastic process in mind, the following method for finding the rate at which a star may be expected to acquire a given velocity naturally suggests itself.

First, we find the probability, $p(v_0, t)\,dt$, that a star with an initial velocity $|\boldsymbol{u}| = v_0$ will acquire for the *first time* a certain preassigned velocity, $|\boldsymbol{u}| = v_\infty$, say, between t and $t + dt$. We then integrate $p(v_0, t)$ over t from 0 to t, to obtain the total probability, $Q(v_0, t)$, that the star will have acquired the velocity v_∞ during the entire interval from 0 to t. Finally, we average $Q(v_0, t)$ over the relevant range of the initial velocities v_0, to obtain the *expectation*, $Q(t)$, that a star will have acquired the velocity v_∞ during a time t.

The advantage of formulating the problem in the manner described is that the function $p(v_0, t)$ can be determined in terms of a spherically symmetric solution of equation (1) which satisfies the boundary conditions

$$W(|\boldsymbol{u}|, t) = 0 \quad \text{for} \quad |\boldsymbol{u}| = v_\infty \text{ for all } t > 0 \tag{4}$$

and

$$W(|\boldsymbol{u}|, t) \to \frac{1}{4\pi v_0^2} \delta(|\boldsymbol{u}| - v_0) \text{ as } t \to 0 , \tag{5}$$

where δ stands for Dirac's δ-function. If W is such a solution, the required probability function $p(v_0, t)$ is given by (cf. the interpretation of eq. [1] in an earlier paragraph)

$$p(v_0, t) = -\left(4\pi q |\boldsymbol{u}|^2 \frac{\partial W(|\boldsymbol{u}|, t)}{\partial |\boldsymbol{u}|} \right)_{|\boldsymbol{u}| = v_\infty} \tag{6}$$

The probability $Q(v_0, t)$ that a star having an initial velocity v_0 will have acquired the velocity v_∞ during a time t is then given by

$$Q(v_0, t) = \int_0^t p(v_0, t)\, dt . \tag{7}$$

And, finally, the expectation $Q(t)$ that a star will have acquired the velocity v_∞ during a time t is given by

$$Q(t)=\int_0^{v_\infty} Q(v_0, t)\, f(v_0)\, d v_0\,, \tag{8}$$

where $f(v_0)$ governs the frequency of occurrence of an initial velocity v_0.

Now the coefficient of dynamical friction η, as derived on the basis of the two-body approximation for stellar encounters, is (cf. I, eq. [32] for the case $m_1 = m_2 = m$)

$$\eta = 8\pi N m^2 G^2 \log\left[\frac{D_0\overline{|\boldsymbol{u}|^2}}{2Gm}\right]\frac{1}{|\boldsymbol{u}|^3}\int_0^{|\boldsymbol{u}|} f(v)\, d v\,. \tag{9}$$

According to this formula, η tends to a constant limiting value as $|\boldsymbol{u}|\to 0$. But, as $|\boldsymbol{u}|\to\infty$, $\eta\to 0$; however, according to the relation (2), q also tends to zero simultaneously with η. Consequently, by allowing q and η to be constants and equal to their respective average values, we shall be compensating for the overestimation of η for large values of $|\boldsymbol{u}|$ by a corresponding overestimation in the diffusion coefficient q. In this paper we shall accordingly restrict ourselves, for the sake of simplicity, to the case where q and η are constants. In a later paper we shall present the results of a similar calculation in which due allowance will be made for the dependence of q and η on $|\boldsymbol{u}|$.

3. *The rate of escape of stars from galactic clusters.*—For the reasons explained toward the end of the last section we shall suppose in this investigation that q and η are both constants and independent of $|\boldsymbol{u}|$. Equation (1) can then be re-written as (cf. eq. [2])

$$\frac{\partial W}{\partial t}=\tfrac{1}{3}\overline{|\boldsymbol{u}|^2}\eta\nabla_{\boldsymbol{u}}^2 W+\eta\,\mathrm{div}_{\boldsymbol{u}}(W\boldsymbol{u})\,. \tag{10}$$

Let

$$\eta t=\tau\,;\qquad \boldsymbol{u}=\left(\tfrac{2}{3}\overline{|\boldsymbol{u}|^2}\right)^{1/2}\boldsymbol{\rho}\,; \tag{11}$$

or, in words, τ measures the time in units of the time of relaxation; and, if a Gaussian distribution of the velocities

$$\frac{j^3}{\pi^{3/2}}\,e^{-j^2|\boldsymbol{u}|^2}d\boldsymbol{u} \tag{12}$$

be assumed, $\boldsymbol{\rho}$ measures the velocity $\boldsymbol{u}$ in units of j^{-1}. With the transformation of the variables (11) equation (10) becomes

$$\frac{\partial W}{\partial \tau}=\tfrac{1}{2}\nabla_{\boldsymbol{\rho}}^2 W+\mathrm{div}_{\boldsymbol{\rho}}\,(W\boldsymbol{\rho})\,. \tag{13}$$

It should be noted that in our present choice of the units the diffusion coefficient has the value $\frac{1}{2}$.

For a spherically symmetric solution, equation (12) reduces to

$$\frac{\partial W}{\partial \tau}=\frac{1}{2\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial W}{\partial\rho}\right)+\left(3W+\rho\frac{\partial W}{\partial\rho}\right), \tag{14}$$

where we have used ρ to denote $|\boldsymbol{\rho}|$. And, according to our remarks in § 2, we have to seek a solution of equation (14) which satisfies the boundary conditions

$$W(\rho,\tau)=0 \qquad \text{for} \qquad \rho_0=\rho_\infty \text{ (say) for } \tau>0 \tag{15}$$

and

$$W(\rho,\tau)\to\frac{1}{4\pi\rho_0^2}\,\delta(\rho-\rho_0)\ \text{ as } \tau\to 0\,. \tag{16}$$

i) *The rate of escape of stars from clusters when dynamical friction is ignored.*—When dynamical friction is ignored, equation (14) further simplifies to

$$\frac{\partial W}{\partial \tau}=\frac{1}{2\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2\frac{\partial W}{\partial \rho}\right), \tag{17}$$

and the solution of this equation satisfying the boundary conditions (15) and (16) is[4]

$$W=\frac{1}{2\pi\rho\rho_\infty\rho_0}\sum_{n=1}^{\infty}e^{-n^2\pi^2\tau/2\rho_\infty^2}\sin\left(\frac{n\pi}{\rho_\infty}\rho\right)\sin\left(\frac{n\pi}{\rho_\infty}\rho_0\right). \tag{18}$$

In terms of the foregoing solution we can determine the probability $p(\rho_0, \tau)\,d\tau$ that a star with an initial velocity corresponding to ρ_0 will acquire for the first time a velocity corresponding to ρ_∞ during τ and $\tau + d\tau$. Remembering that in our present units the coefficient of diffusion has the value $\frac{1}{2}$, we have (cf. eq. [6])

$$p(\rho_0, \tau)=-2\pi\rho_\infty^2\left(\frac{\partial W}{\partial \rho}\right)_{\rho=\rho_\infty}; \tag{19}$$

or, using the solution (18), we have

$$p(\rho_0, \tau)=\frac{\pi}{\rho_0\rho_\infty}\sum_{n=1}^{\infty}n(-1)^{n+1}e^{-n^2\pi^2\tau/2\rho_\infty^2}\sin\left(\frac{n\pi}{\rho_\infty}\rho_0\right). \tag{20}$$

The total probability $Q(\rho_0, \tau)$ that the star would have acquired the velocity ρ_∞ during the interval $(0, \tau)$ is therefore given by

$$Q(\rho_0, \tau)=\int_0^\tau p(\rho_0,\tau)\,d\tau=\frac{2\rho_\infty}{\pi\rho_0}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\left(1-e^{-n^2\pi^2\tau/2\rho_\infty^2}\right)\sin\left(\frac{n\pi}{\rho_\infty}\rho_0\right). \tag{21}$$

Finally, to obtain the expectation that an "average" star will have acquired the velocity ρ_∞ in a time τ, we must average the foregoing expression over all ρ_0. For this purpose we shall use for the distribution over ρ_0 the radial Gaussian function

$$\frac{4}{\pi^{1/2}}e^{-\rho_0^2}\rho_0^2 \tag{22}$$ [5]

and extend the range of integration from 0 to ∞. Strictly speaking, this is not a valid procedure, particularly the extending of the range of integration beyond ρ_∞. However, for the values of ρ_∞ we shall be normally interested in (cf. eqs. [25] and [26], below), the number of stars with $\rho > \rho_\infty$ forms a negligible fraction of the total number (see, e.g., *Stellar Dynamics*, p. 207, eq. [5.311]). With this understanding, the averaging of $Q(\rho_0, \tau)$ over ρ_0 leads to the formula

$$Q(\tau)=2\sum_{n=1}^{\infty}(-1)^{n+1}\left(1-e^{-n^2\pi^2\tau/2\rho_\infty^2}\right)e^{-n^2\pi^2/4\rho_\infty^2}. \tag{23}$$

[4] See, e.g., H. S. Carslaw and J. C. Jaeger, *Operational Methods in Applied Mathematics*, p. 235 (Ex. 16), Oxford, England, 1941.

[5] Remembering that in our present choice of the units $j = 1$.

Now, in a star cluster we have the following relation between the mean square velocity of escape and the mean square velocity of the stars in the system (cf. *ibid.*, pp. 206–207, eqs. [5.306] and [5.311])

$$\overline{v_\infty^2} = 4\overline{|u|^2}\,; \tag{24}$$

or, in our present choice of the units (cf. eq. [11]), we have

$$\overline{\rho_\infty^2} = 6\,. \tag{25}$$

However, in view of the circumstance that a star acquiring a velocity $2(\overline{|u|^2})^{1/2}$ does not necessarily imply its leaving the cluster unless it acquires a somewhat higher velocity (cf. *ibid.*, pp. 208–209), we shall suppose that

$$\rho_\infty^2 = 8\,, \tag{26}$$

to allow a reasonable margin. Table 1 gives the values of $Q(\tau)$ both for $\rho_\infty = \sqrt{6}$ and for $\rho_\infty = \sqrt{8}$.

TABLE 1

THE EXPECTATION $Q(\tau)$ FOR A STAR TO ESCAPE FROM A CLUSTER DURING A TIME τ (MEASURED IN UNITS OF THE TIME OF RELAXATION) WHEN DYNAMICAL FRICTION IS IGNORED

τ	$Q(\tau)$		τ	$Q(\tau)$	
	$\rho_\infty^2=6$	$\rho_\infty^2=8$		$\rho_\infty^2=6$	$\rho_\infty^2=8$
0.25	0.069	0.023	2.5	0.82	0.68
0.5	.19	.081	3.0	.87	.77
1.0	.42	.25	4.0	.94	.87
1.5	.60	.43	5.0	0.97	0.93
2.0	0.73	0.57			

Remembering that the time of relaxation of galactic clusters is of the order of 6×10^7 years, an examination of Table 1 reveals that the rates of escape predicted (when dynamical friction is ignored) are far too rapid to be compatible even with lives for these clusters of the order of 3×10^8 years. This can also be seen directly from equation (23). For, according to this equation,

$$Q(\tau) \sim 2e^{-\pi^2/4\rho_\infty^2}\left(1 - e^{-\pi^2\tau/2\rho_\infty^2}\right) \qquad (\tau \gtrsim 1)\,; \tag{27}$$

or, for $\rho_\infty^2 = 6$, respectively 8, we have the approximate formulae

$$\left.\begin{aligned} Q(\tau) &\sim 1.3\,(1 - e^{-0.82\tau}) \qquad (\rho_\infty^2 = 6)\,, \\ Q(\tau) &\sim 1.5\,(1 - e^{-0.62\tau}) \qquad (\rho_\infty^2 = 8)\,. \end{aligned}\right\} \tag{28}$$

However, as we shall presently see, the rates of escape are drastically reduced from what we have just now found when proper allowance is made for dynamical friction.

ii) *The rate of escape of stars from clusters when allowance is made for dynamical friction.*—Passing now to the case when dynamical friction is not ignored, we have to solve equation (14), together with the boundary conditions (15) and (16). Introducing the variable

$$w = W\rho\,, \tag{29}$$

equation (14) simplifies to

$$\frac{\partial w}{\partial \tau} = \frac{1}{2} \frac{\partial^2 w}{\partial \rho^2} + 2w + \rho \frac{\partial w}{\partial \rho} . \tag{30}$$

The boundary conditions (15) and (16) now become

$$w(\rho, \tau) = 0 \text{ for both } \rho = \rho_\infty \quad \textit{and} \quad \rho = 0 \text{ for all } \tau > 0 \tag{31}$$

and

$$w(\rho, \tau) \to \frac{1}{4\pi\rho_0} \delta(\rho - \rho_0) \text{ as } \tau \to 0 . \tag{32}$$

We shall now show how the solution of equation (30), together with the boundary conditions (31), can be reduced to a problem in characteristic values.

First we notice that a separation of the variables can be effected by the substitution

$$w = e^{-\lambda\tau} \phi(\rho) , \tag{33}$$

where λ is, for the present, an unspecified constant. Equation (30) now leads to the differential equation

$$\frac{d^2\phi}{d\rho^2} + 2\rho \frac{d\phi}{d\rho} + (2\lambda + 4)\phi = 0 . \tag{34}$$

Again, writing

$$\phi = e^{-\rho^2/2}\psi , \tag{35}$$

we have for ψ the differential equation

$$\frac{d^2\psi}{d\rho^2} + (2\lambda + 3 - \rho^2)\psi = 0 ; \tag{36}$$

or, putting

$$\lambda = \mu - 1 , \tag{37}$$

we have

$$\frac{d^2\psi}{d\rho^2} + (2\mu + 1 - \rho^2)\psi = 0 . \tag{38}$$

It is seen that the differential equation (38) for ψ is the same as the familiar wave equation for a simple harmonic oscillator. However, the boundary conditions with which we have now to solve equation (38) are different from those customary in solving the problem of the simple harmonic oscillator in the quantum theory, for the solution we are now looking for must satisfy the boundary conditions

$$\psi = 0 \quad \text{for} \quad \rho = 0 \quad \text{and also for} \quad \rho = \rho_\infty . \tag{39}$$

In other words, the ψ's of our problem are the characteristic functions of a simple harmonic oscillator bounded at the origin and at $\rho = \rho_\infty$, i.e., an oscillator in a "box." It is, therefore, clear that the ψ's which satisfy the boundary conditions (39) form a complete set of orthogonal functions which can be further normalized.

Let

$$\psi_1, \psi_2, \ldots, \psi_n, \ldots \tag{40}$$

represent the normalized characteristic functions of our problem belonging respectively to the characteristic values

$$\mu_1, \mu_2, \ldots, \mu_n, \ldots . \tag{41}$$

The general solution of equation (30) satisfying the boundary conditions (31) can therefore be expressed in the form

$$w = \sum_{n=1}^{\infty} A_n e^{-(\mu_n - 1)\tau} e^{-\rho^2/2} \psi_n(\rho) , \tag{42}$$

where the A_n's are certain constants which should be so chosen that the boundary condition for $\tau = 0$ is satisfied.

Now, since a δ-function can always be built up from any complete set of normalized orthogonal functions according to

$$\delta(\rho - \rho_0) = \sum_{n=1}^{\infty} \psi_n(\rho) \psi_n(\rho_0) , \tag{43}$$

it follows that the solution which satisfies the boundary conditions (31) *and* (32) is

$$w = \frac{e^{-(\rho^2 - \rho_0^2)/2}}{4\pi\rho_0} \sum_{n=1}^{\infty} e^{-(\mu_n - 1)\tau} \psi_n(\rho) \psi_n(\rho_0) . \tag{44}$$

Thus our solution for W takes the form

$$W = \frac{e^{-(\rho^2 - \rho_0^2)/2}}{4\pi\rho\rho_0} \sum_{n=1}^{\infty} e^{-(\mu_n - 1)\tau} \psi_n(\rho) \psi_n(\rho_0) . \tag{45}$$

Using the foregoing solution for W, we find that (cf. eq. [67])

$$p(\rho_0, \tau) = \frac{\rho_\infty}{2\rho_0} e^{-(\rho_\infty^2 - \rho_0^2)/2} \sum_{n=1}^{\infty} e^{-(\mu_n - 1)\tau} \left(-\frac{d\psi_n}{d\rho} \right)_{\rho = \rho_\infty} \psi_n(\rho_0) ; \tag{46}$$

or, for the probability $Q(\rho_0, \tau)$, we have

$$Q(\rho_0, \tau) = \frac{\rho_\infty}{2\rho_0} e^{-(\rho_\infty^2 - \rho_0^2)/2} \sum_{n=1}^{\infty} \frac{1}{\mu_n - 1} [1 - e^{-(\mu_n - 1)\tau}] \left(-\frac{d\psi_n}{d\rho} \right)_{\rho = \rho_\infty} \psi_n(\rho_0) . \tag{47}$$

Finally, to obtain $Q(\tau)$ we must further average the foregoing expression over the relevant range of ρ_0. With this we have formally solved the problem. To make the solution explicit, it remains only to specify the characteristic functions ψ_n and the corresponding characteristic values μ_n.

The nature of the dependence of the characteristic values μ_n on the length of the "box" ρ_∞ can be obtained by following a procedure developed by Sommerfeld in his studies of the Kepler problem and the problem of the rotator in the quantum theory with "artificial" boundary conditions.[6]

First, it is clear that when

$$\rho_\infty \to \infty , \quad \mu_n \to n \quad (n = 1, 3, 5, \ldots) . \tag{48}$$

(Only the odd integral values of n need concern us here, since the wave function has to vanish at the origin.) It is further evident that the functions

$$\Psi_n = e^{-\rho^2/2} H_n(\rho) , \tag{49}$$

where the H_n's are the various Hermite polynomials, formally solve equation (38) with $\mu_n = n$; and, if n is an odd integer, these functions Ψ_n satisfy also the boundary condition

[6] A. Sommerfeld and H. Welker, *Ann. d. Phys.*, **32**, 56, 1938, and A. Sommerfeld and H. Hartmann, *Ann. d. Phys.*, **37**, 333, 1940.

at the origin. If it should now happen that ρ_∞ coincides with a zero of one of the odd Hermite polynomials, then the corresponding wave function ψ_n will satisfy the boundary condition at ρ_∞ as well. Thus,

$$H_3 = 8\rho^3 - 12\rho \tag{50}$$

has a zero at $\rho = (1.5)^{1/2}$. Accordingly, if $\rho_\infty = (1.5)^{1/2}$, $\mu = 3$ is a characteristic value of our problem, and Ψ_3 for $\rho \leqslant (1.5)^{1/2}$ is the characteristic function which belongs to it. This represents, then, a special solution to our problem. Similarly, the higher-order Hermite polynomials will further provide such special solutions. The advantage in obtaining these special solutions is that by plotting the zeros of the various Hermite polynomials in a (μ, ρ_∞) diagram (as in Fig. 1) we obtain at once a general indication of how the various characteristic values are modified by the "artificial" boundary condition at $\rho = \rho_\infty$.

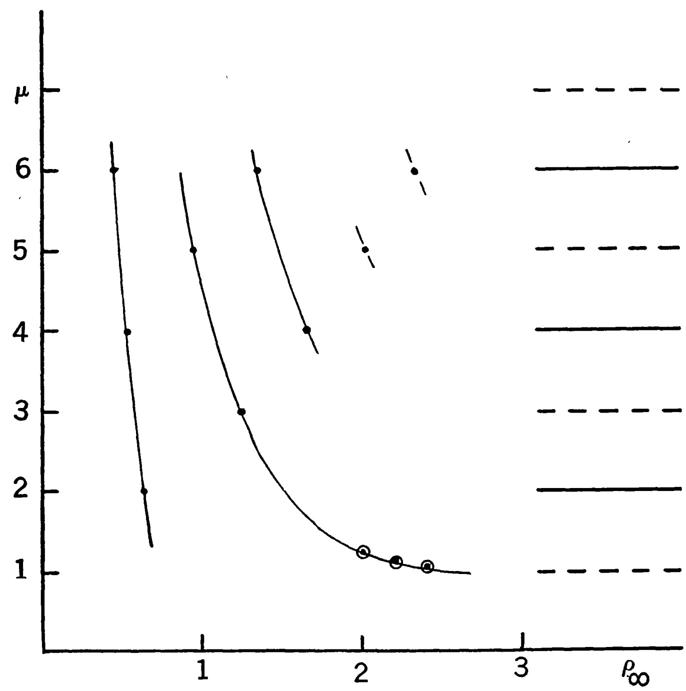

FIG. 1

Now an examination of Figure 1 shows that for $\rho_\infty > 2$ the first characteristic value of our problem must be extremely close to unity, so that $\mu_1 - 1$ must be a very small quantity. On the other hand, the higher characteristic values will lead to values of $(\mu_n - 1) \sim (n - 1)$ ($n > 1$, but an odd integer). Accordingly, for values of τ of the order of unity and greater, the first term in the series on the right-hand side of equation (47) will provide ample accuracy. Thus,

$$\left.\begin{aligned} &Q(\rho_0, \tau) \\ &\simeq \frac{\rho_\infty}{2\rho_0(\mu_1 - 1)} e^{-(\rho_\infty^2 - \rho_0^2)/2}\left[1 - e^{-(\mu_1 - 1)\tau}\right]\left(-\frac{d\psi_1}{d\rho}\right)_{\rho=\rho_\infty} \psi_1(\rho_0) \quad (\tau \gtrsim 1) . \end{aligned}\right\} \tag{51}$$

Finally, to determine $(\mu_1 - 1)$ corresponding to the "lowest state" of our artificially limited simple harmonic oscillator, we proceed as follows:

Writing

$$\psi = e^{-\rho^2/2} f \tag{52}$$

in equation (38), we obtain the differential equation

$$\frac{d^2 f}{d\rho^2} - 2\rho \frac{df}{d\rho} + 2\mu f = 0 . \tag{53}$$

Substituting for f the series

$$f = \Sigma c_s \rho^s , \tag{54}$$

where s runs through all the odd integers, we obtain the recursion formula

$$c_{s+2} = -\frac{2(\mu - s)}{(s+2)(s+1)} . \tag{55}$$

We already know that the particular characteristic value we are interested in must be very close to unity. Accordingly, writing

$$\mu_1 = 1 + \epsilon \tag{56}$$

and treating ϵ as a small quantity, we find that all the coefficients $c_3, c_5, \ldots$, contain ϵ as a factor. Retaining only the first-order terms in ϵ and letting $c_1 = 1$, we readily find that we can write f in the form

$$f = \rho (1 - \epsilon\chi) , \tag{57}$$

where

$$\chi = \tfrac{1}{3}\rho^2 + \tfrac{1}{15}\rho^4 + \tfrac{4}{315}\rho^6 + \tfrac{2}{945}\rho^8 + \ldots . \tag{58}$$

The condition that f has to vanish at some specified ρ_∞ will determine ϵ. Thus it was found that

$$\left.\begin{aligned} \epsilon &= 0.059 \qquad (\rho_\infty^2 = 6) , \\ \epsilon &= 0.013 \qquad (\rho_\infty^2 = 8) ; \end{aligned}\right\} \tag{59}$$

and, as was expected, ϵ is in fact a very small quantity.

In a first approximation ψ_1 can therefore be written as

$$\psi_1 = a e^{-\rho^2/2} \rho (1 - \epsilon\chi) , \tag{60}$$

where a denotes the normalizing factor, which can be determined numerically in any given case.

Substituting for ψ_1 from equation (60) in equation (51), we obtain

$$\left.\begin{aligned} &Q(\rho_0, \tau) \\ &\simeq \frac{a^2 \rho_\infty e^{-\rho_\infty^2}}{2\epsilon} \left[-\frac{d}{d\rho} (\rho - \epsilon\rho\chi) \right]_{\rho=\rho_\infty} [1 - \epsilon\chi(\rho_0)] (1 - e^{-\epsilon\tau}) \qquad (\tau \gtrsim 1) . \end{aligned}\right\} \tag{61}$$

It is found that for the cases $\rho_\infty^2 = 6$, respectively 8, the foregoing equation (after averaging over ρ_0) takes the simple numerical forms

$$\left.\begin{aligned} Q(\tau) &\simeq (1 - e^{-0.059\tau}) \qquad (\rho_\infty^2 = 6;\ \tau \gtrsim 1) , \\ Q(\tau) &\simeq (1 - e^{-0.013\tau}) \qquad (\rho_\infty^2 = 8;\ \tau \gtrsim 1) . \end{aligned}\right\} \tag{62}$$

Comparing the formulae (28) and (62), we see that when allowance is made for dynamical friction the mean life of a cluster is increased by factors ranging from 15 ($\rho_\infty \sim 2.5$) to 50 ($\rho_\infty \sim 2.8$). More particularly, the rates of escape given in Table 2 should be compared with those of Table 1.

It is seen that the rates of escape are sufficiently reduced to be compatible with a time scale of the order of 3×10^9 years. Physically, this drastic reduction in the rates of escape when dynamical friction is allowed for is readily understood, for dynamical friction operates essentially in the direction of preventing a star from being accelerated by too large amounts with any appreciable probability (cf. the remarks in I, § 1), and it is clearly on this account that the probability that a star will acquire the necessary high velocities for escape is so small. Further, it is to be noticed that in the mathematical analysis this reduction is brought about by the small numerical values of $(\mu_1 - 1)$, where μ_1 corresponds to the lowest quantum state of an artificially restricted simple harmonic oscillator; and, as we have seen (cf. Fig. 1) for the values of ρ_∞ which come under discussion, $(\mu_1 - 1)$ is not only a small quantity but it also depends very sensitively on the precise value of ρ_∞ (cf. the values of $[\mu_1 - 1]$ for the cases $\rho_\infty^2 = 6$ and $\rho_\infty^2 = 8$ given in eq. [62]). We may

TABLE 2

THE EXPECTATION $Q(\tau)$ FOR A STAR TO ESCAPE FROM A CLUSTER DURING A TIME τ (MEASURED IN UNITS OF THE TIME OF RELAXATION) WHEN ALLOWANCE IS MADE FOR DYNAMICAL FRICTION

τ	$Q(\tau)$		τ	$Q(\tau)$	
	$\rho_\infty^2=6$	$\rho_\infty^2=8$		$\rho_\infty^2=6$	$\rho_\infty^2=8$
5	0.26	0.064	20	0.95	0.23
10	0.44	0.12	100	....	0.73

therefore conclude that dynamical friction provides exactly the right kind of agency for preventing too rapid a disintegration of an isolated cluster; and thus, in the very existence of galactic clusters like the Pleiades, we can look for evidence not only for the operation of dynamical friction but also for the now generally adopted time scale of the order of 3×10^9 years.

4. *Remarks on further developments.*—Our discussion of the rate of escape of stars from clusters has shown that dynamical friction must be a dominating factor in the dynamics of these systems. The question now arises as to how we can incorporate in a rational system of dynamics the stochastic variations in the velocity which a star suffers on account of the fluctuating force acting on it. It is evident that to build such a system of dynamics what we need is essentially a differential equation which will be appropriate for discussing the probability distribution in *phase space* in contrast to equations of the Fokker-Planck type, which describe the situation only in the velocity space. In other words, we need a proper generalization of Liouville's equation of classical dynamics to include terms corresponding to the stochastic variations in $\boldsymbol{u}$. Such a generalized Liouville equation can be readily found.

Quite generally we may write (cf. I, eq. [6])

$$\left.\begin{aligned} \Delta \boldsymbol{u} &= \boldsymbol{K}\Delta t + \delta \boldsymbol{u}\,(\Delta t) - \eta \boldsymbol{u}\Delta t\,, \\ \Delta \boldsymbol{r} &= \boldsymbol{u}\Delta t\,, \end{aligned}\right\} \tag{63}$$

where $\boldsymbol{K}$ denotes the external force per unit mass acting on a star and the rest of the symbols have the same meanings as in I, § 1. Also, analogous to the integral equation in the velocity space (I, eq. [8]), we now have

$$\left.\begin{aligned} &W(\boldsymbol{r}, \boldsymbol{u}, t+\Delta t) \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} W(\boldsymbol{r}-\Delta \boldsymbol{r}, \boldsymbol{u}-\Delta \boldsymbol{u}, t)\Psi(\boldsymbol{r}-\Delta \boldsymbol{r}, \boldsymbol{u}-\Delta \boldsymbol{u}; \Delta \boldsymbol{r}, \Delta \boldsymbol{u})\, d(\Delta \boldsymbol{r})\, d(\Delta \boldsymbol{u})\,, \end{aligned}\right\} \tag{64}$$

where $\Psi(\boldsymbol{r}, \boldsymbol{u}; \Delta\boldsymbol{r}, \Delta\boldsymbol{u})$ denotes the transition probability in the phase space. We have (cf. I, eq. [9])

$$\left.\begin{aligned}\Psi(\boldsymbol{r}, \boldsymbol{u}; \Delta\boldsymbol{r}, \Delta\boldsymbol{u}) = \frac{1}{(4\pi q\Delta t)^{3/2}}\, & e^{-|\Delta\boldsymbol{u}-\mathrm{grad}_{\boldsymbol{u}} q\Delta t+\eta\boldsymbol{u}\Delta t-\boldsymbol{K}\Delta t|^2/4q\Delta t} \\ & \times \delta(\Delta x - u_x\Delta t)\,\delta(\Delta y - u_y\Delta t)\,\delta(\Delta z - u_z\Delta t)\,.\end{aligned}\right\} \tag{65}$$

Expanding the various terms in equation (64) in the form of Taylor series and proceeding as in usual deviation of the Fokker-Planck equation, we obtain[7]

$$\frac{\partial W}{\partial t} + \boldsymbol{u}\cdot\mathrm{grad}_{\boldsymbol{r}} W + \boldsymbol{K}\cdot\mathrm{grad}_{\boldsymbol{u}} W = \mathrm{div}_{\boldsymbol{u}}\,(q\,\mathrm{grad}_{\boldsymbol{u}} W + \eta W\boldsymbol{u})\,. \tag{66}$$

In the foregoing equation q and η can be functions of $\boldsymbol{r}$ and $\boldsymbol{u}$; they should, however, be related according to

$$\frac{q}{\eta} = \tfrac{1}{3}\overline{|\boldsymbol{u}|^2} \tag{67}$$

at all points of the phase space.

Equation (66) is the required generalization of Liouville's equation of classical dynamics, and it is on the basis of this equation that the dynamics of the galactic and the globular clusters should be developed. We shall return to these further developments on a future occasion.

[7] For details of the derivation see a forthcoming article by the writer in the *Reviews of Modern Physics.*